On the Projective Deformation of Curved Surfaces. II.

By

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Introduction.

In a preceding paper[1] we proved that for a curved surface (S) which is given by the system

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial u^2}&=b\frac{\partial x}{\partial v}+f x,\\ \frac{\partial^2 x}{\partial v^2}&=a\frac{\partial x}{\partial u}+g x.\end{aligned}\right\}\cdots\cdots(1)$$

admits the projective deformation, it is necessary and sufficient that by the transformation

$$\bar{x}=C\sqrt{\alpha_u\beta_v}\,x,\qquad \bar{u}=\alpha(u),\qquad \bar{v}=\beta(v)$$

(where C is an arbitrary constant, $\alpha(u)$ and $\beta(v)$ are functions of u only and v only respectively, and the suffixes u and v denote the differentiation by u and v respectively) which does not alter the form of the equations (1), a and b are reduced so as to satisfy one of the relations

$$1^\circ \qquad \frac{\partial a}{\partial u}=\frac{\partial b}{\partial v}.$$

$$2^\circ \qquad a \text{ or } b=1.$$

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[1] Mem. Coll. Sci., Kyoto, **5**, 1–28 (1922).

In the former case, the surface (S) is of the first kind, while in latter case the surface is of the second kind.

Consider *a* congruence Γ conjugate to (S), that is, *a* congruence of straight lines such that

1° lines of the congruence are in the one to one correspondence with points on (S). Accordingly, to a surface which is generated by the lines of the congruence corresponds a curve on (S)

2° the lines of the congruence are in the tangent plane of (S) at the corresponding point .

3° to a developpable surfaces of the congruence correspond a conjugate net on (S).

In this paper we shall prove, first, that if the said congruence is *a* W-congruence (that is, if asymptotic lines of two sheets of the focal surfaces correspond) and moreover, asymptotic lines of (S) and the focal surfaces also correspond, then the surface (S), all the Laplace-Darboux transforms of (S) with respect to the conjugate net in 3° and the focal surfaces of the congruence admit the projective deformation and the conjugate net in 3° is that of the projective deformation. Next, we shall prove that if there is an isothermal conjugate net on a surface (S) such that, the asymptotic lines of (S) and the minus-first or first Laplace-Darboux transform of (S) with respect to that conjugate net correspond, then (S) admits projective deformation and the conjugate net is that of projective deformation.

§ I.

Let us take as the parameter curves the conjugate net which corresponds to the developpable surfaces of the congruence. Then, as Green[1] has shown, the homogeneous cordinates (x) of a point (P) on (S) may be regarded as a fundamental system of solutions of completely integrable system of simultaneous partial differential equations of the form

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= a\frac{\partial^2 x}{\partial v^2} + b\frac{\partial x}{\partial u} + c\frac{\partial x}{\partial v} + d\,x, \\ \frac{\partial^2 x}{\partial u \partial v} &= b'\frac{\partial x}{\partial u} + c'\frac{\partial x}{\partial v} + d'x, \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (2)$$

1 Amer. J. Math., **37**, 215–246, (1915).

From (2) we have

$$\frac{\partial^3 x}{\partial u^3} = \alpha^{(1)} \frac{\partial^2 x}{\partial v^2} + \beta^{(1)} \frac{\partial x}{\partial v} + \gamma^{(1)} \frac{\partial x}{\partial v} + \delta^{(1)} x,$$

$$\frac{\partial^3 x}{\partial u^2 \partial v} = \alpha^{(2)} \frac{\partial^2 x}{\partial v^2} + \beta^{(2)} \frac{\partial x}{\partial v} + \gamma^{(2)} \frac{\partial x}{\partial v} + \delta^{(2)} x,$$

$$\frac{\partial^3 x}{\partial u \partial v^2} = \alpha^{(3)} \frac{\partial^2 x}{\partial v^2} + \beta^{(3)} \frac{\partial x}{\partial u} + \gamma^{(3)} \frac{\partial x}{\partial v} + \delta^{(3)} x,$$

$$\frac{\partial^3 x}{\partial v^3} = \alpha^{(4)} \frac{\partial^2 x}{\partial v^2} + \beta^{(4)} \frac{\partial x}{\partial u} + \gamma^{(4)} \frac{\partial x}{\partial v} + \delta^{(4)} x.$$

