On the Projective Deformation of Curved Surfaces. III.

By

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§ 1.

In a preceding paper[1] we proved that if a congruence Γ conjugate to a surface (S) is a W-congruence and the asymptotic lines of the focal surfaces of that congruence and the surface (S) correspond, the surface (S) and the focal surfaces admit the projective deformation and, moreover, the conjugate net on (S) which corresponds to the developable surfaces of the congruence is that of projective deformation.

In this paragraph, we shall prove the same for congruence Γ' which is conjugate to a surface (S), that is, for a congruence which has the following relations with (S).

1° The lines of the congruence are in a one to one correspondence to the points of (S), accordingly to a surface of the congruence correspond a curve on (S).

2° A line of the congruence passes through the corresponding point on (S).

3° To the developable surfaces of the congruence correspond a conjugate net on (S), in other words, the developable surfaces of the congruence cut (S) in a conjugate net.

Let us take as parameter curves the conjugate net which corresponds to the developable surfaces of the congruence and suppose that (S) is given by a completely integrable system of partial differential equations

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[1] J. Kanitani, Mem. Coll. Sci., Kyoto, **6**, 129 (1922).

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial u^2} &= a\frac{\partial^2 x}{\partial v^2} + b\frac{\partial x}{\partial v} + c\frac{\partial x}{\partial v} + dx,\\ \frac{\partial^2 x}{\partial u \partial v} & \qquad b'\frac{\partial x}{\partial u} + c'\frac{\partial x}{\partial v} + d'x,\end{aligned}\right\}\quad\text{..................}(1)$$

of which one of the conditions of complete integrability is

$$\frac{\partial}{\partial v}(b + 2c') = \frac{\partial}{\partial u}\left(\frac{2ab' - c - a_v}{a}\right).$$

Then the congruences whose developables cut (S) in the curves C_u (v-const.), C_v (u-const.) are those which are formed by the lines which connect a point (x) and a point (z) such that[1]

$$z = \theta\frac{\partial^2 x}{\partial v^2} - \frac{1}{a}\left(\frac{\partial \theta}{\partial u} + c'\theta\right)\frac{\partial x}{\partial u} - \left(\frac{\partial \theta}{\partial v} + \frac{ab' - c - a_v}{a}\theta\right)\frac{\partial x}{\partial v},$$

where θ is any one of the solutions of the partial differential equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = -\left(b' - \frac{a_v}{a}\right)\frac{\partial \theta}{\partial u} - c'\frac{\partial \theta}{\partial v} + \left(d' + \frac{\partial b'}{\partial u} - \frac{\partial c'}{\partial v} - \frac{\partial b}{\partial v} + \frac{c'}{a}\frac{\partial a}{\partial v}\right)\theta.$$

$$\text{..............................}(2)$$

Focal surfaces are those which are described by the points ω and ν such that

$$\omega = z - \xi x,$$
$$\nu = z - \xi' x.$$

where

$$\xi = -\frac{1}{a}\frac{\partial^2 \theta}{\partial u^2} - \frac{1}{a}\left(b + c' - \frac{1}{a}\frac{\partial a}{\partial u}\right)\frac{\partial \theta}{\partial u} - b'\frac{\partial \theta}{\partial v}$$

$$+\left[\frac{\partial b'}{\partial v} - \frac{\partial}{\partial u}\left(\frac{c'}{a}\right) - \frac{1}{a}(bc' - b'c) + \frac{b'}{a}\frac{\partial a}{\partial v}\right]\theta$$

$$\xi' = -\frac{\partial^2 \theta}{\partial v^2} - \frac{c'}{a}\frac{\partial \theta}{\partial u} - \frac{ab' - c - a_v}{a}\frac{\partial \theta}{\partial v}$$

$$+\left[\frac{1}{a}\frac{\partial c'}{\partial u} - \frac{\partial b'}{\partial v} - \frac{1}{a}(bc' - b'c) - \frac{ca_v}{a^2} + \frac{\partial^2}{\partial v^2}\log a - d\right]\theta.$$

From the above formulas we have

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[1] J. Kanitani, Mem. Coll. Sci., **5**, 329 (1922).

