On the Projective Deformation of Curved Surfaces. I.

By

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(Received January 27, 1922.)

Introduction.

In the year 1916 M. G. Fubini has published a memoir on the projective deformation and discussed the condition that two surfaces may be projectively deformable and the relations between such surfaces.[1]

Afterwards a great advance was made by Cartan in the theory of the projective deformation.[2]

He noticed that the surface which admits the projective deformation is exeptional one.[3] In another words for a surface admits a projective deformation (which is not a projective transformation) certain conditions must be satisfied.

Cartan has proved this fact by aid of the theory of the differential equation, we shall express this condition in terms of the quantities which define the surface. This can be expressed in a simple form. And then we will discuss, by means of it, some geometrical characteristic properties of the surface which admits the projective deformation.

§ I. Cartan's conditions that two surfaces may be projectively deformable.

We say two surfaces (S) and (T) are projectively deformable when they are in the following relations:

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1 Rendiconti del circol matematico di Palermo. 41, 135, (1916).

2 Annales scientifiques de l'école normale supérieure. (3), 37, 259 (1920).

3 Loc. cil cit.

1° There is a one to one correspondence between the points of the surfaces (S) and (T).

2° If M and P be a pair of the corresponding points respectively on (S) and (T), we can displace (T) in the projective space (that is, apply a projective transformation) so as P coincides with M, and (T) has a contact of the second order with (S) at M.

Let us now confine ourselves to the three dimensions, and start from Cartan's conditions for the projective deformability of two surfaces.

Take the following four points:

1° A point A on the surface (S)

2° Two points A_1 and A_2 on the tangent plane at A of the surface

3° A point A_3 outside of the tangent plane.

We can regard the above four points as the vertices of the moving coordinate tetrahedron.

Let (x), (y), (z), (t), (p) be homogeneous coordinates of the points A, A_1, A_2, A_3, and any point P referring to the fixed axes.

Then the coordinates (w) of the point P referring to the moving axes are given by the equations

$$p_i = w_0 x_i + w_1 y_i + w_2 z_i + w_3 t_i. \qquad i=0,\ 1,\ 2,\ 3.$$

Referring to the fixed axes, the coordinates of the points A, A_1, A_2, A_3 will generally depend, besides the parameters u and v which define the coordinates of the point on the surface (S), on some other parameters.

When we give these parameters infinitely small variations, the coordinates referred the moving axes of the points (dx), (dy), (dz), (dt) are given by the equations

$$\begin{aligned} dx_i &= w_{00}x_i + w_{01}y_i + w_{02}z_i + w_{03}t_i, \\ dy_i &= w_{10}x_i + w_{11}y_i + w_{12}z_i + w_{13}t_i, \\ dz_i &= w_{20}x_i + w_{21}y_i + w_{22}z_i + w_{23}t_i, \\ dt_i &= w_{30}x_i + w_{31}y_i + w_{32}z_i + w_{33}t_i, \end{aligned} \qquad (i=0,\ 1,\ 2,\ 3)$$

w_{0i}, w_{1i}, w_{2i}, w_{3i} $(i=0, 1, 2, 3)$ being required coordinates.

As the points (dx) is evidently on the tangent plane, we have

$$w_{03} = o.$$

Evidently, w's are Pfaff's expressions.

For a surface (T) is projectively deformable to (S), it is necessary and sufficient that corresponding to A, A_1, A_2, A_3, we can take a point $B(\xi)$ on (T), two points $B_1(\eta)$, $B_2(\varsigma)$ on the tangent plane at B of (T), and a point $B_3(\tau)$ outside the tangent plane so as to satisfy the following system of simultaneous total differential equations

$$\left.\begin{array}{ll} W_{01}=w_{01}, & W_{02}=w_{02}, \\ W_{12}=w_{12}, & W_{21}=w_{21}, \\ W_{13}=w_{13}, & W_{23}=w_{23}, \\ \multicolumn{2}{l}{W_{00}-w_{00}=W_{11}-w_{11}=W_{22}-w_{22}.} \end{array}\right\} \qquad \text{(I)}$$

where W_{0i}, W_{1i}, W_{2i}, W_{3i} $(i=0, 1, 2, 3)$ are coordinates of the points $(d\xi)$, $(d\eta)$, $(d\varsigma)$, $(d\tau)$ respectively referring to the moving coordinate tetrahedron formed by the points $B(\xi)$, $B(\eta)$, $B_2(\varsigma)$, $B_3(\tau)$.

These are the Cartan's conditions.[1]

§ 2. Particularization of the moving coordinate axes.

Let us suppose the surface (T) is projectively deformable to the surface (S), and the parameters u, v are taken so as the corresponding points on (S) and (T) correspond to the same values of u, v, and on the surface (S) the curves u-const. and v-const. are the asymptotic lines.

Then if the surface (S) is not developpable, as showed by Wilzynski, the coordinates of the point on (S) may be regarded as the fundamental system of the solutions of the system of the simultaneous partial differential equations[2]

$$\frac{\partial^2 x}{\partial u^2}=a_1\frac{\partial x}{\partial u}+b_1\frac{\partial x}{\partial v}+c_1 x,$$

$$\frac{\partial^2 x}{\partial u^2}=a_2\frac{\partial x}{\partial u}+b_2\frac{\partial x}{\partial v}+c_2 x,$$

which are transformed by the transformation of the form

$$\bar{x}=\lambda x \qquad \text{(I)}$$

to the equations

$$\left.\begin{array}{l} \dfrac{\partial^2 x}{\partial u^2}=b\dfrac{\partial x}{\partial v}+fx \\ \dfrac{\partial^2 x}{\partial v^2}=a\dfrac{\partial x}{\partial u}+gx \end{array}\right\} \qquad \text{(II)}$$

1 Loc. cit. 272.

2 Traus. Amer. Math. Soc., **8**, 244 (1917).

where $b=b_1, \quad a=a_2.$

The functions, a, b, f, g of u, v are restricted by the conditions of the integrability:

$$\left.\begin{aligned} 2\frac{\partial g}{\partial u}&=\frac{\partial(ab)}{\partial v}+a\frac{\partial b}{\partial v}-\frac{\partial^2 a}{\partial u^2} \\ 2\frac{\partial f}{\partial v}&=\frac{\partial(ab)}{\partial u}+b\frac{\partial a}{\partial u}-\frac{\partial^2 b}{\partial v^2} \\ \frac{\partial^2 f}{\partial v^2}+b\frac{\partial g}{\partial v}+2g&=\frac{\partial^2 g}{\partial u^2}+a\frac{\partial f}{\partial u}+2f\frac{\partial a}{\partial u} \end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots \text{(III)}$$

As the transformation (1) does not alter the surface, there is no loss of generality, if we suppose the coordinates (x) of the point on (S) to satisfy the equations (II).