where

$$\left.\begin{aligned}
&\alpha^{(1)} = a(b + c') + a_u, \\
&\beta^{(1)} = b_u + ab'_v + d + b^2 + b'(c + ab'), \\
&\gamma^{(1)} = c_u + ac'_v + ad' + bc + c'(c + ab'), \\
&\delta^{(1)} = d_u + ad'_v + bd + d'(c + ab'), \\
&\alpha^{(2)} = ab', \qquad \beta^{(2)} = b'(b + c') + b'_u + d', \\
&\gamma^{(2)} = b'c + c'^2 + c'_u, \qquad \delta^{(2)} = b'd + c'd' + d'_u, \\
&\alpha^{(3)} = c', \qquad \beta^{(3)} = b'^2 + b'_v, \\
&\gamma^{(3)} = b'c' + c'_v + d', \qquad \delta^{(3)} = b'd' + d'_v, \\
&\alpha^{(4)} = \frac{1}{a}\left(ab' - c - a_v\right), \qquad \beta^{(4)} = \frac{1}{a}\left(b'c' + b'_u - b_v + d'\right), \\
&\gamma^{(4)} = \frac{1}{a}\left[b'c + c'(c' - b) + c'_u - c_v - d\right], \\
&\delta^{(4)} = \frac{1}{a}\left[b'd + d'(c' - b) + d'_u - d_v\right].
\end{aligned}\right\} \quad (3)$$

The conditions of complete integrability are

$$\left.\begin{aligned}
&\gamma^{(3)} + \alpha_v^{(3)} = a\beta^{(4)} + \alpha_u^{(4)}, \\
&(\alpha^{(3)} - b)\beta^{(4)} + b'\beta^{(4)} + \beta_v^{(3)} = \alpha^{(4)}\beta^{(3)} + b'\gamma^{(4)} + \beta_u^{(4)} + \delta^{(4)}, \\
&c'\beta^{(3)} + \gamma_v^{(3)} + \delta^{(3)} = \alpha^{(4)}\gamma^{(3)} + c\gamma^{(4)} + \gamma_u^{(4)}, \\
&\alpha^{(3)}\delta^{(4)} + d'\beta^{(3)} + \delta_v^{(3)} = \alpha^{(4)}\delta^{(3)} + d\beta^{(4)} + d'\gamma^{(4)} + \delta_u^{(4)}.
\end{aligned}\right\} \quad (4)$$

In these formulas the suffixes u and v denote the differentiations by u and v respectively.

The first of (4), in virtue of equation (3), becomes

$$\frac{\partial}{\partial v}\left(b+2c'\right)=\frac{\partial}{\partial u}\left(2b'-\frac{c}{a}-\frac{\partial \log a}{\partial v}\right).$$

Hence we can find a function $p(u, v)$ by a quadrature such that

$$\left.\begin{aligned} b+2c' &= \frac{\partial p}{\partial u}, \\ 2b'-\frac{c}{a}-\frac{\partial \log a}{\partial v} &= \frac{\partial p}{\partial v}. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (5)$$

By the transformation

$$x=\lambda\,\bar{x},$$

where λ is subjected to the conditions

$$\frac{1}{\lambda}\frac{\partial\lambda}{\partial u}=\frac{1}{4}\frac{\partial p}{\partial u}, \qquad \frac{1}{\lambda}\frac{\partial\lambda}{\partial v}=\frac{1}{4}\frac{\partial p}{\partial v},$$

the system (2) is transformed to a system of the same from with the conffficients

$$\left.\begin{aligned} &A=a, \qquad B=b-\frac{1}{2}\frac{\partial p}{\partial u}, \qquad C=c+\frac{a}{2}\frac{\partial p}{\partial v}, \\ &D=d+\frac{1}{4}p\frac{\partial p}{\partial u}+\frac{1}{4}c\frac{\partial p}{\partial v}-\frac{1}{4}\frac{\partial^2 p}{\partial u^2}+\frac{1}{4}a\frac{\partial^2 p}{\partial v^2}-\frac{1}{16}\left(\frac{\partial p}{\partial u}\right)^2 \\ &\qquad\qquad +\frac{1}{16}a\left(\frac{\partial p}{\partial v}\right)^2, \\ &B'=b'-\frac{1}{4}\frac{\partial p}{\partial v}, \qquad C'=c'-\frac{1}{4}\frac{\partial p}{\partial u}, \\ &D'=d'+\frac{1}{4}b'\frac{\partial p}{\partial u}+\frac{1}{4}c'\frac{\partial p}{\partial v}-\frac{1}{4}\frac{\partial^2 p}{\partial u\partial v}-\frac{1}{16}\frac{\partial p}{\partial u}\frac{\partial p}{\partial v}. \end{aligned}\right\} \quad (7)$$