$$\left.\begin{aligned}\frac{\partial\omega}{\partial u}&=-\left(\frac{\partial\zeta}{\partial u}+c'\zeta\right)x\\ \frac{\partial\omega}{\partial v}&=-\zeta\frac{\partial x}{\partial v}+b'\zeta x.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\ (3)$$

$$\left.\begin{aligned}\frac{\partial\nu}{\partial u}&=\zeta\frac{\partial x}{\partial u}-c'\zeta x,\\ \frac{\partial\nu}{\partial v}&=\left(\frac{\partial\zeta}{\partial v}+b'\zeta\right)x.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\ (4)$$

$$\frac{\partial^2\zeta}{\partial u\partial v}=-b'\frac{\partial\zeta}{\partial v}-c'\frac{\partial\zeta}{\partial v}+(d'-b'_u-c'_v)\zeta.\quad\cdots\cdots\cdots\cdots\ (5)$$

where

$$\zeta=\xi-\xi'.$$

From (4) we have

$$x=\frac{\nu_v}{\zeta_v+b'\zeta},$$

$$x_v=\frac{\nu_{vv}}{\zeta_v+b'\zeta}-\nu_v\frac{\partial}{\partial v}\left(\frac{1}{\zeta_v+b'\zeta}\right),$$

$$x_u=\frac{\nu_u}{\zeta}+\frac{c'\nu_v}{\zeta_v+b'\zeta},$$

$$x_{uu}=\frac{\nu_{uu}}{\zeta}+\nu_u\frac{\partial}{\partial_u}\left(\frac{1}{\zeta}\right)+\frac{\partial}{\partial u}\left(\frac{c'\nu_v}{\zeta_v+b'\zeta}\right),$$

$$\frac{\partial^2\nu}{\partial u\partial v}=\left(b'+\frac{\zeta_v}{\zeta}\right)\frac{\partial\nu}{\partial u}+\left[c'+\frac{1}{b'\zeta+\zeta_v}\frac{\partial}{\partial u}(b'\zeta+\zeta_v)\right]\frac{\partial\nu}{\partial v}.$$

Substituting these values of x_{uu}, x_u, x_v, x in the equations

$$\nu=\theta\frac{\partial^2x}{\partial v^2}-\frac{1}{a}\left(\frac{\partial\theta}{\partial u}+c'\theta\right)\frac{\partial x}{\partial u}-\left[\frac{\partial\theta}{\partial v}+\left(b'-\frac{c}{a}-\frac{a_v}{a}\right)\theta\right]\frac{\partial x}{\partial v}-\xi'x$$

$$=\frac{\theta}{a}\frac{\partial^2x}{\partial u^2}-\frac{1}{a}\left(\frac{\partial\theta}{\partial u}+(b+c')\theta\right]\frac{\partial x}{\partial u}+\left(b'-\frac{a_v}{a}\right)\theta\right]\frac{\partial x}{\partial v}-\left(\xi'+\frac{d}{a}\right)x.$$

we have the equations of the form

$$\frac{\partial^2\nu}{\partial u^2}=A\frac{\partial^2\nu}{\partial v^2}+B\frac{\partial\nu}{\partial u}+C\frac{\partial\nu}{\partial v}+D\nu,$$

where

$$A=\frac{a\left(\frac{\theta_v}{\theta}+b'+\frac{a_v}{a}\right)}{\left(\frac{\zeta_v}{\zeta}+b'\right)}.$$

Similarly, we have an equation of the form

$$\frac{\partial^2 w}{\partial u^2}=A_1\frac{\partial^2 w}{\partial v^2}+B_1\frac{\partial^2 w}{\partial u}+C_1\frac{\partial w}{\partial v}+D_1 w,$$

where

$$A_1=\frac{a\left(c'+\frac{\zeta_u}{\zeta}\right)}{\left(\frac{\theta_u}{\theta}+c'\right)}.$$

If the congruence in question is a W-congruence and the asymptotic lines of the focal surfaces of that congruence and (S) correspond, we must have

$$a=A=A_1,$$

which is equivalent to

$$\left.\begin{aligned}\frac{\zeta_u}{\zeta}&=\frac{\theta_u}{\theta},\\ \frac{\zeta_v}{\zeta}&=\frac{\theta_v}{\theta}-\frac{a_v}{a}.\end{aligned}\right\} \qquad (6)$$

From (6) we have

$$\frac{\partial^2}{\partial u\partial v}\log a=0,$$

or

$$a=\frac{U(u)}{V(v)}, \qquad (7)$$

where $U(u)$ and $V(v)$ are functions of u only and v only respectively.