Let us now take $\left(\frac{\partial x}{\partial u}\right)$ for (y), $\left(\frac{\partial x}{\partial v}\right)$ for (z) and $\left(\frac{\partial^2 x}{\partial u\partial v}\right)$ for (t).

Then from the equation (II), we have

$$d\begin{vmatrix} x^0 & \frac{\partial x_0}{\partial u} & \frac{\partial x_0}{\partial v} & \frac{\partial^2 x_0}{\partial u\partial v} \\ x_1 & \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} & \frac{\partial^2 x_1}{\partial u\partial v} \\ x_2 & \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} & \frac{\partial^2 x_2}{\partial u\partial v} \\ x_3 & \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} & \frac{\partial^2 x_3}{\partial u\partial v} \end{vmatrix}=0,$$

$$\therefore \quad \begin{vmatrix} x & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial^2 x}{\partial u\partial v} \end{vmatrix}=\text{const.}$$

As we have assumed the surface (S) is not developpable, the value of the above determinant is not zero.

From equation (II), we have

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv,$$

$$d\left(\frac{\partial x}{\partial u}\right)=\frac{\partial^2 x}{\partial u^2}du+\frac{\partial^2 x}{\partial u\partial v}dv=fxdu+b\frac{\partial x}{\partial v}du+\frac{\partial^2 x}{\partial u\partial v}dv,$$

$$d\left(\frac{\partial x}{\partial v}\right)=\frac{\partial^2 x}{\partial u\partial v}du+\frac{\partial^2 x}{\partial v^2}dv=gxdv+a\frac{\partial x}{\partial u}dv+\frac{\partial^2 x}{\partial u\partial v}du,$$

$$d\left(\frac{\partial^2 x}{\partial u\partial v}\right)=\frac{\partial^3 x}{\partial u\partial v}du+\frac{\partial^3 x}{\partial u\partial v^2}dv$$

$$=\Big\{(bg+f_v)du+(af+g_u)dv\Big\}x$$

$$+\Big\{abdu+(a_u+g)dv\Big\}\frac{\partial x}{\partial u}+\Big\{(b_v+f)\,du+abdv\Big\}\frac{\partial x}{\partial v}.$$

Hence we have

$$w_{00}=w_{11}=w_{22}=w_{33}=0,$$
$$w_{01}=du,\quad w_{02}=dv,\quad w_{03}=0,$$
$$w_{10}=fdu,\quad w_{12}=bdu,\quad w_{13}=dv,$$
$$w_{20}=gdv,\quad w_{21}=adv,\quad w_{23}=du,$$
$$w_{31}=abdu+(a_u+g)dv,$$
$$w_{32}=(b_v+f)\,du+abdv,$$
$$w_{30}=(bg+f_v)\,du+(af+g_u)dv.$$

The conditions (I) become

$$\left.\begin{aligned}&W_{01}=W_{23}=du,\\&W_{02}=W_{13}=dv,\\&W_{12}=bdu,\qquad W_{21}=adv,\\&W_{00}=W_{11}=W_{22}.\end{aligned}\right\}\quad\text{...............(IV)}$$

And for the same reason as $w_{03}=0$, we have $W_{03}=0$.

§ 4. Determination of B_1, B_2, B_3.

Equation (IV) is a system of simultaneous total differential equations.

By the theorem of Frobenius[1] the conditions of integrability of (4) are that, denoting with $'$ the bilinear covariant of the original expression, the equations

$$W'_{03}=0$$
$$W'_{01}=W'_{23}=(du)'=0,$$
$$W'_{02}=W'_{13}=(dv)'=0,$$
$$W'_{12}=(bdu)',\qquad W'_{21}=(adv)',$$
$$W'_{00}=W'_{11}=W'_{22}$$

1 Crel. J., **82**, 276, (1877) Crelle journal t. 82 1877 p. 276.

hold as the consequence of (IV).

Now let us consider two expressions of Pfaff w_d, W_d such that

$$w_d = a_0 dx_0 + a_1 dx_1 + \ldots\ldots + a_n dx_n,$$
$$W_d = b_0 dx_0 + b_1 dx_1 + \ldots\ldots + b_n dx_n,$$

and denote with w_δ, W_δ the following expressions

$$w_\delta = a_0 \delta x_0 + a_1 \delta x_1 + \ldots\ldots + a_n \delta x_n,$$
$$W_\delta = b_0 \delta x_0 + b_1 \delta x_1 + \ldots\ldots + b_n \delta x_n,$$

where δx_i means another variation of x_1.

Denote with Cartan the expression $w_d W_\delta - w_\delta W_d$ with the symbol $[wW]$.

Then the bilinear covariant of W_{ij} is given by the following formulas.[1]

$$W'_{ij} = [W_{i0} W_{0j}] + [W_{i1} W_{1j}] + [W_{i2} W_{2j}] + [W_{i3} W_{3j}]. \quad \ldots\ldots\ldots (\mathrm{V})$$

By the above theorem, from the equations.

$$W'_{13} = 0,$$
$$W'_{23} = 0,$$

we have

$$[du(W_{33} - W_{22})] = 0,$$
$$[dv(W_{33} - W_{11})] = 0.$$

But $W_{22} = W_{11}$ and du, dv are independent.

Hence we conclude

$$W_{00} = W_{11} = W_{22} = W_{33}.$$

Let us suppose we have taken the coordinates (ξ), (η), (ζ), (τ) of B, B_1, B_2, B_3 respectively so as to satisfy

$$\begin{vmatrix} \xi_0 & \eta_0 & \zeta_0 & \tau_0 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 \\ \xi_3 & \eta_3 & \zeta_3 & \tau_3 \end{vmatrix} = 1,$$

this being always possible.

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[1] Cartan: Bull. math. France, **48**, 146 (1920) t. XLVIII 1920 p. 146.

Then we have

$$0=d|\ \xi\ \ \eta\ \ \zeta\ \ \tau\ |$$
$$=W_{00}+W_{11}+W_{22}+W_{33}$$
$$\therefore\quad W_{00}=W_{11}=W_{22}=W_{33}=0.$$

From $W_{00}=0$ follows by the formulas (V)

$$[duW_{10}]+[dvW_{20}]=0.$$

Hence by the theorem of Cartan,[1] W_{10}, W_{20} must be linear combinations of du and dv such that

$$W_{10}=hdu+kdv,$$
$$W_{20}=kdu+ldv,$$

where h, k, l, are certain functions of the parameters which define the coordinates of the points B, B_1, B_2, B_3.

If $k\neq 0$, we consider B_3+kB as new B_3.

Evidently by this transformation, the expression W_{ij} which appear in the conditions (I) do not alter, but W_{10} and W_{20} become $W_{10}-kW_{13}$ and $W_{20}-kW_{23}$ respectively.