These coefficients are seminvariants of (2), and the new system is said to be in its canonical form. The relations

$$B+2C'=0, \qquad 2AB'-C-\frac{\partial A}{\partial v}=0,$$

which follow from (6), characteristic of this canonical form.

Any proper transformation of the form

$$\bar{u}=\varphi(u), \qquad \bar{v}=\psi(v). \quad \cdots\cdots\cdots\cdots (7)$$

affects only the parametric representation of the conjugate net given (2), but leaves the net itself unchanged. The invariants of the net are those functions of the seminvariants, which remain unchanged by transformations of the form (7), except for a factor. The fundamental invariants are

$$a, \qquad \mathfrak{B}' = B' - \frac{3}{8}\frac{1}{a}\frac{\partial a}{\partial v}, \qquad \mathfrak{C}' = C' + \frac{1}{8}\frac{1}{a}\frac{\partial a}{\partial u},$$

$$\mathfrak{D}' = D' + B'C', \qquad \mathfrak{D} = D - \left(B'\frac{\partial a}{\partial v} - a\frac{\partial B'}{\partial v}\right) - \frac{\partial C'}{\partial u} + 3\left(aB'^2 - C'^2\right).$$

These satisfy the equations

$$\bar{a} = \frac{\psi_v^2}{\varphi_u^2}, \qquad \overline{\mathfrak{B}'} \frac{1}{\psi_v}\mathfrak{B}', \qquad \overline{\mathfrak{C}'} = \frac{1}{\varphi_u}\mathfrak{C}',$$

$$\overline{\mathfrak{D}} = \frac{1}{\varphi_u^2}\mathfrak{D}, \qquad \overline{\mathfrak{D}'} = \frac{1}{\varphi_u\psi_v}\overline{\mathfrak{D}'}.$$

The second and third of the conditions (4) may be expressed in terms of the invariants as following.

$$\left.\begin{aligned}
&\frac{\partial \mathfrak{D}}{\partial v} - 2\frac{\partial \mathfrak{D}'}{\partial u} + \frac{\mathfrak{D}'}{a}\frac{\partial a}{\partial u} \\
&\quad = \frac{\partial}{\partial u}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} + \frac{1}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right) - 4\mathfrak{C}'\left(\frac{\partial \mathfrak{B}'}{\partial u} + 3\frac{\partial \mathfrak{C}'}{\partial v}\right) \\
&\quad\quad - \frac{1}{2a}\frac{\partial a}{\partial u}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} + \frac{1}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right), \\
&\frac{\partial}{\partial u}\left(\frac{\mathfrak{D}}{a}\right) + 2\frac{\partial \mathfrak{D}'}{\partial v} + \frac{\mathfrak{D}'}{a}\frac{\partial a}{\partial v} \\
&\quad = -\frac{\partial}{\partial v}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} - \frac{3}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right) + 4\mathfrak{B}'\left(3\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v}\right) \\
&\quad\quad - \frac{1}{2a}\frac{\partial a}{\partial u}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} - \frac{3}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right).
\end{aligned}\right\} \quad (8)$$

§ 2.

Consider two points ρ and σ such that

$$\left.\begin{aligned}
\rho &= \theta\frac{\partial x}{\partial u} - \frac{\partial \theta}{\partial u}x, \\
\sigma &= \theta\frac{\partial x}{\partial v} - \frac{\partial \theta}{\partial v}x,
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (9)$$

where θ is a solution of the partial differential equation

$$\frac{\partial^2\theta}{\partial u\partial v} = b'\frac{\partial\theta}{\partial u} + c'\frac{\partial\theta}{\partial v} + d'\theta. \qquad \text{(10)}$$

Then we have seen that[1] congruences whose developpables correspond to the curves C_u (v-const.) and C_v (u-const.) are those which are formed by such lines as $\rho\sigma$ and the surfaces which are described by ρ and σ are focal surfaces.