By the transformation

$$\bar{u}=\int\sqrt{U(u)}\,du,$$

$$\bar{v}=\int\sqrt{-V(v)}\,dv,$$

we can reduce a to -1.

Let us suppose that we have performed the above transformation. Then we have

$$\zeta = k\theta \qquad (k\text{: const.})$$

$$\xi = \frac{\partial^2\theta}{\partial n^2} + (b + c')\frac{\partial\theta}{\partial v} - b'\frac{\partial\theta}{\partial v} + \left[b'_v + c'_u + (bc' - b'c)\right]\theta,$$

$$\xi' = -\frac{\partial^2\theta}{\partial v^2} + c'\frac{\partial\theta}{\partial v} - (b' + c)\frac{\partial\theta}{\partial v} + \left[bc' - bc' - (b'_v + c'_u) + d\right]\theta,$$

and from (2) and (5)

$$\frac{\partial b}{\partial v} = 2\frac{\partial b'}{\partial u}. \qquad \qquad (8)$$

Accordingly, θ must satisfy the system of the equations.

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = +b\frac{\partial\theta}{\partial u} - c\frac{\partial\theta}{\partial v} - \left[d + 2(b'_v + c'_u)\right]\theta,$$

$$\frac{\partial^2\theta}{\partial u\partial v} = -b'\frac{\partial\theta}{\partial v} - c\frac{\partial\theta}{\partial v} + \left[d' - b'_u - c'_v\right]\theta.$$

This system is completely integrable in virtue of (7), (8) and the conditions of the complete integrability of (1).

From (7) and (8) we know that the surface (S) admits the projective transformation and the curves C_u and C_v form the conjugate net of the projective deformation.

The focal surface which is described by ν is given by the system

$$\frac{\partial^2\nu}{\partial u^2} + \frac{\partial^2\nu}{\partial v^2} = B\frac{\partial\nu}{\partial u} + C\frac{\partial\nu}{\partial v} + D\nu,$$

$$\frac{\partial^2\nu}{\partial u\partial v} = \left(b' + \frac{\theta_u}{\theta}\right)\frac{\partial\nu}{\partial u} + \left[c' + \frac{1}{b'\theta + \theta_v}\frac{\partial}{\partial u}(b'\theta + \theta_v)\right]\frac{\partial\nu}{\partial v},$$

where

$$B = b + \frac{2\theta_u}{\theta}.$$

Therefore we know that this surface also admits the projective deformation.

Similarly, we can see that the other focal surface also admits the projective deformation.

§ 2.

We have, hitherto, supposed that a surface is given and considered congruence Γ and Γ' conjugate to that surface.

We shall, hereafter, suppose that a congruence is given and consider surfaces which have the following relations with that congruence.

1° There are one to one correspondences between the lines of given congruence and points on a surface (S).

2° A tangent plane at a point on (S) passes through the corresponding line of the congruence.

3° To the developable surfaces of the congruence correspond a conjugate net on (S). ......(8)

Wilcznski[1] has showed that the general theory of the congruence may be based upon the theory of the following completely integrable system of the partial differential equations.

$$\left.\begin{aligned} &\frac{\partial y}{\partial v} = mz, \qquad \frac{\partial z}{\partial u} = ny. \\ &\frac{\partial^2 y}{\partial u^2} = ay + bz + c\frac{\partial y}{\partial u} + d\frac{\partial z}{\partial v}, \\ &\frac{\partial^2 z}{\partial v^2} = a'y + b'z + c'\frac{\partial y}{\partial u} + d'\frac{\partial z}{\partial v}. \end{aligned}\right\} \qquad (9)$$

The system (8) has precisely four pairs of linearly independent solutions (y_k, z_k), $(k = 1, 2, 3, 4)$, such that the general solution will be of the form

$$y = \sum_{k=1}^{4} C_k y_k \qquad z = \sum_{k=1}^{4} C_k y_k$$

Let $y_1, \ldots\ldots, y_4$ and $z_1, \ldots\ldots, z_4$ be interpreted as the homogeneous coordinates of two points P_y and P_z.