Hence we can reduce W_{10}, W_{20} to the forms

$$W_{10}=\lambda du,$$
$$W_{20}=\mu dv.$$

Now we have by differentiation

$$d\xi_i=\frac{\partial\xi_i}{\partial u}du+\frac{\partial\xi_i}{\partial v}dv,$$

On the other hand, we have

$$d\xi_i=\eta_i du+\zeta_i dv.$$
$$\therefore\quad (\eta)=\left(\frac{\partial\xi}{\partial u}\right),$$
$$(\zeta)=\left(\frac{\partial\xi}{\partial v}\right).$$

From the relations

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[1] Loc. cit.

$$W_{10}=\lambda du,$$
$$W_{12}=bdu,$$
$$W_{13}=dv,$$
$$W_{11}=0,$$

we have

$$d\left(\frac{\partial \xi_i}{\partial u}\right)=\frac{\partial^2 \xi_i}{\partial u^2}du+\frac{\partial^2 \xi_i}{\partial u \partial u}dv$$
$$=\left(\lambda \xi_i+b\frac{\partial \xi_i}{\partial v}\right)du+\tau_i dv.$$
$$\therefore \qquad (\tau)=\left(\frac{\partial^2 \xi}{\partial u \partial v}\right),$$
$$\frac{\partial^2 \xi_i}{\partial u^2}=b\frac{\partial \xi_i}{\partial v}+\lambda \xi_i.$$

From the relations

$$W_{20}=dv,$$
$$W_{21}=adv,$$
$$W_{22}=0,$$
$$W_{23}=du,$$

we have

$$d\left(\frac{\partial \xi_i}{\partial v}\right)=\frac{\partial^2 \xi_i}{\partial u \partial v}du+\frac{\partial^2 \xi_i}{\partial v^2}dv$$
$$=\left(\mu\, \xi_1+a\frac{\partial \xi_1}{\partial u}\right)dv+\frac{\partial^2 \xi_9}{\partial u \partial v}du.$$
$$\therefore \qquad \frac{\partial^2 \xi_i}{\partial v^2}=a\frac{\partial \xi_i}{\partial v}+\mu\, \xi_i.$$

Hence we know that (ξ) must be a system of the fundamental solutions of a partial differential equation

$$\left.\begin{aligned}\frac{\partial^2 y}{\partial u^2}&=b\frac{\partial y}{\partial v}+\lambda y,\\ \frac{\partial^2 y}{\partial v^2}&=a\frac{\partial y}{\partial u}+\mu y.\end{aligned}\right\}\dots\dots\dots\dots\dots\dots \text{(VI)}$$

The conditions of integrability of the system (VI) are given by

$$\left.\begin{aligned} 2\frac{\partial\lambda}{\partial v}&=\frac{\partial(ab)}{\partial u}+b\frac{\partial a}{\partial u}-\frac{\partial^2 b}{\partial v^2},\\ 2\frac{\partial\mu}{\partial u}&=\frac{\partial(ab)}{\partial v}+a\frac{\partial b}{\partial v}-\frac{\partial^2 a}{\partial u^2},\end{aligned}\right\}\text{..............} \quad \text{(VII)}$$

$$\frac{\partial^2\lambda}{\partial v^2}+b\frac{\partial\mu}{\partial v}+2\mu\frac{\partial b}{\partial v}=\frac{\partial^2\mu}{\partial u^2}+a\frac{\partial\lambda}{\partial u}+2\lambda\frac{\partial a}{\partial u}.$$

We can easily prove by preceding method that (VII) are also the conditions of integrability of the system of simultaneous total differential equations (IV).

Reciprocally, if (ξ) be a system of the fundamental solutions of the system (VI), we can, by taking (ξ) for B, $\left(\frac{\partial\xi}{\partial u}\right)$ for B_1 and $\left(\frac{\partial\xi}{\partial v}\right)$ for B_2 and $\frac{\partial^2\xi}{\partial u\partial v}$ for B_3, easily prove that the surface generated by the point whose coordinates are (ξ) is projectively deformable to (S).

Hence we know that *for two nondeveloppable surface are projectively deformable to each other, it is necessary and sufficient that Wilczynski's invariant a and b have equal values at the corresponding points of the two surfaces.*

If one of a and b are equal to zero, the surface is a ruled surface.

Hence we know that the surface which are projectively deformable to a curved surface are also curved. We will hereafter consider only the curved surface, that is, we assume $a\neq 0$, $b\neq 0$.

§ 4. Condition that a curved surface may admit projective deformation.

From the first and second equations of (III) and (VII), we have

$$\frac{\partial\lambda}{\partial v}=\frac{\partial f}{\partial v},$$

$$\frac{\partial u}{\partial u}=\frac{\partial g}{\partial u},$$

$$\therefore \quad \lambda=f+\{U(u)\}^2,$$

$$\mu=g+\{V(v)\}^2,$$

where $U(u)$ is a function of u only and $V(v)$ is a function of v only.

Substituting these values in the third equations of (III) and (VII), we have

$$U(u)\left\{U(u)\frac{\partial a}{\partial u}+U'(u)a\right\}=V(v)\left\{V(v)\frac{\partial b}{\partial v}+V'(v)b\right\}.$$

Now we have various cases.

Case I. If

$$U(u)=V(v)=0,$$

we have

$$\lambda=f,$$

$$\mu=g.$$

In this case (T) is a mere projective transform of (S).

Case II. If only one of the $U(u)$ and $V(v)$ are equal to zero, e.g., if $V(v)=0$, then we have

$$U(u)\frac{\partial a}{\partial u}+U'(u)a=0,$$

$$\frac{\partial}{\partial u}(Ua)=0.$$

Hence a takes the form

$$a=\frac{fn(v)}{U(u)}=\frac{U(v)}{U(u)}(say).$$

Similarly if $U(u)=0$, b takes the form

$$b=\frac{fn(u)}{V(v)}=\frac{U(u)}{V(v)}\ (say).$$

By the transformation of the form

$$\bar{x}=C\sqrt{UV}x, \qquad d\bar{u}=U(u)du, \qquad d\bar{v}=V(v)dv,$$

the surface and also the parameter curves do not alter and the system (II) is transformed into another of the same form in which, denoting with $\bar{a}$, $\bar{b}$ the corresponding quantities, we have[1]

$$\bar{a}=\frac{U(u)}{\{V(v)\}^2}, \qquad \bar{b}=\frac{V(v)}{\{U(u)\}^2}.$$

Hence in the former case by the transformations

$$\bar{x}=C\sqrt{U}\sqrt[4]{V}x \qquad d\bar{u}=Udu, \qquad d\bar{v}=\sqrt{V}dv$$

we have

$$\bar{a}=1.$$

In the latter case by the tranformations

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1 Wilczynski, loc. cit.