Put

$$w = a\frac{\partial^2\theta}{\partial v^2} + b\frac{\partial\theta}{\partial u} + d\theta - \frac{\partial^2\theta}{\partial u^2},$$

$$\phi = \alpha^{(1)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(1)}\frac{\partial\theta}{\partial u} + \gamma^{(1)}\frac{\partial\theta}{\partial v} + \delta^{(1)}\theta - \frac{\partial^3\theta}{\partial u^3},$$

$$\psi = \alpha^{(4)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(4)}\frac{\partial\theta}{\partial u} + \gamma^{(4)}\frac{\partial\theta}{\partial \theta} + \delta^{(4)}\theta - \frac{\partial^3\theta}{\partial v^3},$$

Then we have from (3), (4) and (10)

$$\frac{\partial^3\theta}{\partial u^2\partial v} = -b'w + \alpha^{(2)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(2)}\frac{\partial\theta}{\partial u} + \gamma^{(2)}\frac{\partial\theta}{\partial v} + \delta^{(2)}\theta,$$

$$\frac{\partial^3\theta}{\partial u\partial v^2} = \alpha^{(3)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(3)}\frac{\partial\theta}{\partial u} + \gamma^{(3)}\frac{\partial\theta}{\partial v} + \delta^{(3)}\theta,$$

$$\frac{\partial w}{\partial u} = -bw + \phi,$$

$$\frac{\partial w}{\partial v} = b'w - a\psi,$$

$$\frac{\partial\psi}{\partial u} = c'\psi - \beta^{(4)}w,$$

$$\frac{\partial^2 w}{\partial u\partial v} = b'\frac{\partial w}{\partial u} + \left(c' + \frac{a_u}{a}\right)\frac{\partial w}{\partial v} + \left(2b'_u - b_v + d' - \frac{b'a_u}{a}\right)w. \qquad \text{(11)}$$

From (2), (9) and (10) we have

[1] J. Kanitani, Mem. Coll. Sci., Kyoto, 5, 329, (1922).

$$\left.\begin{aligned}\frac{\partial\rho}{\partial v}&=\left(b'+\frac{\theta_v}{\theta}\right)\rho+\left(c'-\frac{\theta_v}{\theta}\right)\sigma,\\ \frac{\partial\sigma}{\partial u}&=\left(b'-\frac{\theta_v}{\theta}\right)\rho+\left(c'+\frac{\theta_u}{\theta}\right)\sigma,\end{aligned}\right\}\qquad\qquad(12)$$

from which follow

$$\left.\begin{aligned}\frac{\partial^2\rho}{\partial u\partial v}&=\left(b'+\frac{\theta_v}{\theta}\right)\frac{\partial\rho}{\partial u}+\left(c'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\frac{\partial\rho}{\partial v}\\ &\qquad+\left[b'_u+b'c'+d'-\left(b'+\frac{\theta_v}{\theta}\right)\left(c'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\right]\rho,\\ \frac{\partial^2\sigma}{\partial u\partial v}&=\left(b'+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\frac{\partial\sigma}{\partial u}+\left(c'+\frac{\theta_u}{\theta}\right)\sigma_v\\ &\qquad+\left[c'_v+b'c'+d'-\left(b'+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\left(c'+\frac{\theta_u}{\theta}\right)\right]\rho,\end{aligned}\right\}\quad(13)$$

where

$$\tau\equiv c'-\frac{\theta_u}{\theta},\qquad\qquad \tau'\equiv b'-\frac{\theta_v}{\theta}.$$

Now we shall prove that ρ and σ satisfy the equations of the form

$$\frac{\partial^2\rho}{\partial u\partial v}=A_1\frac{\partial\rho}{\partial v^2}+B_1\frac{\partial\rho}{\partial u}+C_1\frac{\partial\rho}{\partial v}D_1\rho\qquad\qquad(14)$$

and

$$\frac{\partial^2\sigma}{\partial u\partial v}=A_2\frac{\partial\sigma}{\partial v^2}+B_2\frac{\partial\sigma}{\partial u}+C_2\frac{\partial\sigma}{\partial v}+D_2\sigma\qquad\qquad(15)$$

respectively,

where A_1, B_1, ... ; A_2, B_2, ... are functions of a, b, ...,
θ and their derivatives.