$$\bar{x}=C\sqrt[4]{U}\sqrt{V}x, \qquad d\bar{u}=\sqrt{U}du, \qquad dv=Vd\bar{v},$$

we have

$$b=1.$$

Case III. If $U(u)\neq 0$, $V(v)\neq 0$, we have

$$\frac{U(u)\frac{\partial a}{\partial u}+U'(u)a}{V(v)}=\frac{V(v)\frac{\partial b}{\partial v}+V'(v)b}{U(u)},$$

$$\frac{\partial}{\partial u}\left(\frac{U(u)a}{V(v)}\right)=\frac{\partial}{\partial v}\left(\frac{V(v)b}{U(u)}\right).$$

Hence we can determine by a quadrature a function p which satisfies the relation

$$a=\frac{V(v)}{U(u)}\ \frac{\partial p}{\partial v},$$

$$b=\frac{U(u)}{V(v)}\ \frac{\partial p}{\partial u}.$$

By the transformations

$$\bar{x}=C\sqrt{UV}x, \qquad d\bar{u}=Udu, \qquad d\bar{v}=Vdv,$$

we have

$$\bar{a}=\frac{U(u)}{\{V(v)\}^2}a, \qquad \bar{b}=\frac{V(v)}{\{U(u)\}^2}b.$$

But

$$\frac{\partial p}{\partial v}=\frac{\partial p}{\partial \bar{v}}V(v), \qquad \frac{\partial p}{\partial u}=\frac{\partial p}{\partial \bar{u}}U(u).$$

Hence, we have

$$\bar{a}=\frac{\partial p}{\partial \bar{v}}, \qquad \bar{b}=\frac{\partial p}{\partial \bar{u}}.$$

Therefore, we know that *for the curved surfaces which admit the projective deformation, Wilczynski's invariants, a, b must be reduced so as to satisfy one of the relations.*

(1) $a=1.$

(2) $b=1.$

(3) $\dfrac{\partial a}{\partial u}=\dfrac{\partial b}{\partial v}.$

Reciprocally we will prove in the next paragraph that if a surface satisfies one of the above conditions, there are at least ∞^1 surfaces which are projectively deformable to the surface (S).

Therefore there are three kinds of surfaces which admit the projective transformation. But the surface where $a=1$ and the surface where $b=1$ are of the same kind, for by the interchange of parameters u, v, we can reduce one into the other. We shall call the surface of this kind the surface of the second kind which admits the projective transformation and the surface where $a_u=b_v$ the surface of the first kind which admits the projective transformation.

§ 5. All the surfaces which are projectively deformable to a given surface.

Let us suppose that we have performed the transformations explained in the preceding paragraph.

First, we consider the case where

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v},$$

or

$$a=\frac{\partial p}{\partial v}, \qquad b=\frac{\partial p}{\partial u}.$$

If the equations of the given surface be

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+fx,$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+gx,$$

where a, b, f and g are restricted by the conditions of the integrability, then all the surfaces which are projectively deformable to it, if they exist, are given by

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+\lambda x,$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+\mu x,$$

where λ, u must satisfy the conditions of integrability

$$\frac{\partial \lambda}{\partial v}=\frac{\partial f}{\partial v},$$

$$\frac{\partial \mu}{\partial u}=\frac{\partial g}{\partial u},$$

$$\frac{\partial^2\mu}{\partial u^2}+a\frac{\partial\lambda}{\partial u}+2\lambda\frac{\partial a}{\partial u}=\frac{\partial^2\lambda}{\partial v^2}+b\frac{\partial\mu}{\partial v}+2\mu\frac{\partial b}{\partial v}.$$

If we assume

$$\lambda=f+U(u),$$
$$u=g+V(v),$$

the functions $U(u)$, $V(v)$ must satisfy the equation

$$aU'(u)+2U\frac{\partial a}{\partial u}=bV'(v)+2V\frac{\partial b}{\partial v},$$

or

$$2(U-V)\frac{\partial^2 p}{\partial u\partial v}-V'(v)\frac{\partial p}{\partial u}+U'(u)\frac{\partial p}{\partial v}=0. \quad \text{...........(1)}$$

This equation is satisfied by

$$U=V=k. \qquad k: \text{const.}$$

Hence the surfaces which are given by

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial u^2}&=b\frac{\partial x}{\partial v}+(f+k)x,\\ \frac{\partial^2 x}{\partial v^2}&=a\frac{\partial x}{\partial v}+(g+k)x,\end{aligned}\right\}\text{.................. (2)}$$

are projectively deformable to the given surface.

By Wilczynski's theory,[1] all the surfaces which correspond to the same value of k are projective transforms of a particular one, and moreover we can easily see that two surfaces which correspond to the different values of k can not be projective transforms of the same surface.

Therefore, if we regard the surfaces which may be derived by the projective transformation from the same surface as all the same, the equation (1) gives ∞^1 surfaces which are projectively deformable to the given surface.

If there be other pairs of solutions of $U(u)$ and $V(v)$, p must be a solution of the partial differential equation of the form (1).

If p be a particular solution of

$$2\Big(\alpha(u)-\beta(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(v)\frac{\partial p}{6v}=0,$$

[1] Loc. cit.

$$U(u)=k\alpha(u)+k_1, \qquad k,\ k_1 : \text{const.}$$
$$V(v)=k\beta(v)+k_1,$$

are evidently solutions of (1).

If these exist a further pair of solutions of $U(u)$ and $V(v)$, p must satisfy the two partial differential equations

$$2\Big(\alpha(u)-\beta(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(u)\frac{\partial p}{\partial v}=0,$$
$$2\Big(U(u)-V(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-V'(v)\frac{\partial p}{\partial u}+U'(u)\frac{\partial p}{\partial v}=0,$$

where

$$U(u)\not\equiv k\alpha(u)+k_1, \qquad V(v)\not\equiv k\beta(v)+k_1.$$

By the transformotion

$$x=\alpha(u), \qquad \eta=\beta(v),$$

the above equation is transformed to

$$\left.\begin{aligned} &2(x-y)\frac{\partial^2 p}{\partial x\partial y}-\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}=0,\\ &2\Big(X(x)-Y(y)\Big)\frac{\partial^2 p}{\partial x\partial y}-Y'(x)\frac{\partial p}{\partial x}+X'(x)\frac{\partial p}{\partial y}=0,\end{aligned}\right\}\cdots\cdots\cdots\cdots(3)$$

where

$$X(x)\equiv U(u), \qquad Y(y)\equiv V(v),$$
$$X(x)\not\equiv kx+k_1, \quad Y(y)\not\equiv ky+k_1.$$

From the equations (3), we have

$$-\Big(X-Y-Y'(x-y)\Big)\frac{\partial p}{\partial x}+\Big(X-Y-X'(x-\eta)\Big)\frac{\partial p}{\partial y}=0. \dots(4)$$