In fact, we have form (2), (9) and (10)

$$\rho=\theta\frac{\partial x}{\partial u}-\frac{\partial\theta}{\partial u}x,$$

$$\frac{\partial\rho}{\partial u}=\theta\frac{\partial^2x}{\partial u^2}-\frac{\partial^2\theta}{\partial u^2}x=a\frac{\partial\sigma}{\partial v}+b\rho+c\sigma+wx,$$

$$\frac{\partial\rho}{\partial v}=\left(b'+\frac{\theta_v}{\theta}\right)\rho+\left(c'-\frac{\theta_v}{\theta}\right)\sigma,$$

$$\frac{\partial^2\rho}{\partial u^2}=\frac{\theta_u}{\theta}\frac{\partial\rho}{\partial u}+\alpha^{(1)}\frac{\partial\rho}{\partial v}+\left(\beta^{(1)}-\frac{\theta_{uu}}{\theta}\right)\rho+\gamma^{(1)}\sigma+\phi x,$$

$$\frac{\partial^2\rho}{\partial v^2}=\left(b'+\frac{\theta_v}{\theta}\right)\frac{\partial\rho}{\partial v}+\left(c^1-\frac{\theta_u}{\theta}\right)\frac{\partial\sigma}{\partial v}+\frac{\partial}{\partial v}\left(b^1+\frac{\theta_v}{\theta}\right)\rho$$
$$+\frac{\partial}{\partial v}\left(c^1-\frac{\theta_u}{\theta}\right)\sigma,$$

from which follows

$$\frac{\partial^2\rho}{\partial u^2}+\frac{a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}}{c'-\frac{\theta_v}{\theta}}\frac{\partial^2\rho}{\partial v^2}$$

$$=\left(b+\frac{\theta_u}{\theta}+\frac{w_u}{w}\right)\frac{\partial\rho}{\partial u}+\frac{a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}\left(b'+\frac{\theta_v}{\theta}\right)}{c'-\frac{\theta_u}{\theta}}$$

$$+\left[\frac{\beta^{(3)}-\frac{\partial_{uu}}{\theta}+a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}}{c'-\frac{\theta_u}{\theta}}\frac{\partial}{\partial v}\left(b'+\frac{\theta_v}{\theta}\right)-b\left(b+\frac{w_u}{w}\right)\right]\rho$$

$$+\left[\gamma^{(1)}\frac{a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}}{c'-\frac{\theta_u}{\theta}}\frac{\partial}{\partial v}\left(c'-\frac{\theta_u}{\theta}\right)-c\left(b+\frac{w_u}{w}\right)\right]\sigma. \quad \text{......} (16)$$

Substituting

$$\sigma=\frac{\frac{\partial\rho}{\partial v}}{c'-\frac{\theta_u}{\theta}}-\frac{b'+\frac{\theta_v}{\theta}}{c'+\frac{\theta_u}{\theta}}\rho$$

in (17), we have an equation of the form (14), and

$$A_1=\frac{a\left(c'+\frac{\theta_u}{\theta}-\frac{w_u}{w}\right)}{c'-\frac{\theta_u}{\theta}}. \quad \text{........................} (17)$$

Similarly we can prove for σ and we have

$$A_2=\frac{a\left(b'-\frac{\theta_v}{\theta}\right)}{b'-\frac{w_v}{w}}. \quad \text{................................} (17')$$

Asymptotic lines of (S) and the surfaces which are described by ρ and σ are given by

$$a\,du^2 + dv^2 = 0,$$
$$A_1\,du^2 + dv^2 = 0,$$
$$A_2\,du^2 + dv^2 = 0. \qquad \text{respectively.}$$

If those asymptotic lines correspond, we must have

$$a = A_1 = A_2,$$

or

$$a = \frac{a\left(c' + \dfrac{a_u}{a} - \dfrac{w_u}{w}\right)}{c' - \dfrac{\theta_u}{\theta}} = \frac{b' + \dfrac{\theta_v}{\theta}}{b' - \dfrac{w_v}{w}} \quad \text{..................} \quad (18)$$

from which follow

$$w = f(u)\theta,$$
$$a = \frac{f(u)}{g(v)}, \quad \text{..............................} \quad (19)$$

where $f(u)$ and $g(v)$ are functions of u only and v only respectively.

Substituting the above values of w in the equation (11), we have

$$\frac{\partial^2\theta}{\partial u\partial v} = b'\frac{\partial\theta}{\partial u} + c'\frac{\partial\theta}{\partial v} + (2b'_u - b_v + d')\theta = 0.$$

From this equation and (10) we have

$$\theta(2b'_u - b_v) = 0.$$

If $\theta = 0$, the points (ρ) and (σ) coincide with (x).

Hence we must have

$$2\frac{\partial b'}{\partial u} - \frac{\partial b}{\partial v} = 0. \quad \text{..............................} \quad (20)$$

The conditions (19) and (20) may be expressed in terms of invariants as following.

$$\left.\begin{aligned} &\frac{\partial^2 \log a}{\partial u\partial v} = 0, \\ &\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} = 0. \end{aligned}\right\} \quad \text{........................} \quad (21)$$

Reciprocally, if

$$a = \frac{f(u)}{g(v)}$$

and

$$2\frac{\partial b'}{\partial u} - \frac{\partial b}{\partial v} = 0,$$

the system of equations

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u^2} &= a\frac{\partial^2\theta}{\partial v^2} + b\frac{\partial^2\theta}{\partial u} + c\frac{\partial\theta}{\partial v} + \{d - f(u)\}\theta, \\ \frac{\partial^2\theta}{\partial u\partial v} &= b'\frac{\partial\theta}{\partial\theta} + c'\frac{\partial\theta}{\partial v} + d'\theta \end{aligned}\right\} \quad \cdots\cdots\cdots (22)$$

is completely integrable by (4).

Therefore, if we take a particular solution of the system (22) and consider two points

$$\rho = \theta\frac{\partial x}{\partial u} - \frac{\partial\theta}{\partial u}x,$$

$$\sigma = \theta\frac{\partial x}{\partial v} - \frac{\partial\theta}{\partial u}x,$$

then the congruence formed by the lines $\rho\sigma$ satisfies our conditions.

§ 3.

By the transformation

$$\bar{u} = \int\sqrt{f(u)}\,du, \qquad \bar{v} = i\int\sqrt{g(v)}\,dv,$$

(21) become

$$\bar{a} = -1,$$

$$\frac{\partial\bar{\mathfrak{B}}'}{\partial\bar{u}} + \frac{\partial\bar{\mathfrak{C}}'}{\partial\bar{v}} = 0.$$

Let us suppose that we have performed the above transformation.

Then from (18) we have

$$w = k\theta,$$

where k is a constant.

Next, reduce the thus transformed system into its canonical form and denote by A, B, etc., the coefficients of that canonical form, then we have

$$\left.\begin{aligned} A &= -1, \\ B + 2C' &= 0, \\ 2B' + C &= 0, \\ \frac{\partial B'}{\partial u} + \frac{\partial C'}{\partial v} &= 0. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (23)$$

Asymptotic lines of (S) are given by

$$du^2 - dv^2 = 0.$$

Hence, if we carry out the transformation

$$\xi = u - v, \qquad \mu = u + v,$$

the new parameter curves are asymptotic lines and the system (2) is transformed into

$$\frac{\partial^2 x}{\partial \xi^2} = \beta \frac{\partial x}{\partial \mu} + \gamma x,$$

$$\frac{\partial^2 x}{\partial \mu^2} = \alpha \frac{\partial x}{\partial \xi} + \gamma' x,$$

where

$$\alpha = (B' - C'), \qquad \beta = -(B' + C'),$$

$$\gamma = \frac{D + 2D'}{4}, \qquad \gamma' = \frac{D - 2D'}{4}.$$

Moreover, we have

$$\frac{\partial B'}{\partial u} = \frac{\partial B'}{\partial \xi} + \frac{\partial B'}{\partial \mu},$$

$$\frac{\partial C'}{\partial v} = -\frac{\partial C'}{\partial \xi} + \frac{\partial C'}{\partial \mu},$$

from which follows

$$\frac{\partial}{\partial \xi}\left(B' - C'\right) + \frac{\partial}{\partial \mu}\left(B' + C'\right),$$

which is equivalent to

$$\frac{\partial \alpha}{\partial \xi} = \frac{\partial \beta}{\partial \mu}.$$

Therefore we know that (S) admits the projective deformation and is of the first kind, and the curves $u=$const. and $v=$const. form the conjugate net of the projective deformation.