Consider the function

$$q=2\int Xdx-2\int Ydy-(X+Y)(x-y).$$

Then the equation (4) is equivalent to

$$\frac{\partial(p,\ q)}{\partial(x,\ y)}=0.$$
$$\therefore \qquad p=F(q).$$

Substituting this in the first equation of (3), we have

$$2(x-y)F''(q)\Big\{X-Y-X'(x-\eta)\Big\}\Big\{X-Y-Y'(x-\eta)\Big\}$$
$$+3F'(q)(x-y)(X'-Y')=0. \quad\ldots\ldots\ldots\ldots(5)$$

As x and y are independent $x-y$ is not always zero. If one of the functions $X-Y-X'(x-y)$, and $X-Y-Y'(x-y)$ vanish identically, by the relation (4) the other must vanish, for

$$\frac{\partial p}{\partial x}=\frac{a}{\alpha'(u)}\neq 0,$$
$$\frac{\partial p}{\partial y}=\frac{b}{\beta'(v)}\neq 0.$$

And if both $X-Y-X'(x-y)=0$ and $X-Y-Y'(x-y)=0$ vanish identically, we can easily see that

$$X=kx+k,$$
$$Y=ky+k.$$

This contradicts our assumption.
Hence,

$$X-Y-X'(x-y)\neq 0,$$
$$X-Y-Y'(x-y)\neq 0.$$

Therefore, from (5) we have

$$\frac{2}{3}\frac{F''(q)}{F'(q)}\Big\{X-Y-Y'(x-y)\Big\}=-\frac{X'-Y'}{X-Y-X'(x-y)},$$
$$\frac{2}{3}\frac{F''(q)}{F'(q)}\Big\{X-Y-X'(x-q)\Big\}=-\frac{X'-Y'}{X-Y-X'(x-y)},$$

or

$$\frac{2}{3}\frac{\partial}{\partial y}\log F'(q)=-\frac{\partial}{\partial y}\log\Big(X-Y-Y'(x-y)\Big),$$
$$\frac{2}{3}\frac{\partial}{\partial x}\log F'(q)=-\frac{\partial}{\partial x}\log\Big(X-Y-Y'(x-y)\Big).$$

Integrating we have

$$\Big\{F'(q)\Big\}^{\frac{2}{3}}=\frac{\xi(x)}{X-Y-X'(x-y)}=\frac{\eta(y)}{X-Y-Y'(x-y)} \quad\ldots\ldots\ldots (6)$$

From the equation

$$0=\frac{\partial^2}{\partial x\partial y}\log\left(\frac{\xi(x)}{\eta(y)}\right)=\frac{\partial^2}{\partial x\partial y}\log\left(\frac{X-Y-X'(x-y)}{X-Y-Y'(x-y)}\right),$$

we can easily deduce

$$\begin{aligned}\xi(x)&=-kX''^{\frac{1}{3}},\\ \eta(y)&=+kY''^{\frac{1}{3}}.\end{aligned}\qquad k:\text{const.}$$

If either X'' and Y'' be zero, $F'(q)=0$, accordingly $a=b=0$.
Hence $X''\neq 0$, $Y''\neq 0$.
Substituting these in (6), we have

$$\frac{X-Y}{x-y}=\frac{X'Y''^{\frac{1}{3}}+Y'X''^{\frac{1}{3}}}{X''^{\frac{1}{3}}+Y''^{\frac{1}{3}}}=f(x,\ y),$$

$$X-Y=f(x,\ y)\ (x-y),$$

$$X'=f(x,\ y)+\frac{\partial f}{\partial x}(x-y),$$

$$Y'=f(x,\ y)-\frac{\partial f}{\partial y}(x-y),$$

$$\frac{\partial f}{\partial x}Y''^{\frac{1}{3}}-X''^{\frac{1}{3}}\frac{\partial f}{\partial y}=0. \qquad\qquad (7)$$

Consider the function

$$w=\int X''^{\frac{1}{3}}dx+\int Y^{\frac{1}{3}}dy.$$

Then (7) is equivalent to

$$\frac{\partial(f,\ w)}{\partial(x,\ y)}=0.$$

$$\therefore\quad f=\varphi(w).$$

$$X'=\varphi(w)+\varphi'X''^{\frac{1}{3}}(x-y).$$

Differentiating the above equation by y, we have

$$\frac{X''^{\frac{1}{3}}-Y''^{\frac{1}{3}}}{(x-y)X''^{\frac{1}{3}}Y''^{\frac{1}{3}}}=\frac{\varphi''(w)}{\varphi'(w)}=fn(w).$$

From the equation

$$\frac{\partial(fn(w)\,w)}{\partial(x,y)}=0,$$

we have

$$\frac{1}{3}(x-y)\left(X''^{-\frac{5}{3}}X'''+Y''^{-\frac{5}{3}}Y'''\right)+X''^{-\frac{2}{3}}-Y''^{-\frac{2}{3}}=0. \quad \ldots\ldots(8)$$

Differentiate the above equation by x and then by y, then we have

$$\frac{d}{dy}(Y''^{-\frac{5}{3}}Y''')-\frac{d}{dx}(X''^{-\frac{5}{3}}X''')=0.$$

$$\therefore \qquad \frac{d}{dy}(Y''^{-\frac{5}{3}}Y''')=-3c,$$

c : const.

$$\frac{d}{dx}(X''^{-\frac{5}{3}}X''')=-3c.$$

or

$$X''^{-\frac{5}{3}}X'''=-3(cx+c_1),$$

$$Y''^{-\frac{5}{3}}Y'''=-3(cy+D_1),$$

$$Y''^{-\frac{2}{3}}=cx^2+2c_1x+c_2,$$

$$Y''^{-\frac{2}{3}}=cy^2+2D_1y+D_2.$$

By the equation (8), we know

$$c_1=D_1,$$
$$c_2=D_2.$$

Therefore

$$X''^{\frac{1}{3}}=\frac{\varepsilon}{\sqrt{cx^2+2c_1x+c_2}},$$

$\varepsilon=+1$ or -1.

$$Y''^{\frac{1}{3}}=\frac{\delta}{\sqrt{cy^2+2c_1y+c_2}}.$$

$\delta=+1$ or -1.

First, we assume $cc_2-c_1\neq 0$.
Then we have

$$X'=\frac{\varepsilon}{cc_2-c_1^2}\left(\frac{cx+c_1}{\sqrt{cx^2+2cx+c_2}}+c_3\right)$$

$$Y' = \frac{\delta}{c\,c_2 - c_1^2}\left(\frac{c\,y + c_4}{\sqrt{cy^2 + 2c_1 y + c_2}} + D_3\right),$$

$$X = \frac{\delta}{c\,c_2 - c_2^2}\left(\sqrt{cx^2 + 2c_1 x + c_2} + c_3 x + c_4\right),$$

$$Y = \frac{\varepsilon}{c\,c_2 - c_1^2}\left(\sqrt{cy^2 + 2c_1 y + c_2} + D_3 y + D_4\right).$$

By the equation (6), we know

$$\varepsilon = \delta,$$
$$c_3 = D_3,$$
$$c_4 = D_4.$$

Therefore

$$X - Y - X'(x-y) = \frac{\varepsilon}{cc_2 - c_1^2}\left(-\sqrt{cy^2 + 2c_1 y + c_2} + \frac{cxy + c_1(x+y) + c_2}{\sqrt{cx^2 + 2c_1 x + c_2}}\right),$$

$$X - Y - Y'(x-y) = \frac{\varepsilon}{cc_2 - c_1^2}\left(\sqrt{cx^2 + 2c_1 x + c_2} - \frac{cxy + c_1(x+y) + c_2}{\sqrt{cy_2 + 2c_1 y + c_2}}\right),$$

$$\left(F'(q)\right)^{-\frac{2}{3}} =$$
$$\frac{K}{cc_2 - c_1^2}\left[\sqrt{cy^2 + 2c_1 y + c_2}\,\sqrt{cx^2 + 2c_1 x + c_2} - \{cxy + c_1(x+y) + c_2\}\right].$$

$$p = F(q) = K \log \frac{\sqrt{\sqrt{\xi^2 + a} - \xi} - \sqrt{\sqrt{\eta^2 + a} - \eta}}{\sqrt{\sqrt{\xi^2 + a} - \xi} + \sqrt{\sqrt{\eta^2 + a} - \eta}} \qquad \text{..............(9)}$$

where

$$\xi = cx + c_1, \qquad \eta = cy + c_1,$$
$$a = cc_2 - c_1^2,$$

K : certain constant.

Similarly when $cc_2 - c_1^2 = 0$, we have

$$X = \frac{\varepsilon}{2\sqrt{c}}\frac{1}{cx + c_1} c_3 x + c_4,$$

$$Y = \frac{\varepsilon}{2\sqrt{c}}\frac{1}{cy + c_1} + c_3 y + c_4 \qquad \varepsilon = +1, \text{ or } -1.$$

$$p = K \log \frac{\sqrt{cx + c_1} - \sqrt{cy + c_1}}{\sqrt{cx + c_1} + \sqrt{cy + c_1}}. \qquad K : \text{const.} \text{ ....(9')}$$

Therefore we know that when there are the following relations between a and b,

$$a = \frac{\partial p}{\partial v} \quad ; \qquad b = \frac{\partial p}{\partial u},$$

(1°) if p does not satisfy the partial differential equation of the form

$$2\left(\alpha(u)-\beta(v)\right)\frac{\partial^2 p}{\partial u \partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(u)\frac{\partial p}{\partial u}=0, \quad \text{..................(10)}$$

there are only ∞^1 surfaces which are projectively deformable to the given surface and they are given by the equations

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+(f+k)x,$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+(g+k)x,$$

(2°) if p satisfies the equations (10) and does not satisfy another partial differential equation of the same form, but not identical with it, there are ∞^2 surfaces which are projectively deformable to the given surface and they are given by the equations

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+\left(f+k\alpha(u)+k_1\right),$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+\left(g+k\beta(v)+k_1\right),$$

(3°) if p satisfies the following two partial diffential equations

$$2\left(\alpha(u)-\beta(v)\right)\frac{\partial^2 p}{\partial u \partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(u)\frac{\partial p}{\partial r}=0,$$

$$2\left(U(u)-V(v)\right)V'(v)\frac{\partial^2 p}{\partial u}+U'(u)\frac{\partial p}{\partial v}=0,$$

we must have

$$\begin{cases} p=K\log\dfrac{\sqrt{\sqrt{c^2\alpha^2+2cc_1+cc_2}-(c\alpha+c_1)}-\sqrt{\sqrt{c^2\beta+2cc_1\beta+cc_2}-(c\alpha+c_1)}}{\sqrt{\sqrt{c^2\alpha^2+2c_1c\alpha+c_2c}-(c\alpha+c_1)}+\sqrt{\sqrt{c^2\beta+2a_1c\beta+c_2c}-(c\alpha+c_1)}}, \\ U(u)=\dfrac{\pm 1}{cc_2-c_1^{\,2}}\left(\sqrt{c\alpha+2c_1\alpha+c_2}+c_3\alpha+c_4\right), \\ V(v)=\dfrac{\pm 1}{cc_2-c_1^{\,2}}\left(\sqrt{c\beta+2c_1\beta+c_2}+c_3\beta+c_4\right), \\ \text{where} \quad cc_2-c_1^{\,2}\neq 0 \end{cases}$$

or

$$\begin{cases} p = K \log \dfrac{\sqrt{c\alpha + c_1} - \sqrt{c\beta + c_1}}{\sqrt{c\alpha + c_1} + \sqrt{c\beta + c_1}}, \\ U(u) = \dfrac{\pm 1}{2\sqrt{c}} \dfrac{1}{c\alpha + c_1} + c_3\alpha + c_4, \\ V(v) = \dfrac{\pm 1}{2\sqrt{c}} \dfrac{1}{c\beta + c_1} + c_3\beta + c_4, \end{cases}$$

and all the surfaces which are projectively deformable to the given surfaces are given by the equations

$$\begin{cases} \dfrac{\partial^2 x}{\partial u^2} = b\dfrac{\partial x}{\partial v} + \left(f + k\sqrt{c\alpha^2 + 2c_1\alpha + c_2} + k_1\alpha + k_3\right), \\ \dfrac{\partial^2 x}{\partial v^2} = a\dfrac{\partial x}{\partial u} + \left(g + k\sqrt{c\beta^2 + 2c_1\beta + c_2} + k_1\beta + k_3\right), \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 x}{\partial u^2} = b\dfrac{\partial x}{\partial v} + \left(f + \dfrac{k}{c\alpha + c_1} + k_1 d + k_3\right), \\ \dfrac{\partial^2 x}{\partial v^2} = a\dfrac{\partial x}{\partial u} + \left(g + \dfrac{k}{c\beta + c_1} + k_1\alpha + k_3\right), \end{cases} \qquad \text{respectively}$$

where k, k_1, k_2 are arbitrarly constants.

Next, we consider the case where

$$a = 1 .$$

In this case the equations of the given surface are

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + fx,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + gx.$$

Therefore, if we put

$$\lambda = f + U(u),$$
$$\mu = g + V(v).$$

U, V must satisfy the equation

$$U'(n) = bV'(v) + 2V\frac{\partial b}{\partial v}. \qquad \qquad (4)$$

This equation is satisfied by

$$U(u) = k,$$
$$V(v) = 0.$$

If there is another pair of solutions of U and V, b must satisfy the partial differential equation of the form (4).
If b satisfies the equation

$$\alpha'(u) = b\beta'(v) + 2\beta\frac{\partial b}{\partial v}, \quad \text{.................................}(5)$$

Then
$$U = k\alpha + k_1,$$
$$V = k\beta \qquad\qquad k,\ k_1 : \text{const.}$$

are solutions of (4).