From (13) and (17), we know that when (S) is given by the said canonical form the surface which is described by (ρ) is given by the system

$$\left.\begin{aligned}&\frac{\partial^2\rho}{\partial u^2}+\frac{\partial^2\rho}{\partial v^2}=2\left(\frac{\theta_u}{\theta}-C'\right)\frac{\partial\rho}{\partial u}+2\left(\frac{\theta_v}{\theta}+\frac{\tau_v}{\tau}-B'\right)\frac{\partial\rho}{\partial v}+(\bigstar)\rho,\\&\frac{\partial^2\rho}{\partial u\partial v}=\left(B'+\frac{\theta_v}{\theta}\right)\frac{\partial\rho}{\partial u}+\left(C'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\frac{\partial\rho}{\partial v}\\&\qquad+\left[K-\left(C'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\left(B'+\frac{\theta_v}{\theta}\right)\right]\rho,\end{aligned}\right\}\quad(23)$$

where

$$K=B'C'+D'-\frac{\partial C'}{\partial v},$$

$$\tau=C'-\frac{\theta_u}{\theta}.$$

Similary we know that the surface which is described by σ is given by the system

$$\left.\begin{aligned}&\frac{\partial^2\sigma}{\partial u^2}+\frac{\partial^2\sigma}{\partial v^2}=2\left(\frac{\theta_u}{\theta}+\frac{\tau'_u}{\tau'}-C'\right)\frac{\partial\sigma}{\partial u}+2\left(\frac{\theta_v}{\theta}-B'\right)\frac{\partial\sigma}{\partial v}+(\bigstar)\sigma,\\&\frac{\partial^2\sigma}{\partial u\partial v}=\left(B^1+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\frac{\partial\sigma}{\partial u}+\left(C'+\frac{\theta_u}{\theta}\right)\frac{\partial\sigma}{\partial v}\\&\qquad+\left[H-\left(B'+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\left(C'+\frac{\theta_v}{\theta}\right)\right]\sigma,\end{aligned}\right\}\quad(24)$$

where

$$H=B'C'+D'-\frac{\partial B'}{\partial u},$$

$$\tau=B'-\frac{\theta_v}{\theta}.$$

From (24) and (25) we can easily see that the focal surfaces also admit the projective deformation.

§ 4.

Let (y) be the point on the minus-first Laplace-Darboux transform (S_{-1}) of (S) corresponding to (x).

Then we have from (2)

$$
\left.
\begin{aligned}
y &= \frac{\partial x}{\partial u} - c'x, \\
\frac{\partial y}{\partial u} &= a\frac{\partial^2 x}{\partial v^2} + \left(b - c'\right)\frac{\partial x}{\partial u} + c\frac{\partial x}{\partial v} + (d - c'_u)x, \\
\frac{\partial y}{\partial v} &= \qquad b'\frac{\partial x}{\partial u} \qquad + (d' - c'_v)x, \\
\frac{\partial^2 y}{\partial u^2} &= \left(\alpha^{(1)} - ac'\frac{\partial^2 x}{\partial v^2}\right) + \left(\beta^{(1)} - bc' - 2c'_u\right)\frac{\partial x}{\partial u} + \left(\alpha^{(1)} - cc'\right)\frac{\partial x}{\partial v} \\
&\qquad + \left(\delta^{(1)} - c'd - c'_{uu}\right)x, \\
\frac{\partial^2 y}{\partial v^2} &= \left(b'^2 + b'_v\right)\frac{\partial x}{\partial u} + \left(b'c' + d' - c'_v\right)\frac{\partial x}{\partial v} + \left(d'_v + b'd' - c'_{vv}\right)x,
\end{aligned}
\right\}
$$

from which follows

$$
\begin{aligned}
\frac{\partial^2 y}{u^2} - A\frac{\partial^2 y}{\partial v^2} - \left(b + \frac{a_u}{a}\right)\frac{\partial y}{\partial u}
&= \left[\beta^{(1)} - bc' - 2c'_u - A\left(b'^2 + b'_v\right) - \left(b - c'\right)\right]\frac{\partial x}{\partial u} \\
&\quad + \left[\delta^{(1)} - c'd - c'_{uu} - A\left(d'_v + b'd' - c'_{vv}\right)\left(d - c'_u\right)\left(b + \frac{a_u}{a}\right)\right]x,
\end{aligned}
\tag{26}
$$

where

$$
A = \frac{a\left\{\dfrac{\partial}{\partial u}\left(\dfrac{c}{a}\right) + 2\dfrac{\partial c'}{\partial v} + K\right\}}{K}.
$$