If (4) has a further pair solutions, b must satisfy the two partial differential equations

$$\alpha'(u) = b\beta'(v) + 2\beta\frac{\partial b}{\partial v},$$
$$V'(u) = bV'(v) + 2V\frac{vb}{\partial v},$$

from which we can easily deduce

$$b = \frac{\alpha'\left(\int\frac{dv}{\sqrt{\beta}} + c\right)}{2\sqrt{\beta}},$$

$$U(u) = k_1\alpha(u) + k_2,$$

$$V(v) = \frac{k\beta(v)}{\left(\int\frac{dv}{\sqrt{\beta}} + c\right)^2}.$$

Therefore, we know that, when we have

$$a = 1,$$

(1°) if b does not satisfy the partial differential equation of the form

$$\alpha'(u) = b\beta'(v) + 2\beta(v)\frac{\partial b}{\partial v},$$

there are only ∞^1 surfaces which are projectively deformable to the given surface, and they are given by the equations

$$\frac{\partial^2 x}{\partial u_2} = b\frac{\partial x}{\partial v} + (f + k)x,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + gx,$$

(2°) if b satisfies the above partial differential equation, but not equation of the same form which is not identical with it, there are ∞^2 surfaces which are projectively deformable to the given surface and they are given by the equations

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + (f + ka + k_1)x,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + (g + k\beta)x,$$

(3)° if b satisfies the following two partial differential equations

$$a'(u) = b\beta'(v) + 2\beta\frac{\partial b}{\partial v},$$

$$U'(u) = bV'(v) + 2V\frac{\partial b}{\partial v},$$

we must have

$$b = \frac{a'\left(\int \frac{dv}{\sqrt{\beta}} + c\right)}{2\sqrt{\beta}},$$

$$U(u) = \quad k_1 a(u) + k_2,$$

$$V(v) = \frac{k\beta(v)}{\left(\int \frac{dv}{\sqrt{\beta}} + c\right)^2} + k_1\beta(v),$$

and all the surfaces which are projectively deformable to the given surface are given by the equations

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + (f + k_1 a + k_2)x,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + \left(g + \frac{k\beta}{\left(\int \frac{dv}{\sqrt{\beta}} + c\right)^2} + k_1\beta\right),$$

which are of the manifoldness ∞^3.

§ 6. Geometrical interpretation.

We will now investigate the geometrical properties which are peculiar to the surface which admits projective deformation.

Let us begin from the case where

$$\frac{\partial a}{\partial u} = \frac{\partial b}{\partial v}.$$

In this case the conjugate net of the curves

$$du^2 - dv^2 = 0,$$

has very interesting properties.

We shall call with Carton this conjugate net the conjugate net of the projective deformation.

In the subsequent discussion we suppose the equations of the surface are

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + fx,$$

$$\frac{\partial^2 x}{\partial v^2} = a\frac{\partial x}{\partial u} + gx.$$

Recently Green has published a Memoire on reciprocal congruence.

Let us denote by C_u *a* curve $v = \text{const.}$, by C_v *a* curve $u = \text{const.}$, by R_u the ruled surface formed by the tangents to the curve C_u at the points of a fixed curve C_v and by R_v the similar parametric ruled surfaces formed by tangents to the curve C_v.

The congruence formed by the line l which connects two points

$$(\rho) = \left(\frac{\partial x}{\partial u}\right) - \beta(x), \qquad (\sigma) = \left(\frac{\partial x}{\partial v}\right) - \alpha(x),$$

(where α, β are functions of u, v)
are called the congruence Γ.

The two tangent planes to R_u and R_v at (ρ) and (σ) respectively intersect at the line l' which connects (x) and the point (z)

$$(z) = \frac{\partial^2 x}{\partial u \partial v} = \alpha\left(\frac{\partial x}{\partial u}\right) - \beta\left(\frac{\partial x}{\partial v}\right).$$

Reciprocally the plane determined by (z) and the tangent of C_u at (x) touches R_u at (ρ), and the plane determined by (z) and the tangent of C_v at (α) touches R_v at (σ).

We say one of the lines l, l' is the reciprocal line of the other.

The congruence formed by l' is called the congruence Γ', and we speak of the congruences Γ and Γ' as reciprocal congruences.

Green has proved that when and only when

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1 Trans. Amer. Math. Soc., **20**, 79–153, (1919).

$$\frac{\partial \alpha}{\partial u}=\frac{\partial \beta}{\partial v},$$

the developpables of the congruence Γ correspond to a conjugate net of the surface and the developpables of the congruence Γ' cut the surface in a conjugate net.

For the subsequent use, we shall give a short note on the directrix congruence, ray congruence, and axis congruence.

(1) Directrix congruence.[1]

The two linear complexes which osculate at a point (x) the two asymptotics passing through (x) have in common a linear congruence with directrices d and d'.

The directrix of the first kind, d, lies in the tangent plane of the surface, and connects two points

$$\left(\frac{\partial x}{\partial u}\right)-\frac{a_u}{2a}(x), \qquad \left(\frac{\partial x}{\partial v}\right)-\frac{b_v}{2b}(x).$$

The directrix of the second kind, d', passes through (x) and through the point

$$\left(\frac{\partial^2 x}{\partial u \partial v}\right)-\frac{b_v}{2b}\left(\frac{\partial x}{\partial u}\right)-\frac{a_u}{2a}\left(\frac{\partial x}{\partial v}\right).$$

The congruences formed by d and d' are called the congruences of the first and second kind respectively. They are evidently reciprocal congruences.

(2) Ray congruence.[2]

Wilczynski has called the line joining the minus first and first Darboux-Laplace transforms of the point (x) with respect to a conjugate net the ray of the point (x), and the congruence formed by the ray the ray congruence of the conjugate net.

(3) Axis congruence.

Wilczynski has called the intersection of the osculating planes of the two curves of a conjugate net which meet at (x), the axis of the point (x) with respect to that conjugate net, and the congruence formed by the axis the axis congruence.

We shall call the reciprocal line of the axis as the reciprocal axis.

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1 Wilczynski, projective Trans. Amer. Math. Soc., **9**, 79–120, (1908).

2 Trans. Amer. Math. Soc., **16**, 311–327, (1915).

If

$$\mu\, du^2 - dv^2 = 0.$$

be the differential equation of a conjugate net, the ray of the point (x) with respect to that conjugate net is the line which joins the two points

$$\left(\frac{\partial x}{\partial u}\right) + \frac{1}{4\mu}\left(2a\,\mu^2 + \frac{\partial\mu}{\partial u}\right)(x), \qquad \left(\frac{\partial x}{\partial v}\right) + \frac{1}{4\mu}\left(2b - \frac{\partial\mu}{\mu v}\right)(x),$$

the axis is the line which passes through (x) and through the point

$$\left(\frac{\partial^2 x}{\partial u\,\partial v}\right) - \frac{1}{4\mu}\left(2b + \frac{\partial\mu}{\partial v}\right)\left(\frac{\partial x}{\partial u}\right) - \frac{1}{4\mu}\left(2a\mu^2 - \frac{\partial\mu}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right),$$

and the reciprocal axis is the line which connects the points

$$\left(\frac{\partial x}{\partial u}\right) - \frac{1}{4\mu}\left(2a\mu^2 - \frac{\partial\mu}{\partial u}\right)(x), \qquad \left(\frac{\partial x}{\partial v}\right) - \frac{1}{4\mu}\left(2b + \frac{\partial\mu}{\partial v}\right)(x).$$

For the conjugate net of the projective deformation, we have

$$\mu = 1$$

Therefore the ray of (x) with respect to that net is the line which connects the two points

$$\left(\frac{\partial x}{\partial u}\right) + \frac{a}{2}(x)\ , \qquad \left(\frac{\partial x}{\partial v}\right) + \frac{b}{2}(x)$$

and the reciprocal axis of (x) with respect to that net is the line which connects the two points

$$\left(\frac{\partial x}{\partial u}\right) - \frac{a}{2}(x)\ , \qquad \left(\frac{\partial x}{\partial v}\right) - \frac{b}{2}(x).$$

The line which is *a* harmonic conjugate of the line which joins (x) and the intersection of the reciprocal axis and the ray of (x) with respect to the ray and the reciprocal axis, passes through the points

$$\left(\frac{\partial x}{\partial u}\right)\ , \qquad \left(\frac{\partial x}{\partial v}\right)\ .$$

The developpables of the congruence formed by this line evidently correspond to *a* conjugate net.

Now for the surface of the first kind which admits the projective deformation, we have

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v}.$$

Therefore, in this case the intersection of the said line and the directrix of the first kind is

$$a\left(\frac{\partial x}{\partial u}\right)-b\left(\frac{\partial x}{\partial v}\right).$$

The intersection of the ray and the reciprocal axis with respect to the conjugate net of the projective deformation is

$$b\left(\frac{\partial x}{\partial u}\right)-a\left(\frac{\partial x}{\partial v}\right).$$

Therefore, we know that on the surface of the first kind which admits the projective deformation, there is a conjugate net which has the following properties.

1° Denote by l_1 and l_2 the ray and reciprocal axis of (x) with respect to that conjugate net respectively, by l_3 the line which joins (x) and the intersection of l_1 and l_2, and by l_4 the fourth harmonic line of l_1, l_2, l_3.

Then the developables of the congruence formed by l_4 correspond to a conjugate net.

2° Denote by P_1 and P_2 the points where the asymptotic tangents meet l_4, by P_3 the intersection of l_1 and l_2, by P_4 the intersection of the directrix of the first kind and l_4, and by D_1 and D_2, the points where the tangents at (x) to that conjugate net meet l_4.

Then P_3, P_4 are separated harmonically by D_1 and D_2.

Or we may express it in the following manner:—

The points P_1, P_2 and P_3, P_4 form two pairs of involution of which D_2 and D_1 are double points.

Reciprocally, if there be a conjugate net which has the said properties on a surface, that surface admits the projective deformation and is of the first species, and that conjugate net is that of the projective deformation.

Let

$$\mu\, du^2-dv^2=0.$$

be the differential equation of that conjugate net.

Then the line l_4 with respect to that conjugate net connects the two points

$$\left(\frac{\partial x}{\partial u}\right)+\frac{1}{4\mu}\frac{\partial\mu}{\partial u}(x), \qquad \left(\frac{\partial x}{\partial v}\right)-\frac{1}{4\mu}\frac{\partial\mu}{\partial v}(x).$$

As the developpables of the congruence of l_3 correspond to a conjugate net, we have

$$-\frac{\partial}{\partial u}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial u}\right),$$

or $$\frac{\partial^2}{\partial u\partial v}\log\mu=0,$$

$$\mu=\frac{\{U(u)\}^2}{\{V(v)\}^2}.$$

where $U(u)$ is *a* function of u only, $V(v)$ is *a* function of v only.

Perform the transformation

$$\bar{x}=c\sqrt{UV}, \qquad d\xi=Udu, \qquad d\eta=Vdv.$$

which transforms the differential equation of the above conjugate net into

$$d\xi^2-d\eta^2=0.$$

Let

$$\frac{\partial^2\bar{x}}{\partial\xi^2}=\bar{b}\frac{\partial\bar{x}}{\partial\eta}+\bar{f}\,\bar{x}=0$$

$$\frac{\partial^2\bar{x}}{\partial\eta^2}=\bar{a}\frac{\partial\bar{x}}{\partial\xi}+\bar{g}\,\bar{x}=0$$

be the transformed equations of the surface.

Then by the second property of the above conjugate net, we have

$$\frac{\partial\bar{a}}{\partial\xi}=\frac{\partial\bar{b}}{\partial\eta},$$

that is, the surface is that of the first kind which admits the projective deformation.

Next, we consider the case where $a=I$.

In this case we consider the canonical (or scroll) congruence which is due to Sullivan and Green.

The canonical edge of the first kind is the line which connects the points

$$\left(\frac{\partial x}{\partial u}\right)+\frac{b_u}{4b}(x), \qquad \left(\frac{\partial x}{\partial v}\right)+\frac{a_v}{4a}(x),$$

and that of the second kind is the line which passes through (x) and through the point

$$\left(\frac{\partial^2 x}{\partial u \partial v}\right) + \frac{a_v}{4a}\left(\frac{\partial x}{\partial u}\right) + \frac{b_v}{4b}\left(\frac{\partial x}{\partial v}\right).$$

Now we have

$$\frac{\partial a}{\partial u} = 0, \qquad \frac{\partial a}{\partial v} = 0.$$

Therefore the intersection of the tangent at (x) of the curve C_u and the directrix of the first kind is

$$\left(\frac{\partial x}{\partial u}\right),$$

and the intersection of the tangent at (x) of the curve C_v and the canonical edge of the first kind is

$$\left(\frac{\partial x}{\partial v}\right).$$

The developpables of the congruence formed by the line which passes through the above two points correspond to a conjugate net.

Hence, we have the following theorem.

For a surface of the second kind which admits the projective deformation, the developables of the congruence formed by the lines which pass through the point where one of the asymptotic tangents at (x) meets the directrix of the first kind of (x), and through the point where the other of the asymptotic tangents at (x) meets the canonical edge of the first kind of (x), correspond to a conjugate net.

The inverse of this theorem is also true.

For then, we must have one of the equations

$$\frac{\partial^2}{\partial u \partial v}\log a = 0 \quad \text{and} \quad \frac{\partial^2}{\partial u \partial v}\log b = 0,$$

which we can reduce by the transformations of § 4 to

$$a = 1 \quad \text{and} \quad b = 1 \quad \text{respectively.}$$