Eliminating x, $\dfrac{\partial x}{\partial u}$ from the first, the third and last equations, we have the equation of the form

$$
\frac{\partial^2 y}{\partial u^2} = A\frac{\partial^2 y}{\partial v^2} + B\frac{\partial y}{\partial v^2} + C\frac{\partial y}{\partial v} + Dy,
$$

where

$$
A = \frac{a\left\{\dfrac{\partial}{\partial u}\left(\dfrac{c}{a}\right) + 2\dfrac{\partial c'}{\partial v} + K\right\}}{K}.
$$

And from the equations

$$
y = \frac{\partial x}{\partial u} - c'x,
$$

$$\frac{\partial^2 x}{\partial u \partial v} = b' \frac{\partial x}{\partial v} + c' \frac{\partial x}{\partial v} + d'x.$$

we have

$$\frac{\partial^2 y}{\partial u \partial v} = b' \frac{\partial y}{\partial u} + \left(c' + \frac{K_u}{K}\right)\frac{\partial y}{\partial v} + \left[b'_u + K - b'\left(c' + \frac{K_u}{K}\right)\right] y. \quad \text{......} (27)$$

When the asymptotic lines of (S) and (S_1) correspond, we must have

$$a = A,$$

or

$$\frac{\partial}{\partial u}\left(\frac{c}{a}\right) - \frac{\partial c'}{\partial v} = 0,$$

which is equivalent to

$$2\frac{\partial b'}{\partial u} - \frac{\partial b}{\partial v} - \frac{\partial^2}{\partial u \partial v}\log a = 0. \quad \text{....................} (28)$$

If the curves C_u and C_v form an isothermal conjugate net, we must have[1]

$$\frac{\partial^2}{\partial u \partial v}\log a = 0. \quad \text{..............................} (29)$$

Therefore, from (28) and (29) we know that *if there is an isothermal conjugate net on a surface* (S) *such that, the asymptotic lines of* (S) *and the minus-first Laplace-Darboux transform of it with respect to that conjugate net correspond,* (S) *admits the projective deformation and the conjugate net is that of projective deformation.*

This theorem also hold, if we take the first instead of the minus-first Laplace-Darboux transform of (S).

When (S) admits the projective deformation and given by the system which is its canonical form explained in § 3, from (8), (23) and (26) we can see that the minus-first Laplace-Darboux transform of (S) is given by the system

$$\left.\begin{aligned} \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} &= -2C'\frac{\partial y}{\partial u} + 2\left(\frac{K_v}{K} - B'\right)\frac{\partial y}{\partial v} + (\bigstar)y, \\ \frac{\partial^2 y}{\partial u \partial v} &= B'\frac{\partial y}{\partial u} + \left(C' + \frac{K_u}{K}\right)\frac{\partial y}{\partial v} + \left[B'_u + K - B'\left(C' + \frac{K_u}{K}\right)\right]y. \end{aligned}\right\} (30)$$

Similarly we can see that the first transform is given by the system

[1] Wilcynski, Amer. J. Math., **42**, 211, (1920).

$$\left.\begin{aligned}\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}&=2\left(\frac{H_u}{H}-C'\right)\frac{\partial z}{\partial u}-2B'\frac{\partial z}{\partial v}+(\bigstar)z,\\ \frac{\partial^2 z}{\partial u\partial v}&=\left(B'+\frac{H_v}{H}\right)\frac{\partial z}{\partial v}+C'\frac{\partial z}{\partial v}+\left[C'_v+H-C'\left(B'+\frac{H_v}{H}\right)\right]z.\end{aligned}\right\}\quad(31)$$

From (30) and (31) we know that the minus-first and the first Laplace-Darboux transforms with respect to the conjugate net of the projective deformation admit the projective deformation and, moreover, the conjugate net of the projective deformation is transformed into the conjugate net of the projective deformation, accordingly the same is true for all the Laplace-Darboux transforms with respect to the conjugate net of the projectivc deformation.