As u and v vary, these points will describe two surfaces S_y and S_z respectively, and the line P_yP_z will generate a congruence whose focal surfaces consist of the two surfaces S_y and S_z.

Let us assume S_y and S_z are not degenerate.

By the transformations

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

[1] Trans. Amer. Math. Soc., **16**, 310–327 (1915).

which do not alter the congruence, system (9) is transformed into a system of the same form. Denoting its coefficients by $\bar{m}$, $\bar{n}$, $\bar{a}$, etc., we have

$$\bar{m}=\frac{m}{\psi_v}, \qquad \bar{n}=\frac{n}{\phi_u},$$

$$\bar{d}=\frac{\psi_v}{\phi_u^2}d, \qquad \bar{c}'=\frac{\phi_u}{\psi_v^2}c'.$$

The condition of the complete integrability of the system (9) are

$$\left.\begin{aligned}
&c=f_u, \qquad d'=f_v, \qquad b=-d_v-df_v,\\
&a'=-c'_u-c'f_u, \qquad mn-c'd=f_{uv},\\
&m_{uu}+d_{vv}+df_{vv}+d_vf_v-f_um_u=ma+db',\\
&n_{vv}+c'_{uu}+c'f_{uu}+c'_uf_u-f_vn_v=c'a+nb',\\
&2m_un+mn_u=a_v+f_umn+a'd,\\
&m_vn+2mn_v=b'_u+f_vmn+bc'.
\end{aligned}\right\} \quad (10)$$

By the relation 3° of (8), the curves C_u (v const.) C_v (u-const.) on (S) form a conjugate net and the given congruence is a congruence Γ of (S) with respect to that conjugate net. Therefore, to the line which passes through points (y) and (z), corresponds a point ($\bar{x}$) on (S), and tangents at ($\bar{x}$) to the curves C_u and C_v which meet at ($\bar{x}$), pass through the focal points of this line, *i.e.*, the points (y) and (z) respectively.[1]

Hence ($\bar{x}$) is determined by the equations

$$\frac{\partial\bar{x}}{\partial u}=\lambda\bar{x}+\mu y, \qquad \dots\dots\dots\dots (11)$$

$$\frac{\partial\bar{x}}{\partial v}=\lambda_1\bar{x}+\mu_1 z. \qquad \dots\dots\dots\dots (12)$$

Differentiating (11) by v and (12) by u, we have

$$\frac{\partial}{\partial v}\left(\frac{\partial\bar{x}}{\partial u}\right)=\left(\lambda\lambda_1+\frac{\partial\lambda}{\partial v}\right)\bar{x}+(\lambda\mu_1+\mu m)z+\frac{\partial\mu}{\partial v}y,$$

$$\frac{\partial}{\partial u}\left(\frac{\partial\bar{x}}{\partial v}\right)=\left(\lambda\lambda^1+\frac{\partial\lambda_1}{\partial u}\right)\bar{x}+(\lambda_1\mu+\mu n)y+\frac{\partial\mu_1}{\partial u}z.$$

[1] J. Kanitani, Mem. Coll. Sci., **5**, 329 (1922).

Equating these two values of $\frac{\partial^2 \bar{x}}{\partial u \partial v}$, we have

$$\frac{\partial \lambda}{\partial v} = \frac{\partial \lambda_1}{\partial u},$$

$$\frac{\partial \mu}{\partial v} = \lambda_1 \mu + \mu_1 n,$$

$$\frac{\partial \mu_1}{\partial n} = \lambda \mu_1 + \mu m.$$

From these equations, we see that there is a function θ such that

$$\lambda = -\frac{\theta_u}{\theta}, \qquad \lambda_1 = -\frac{\theta_v}{\theta}.$$

$$\theta \frac{\partial \mu}{\partial v} + \mu \frac{\partial \theta}{\partial v} = \theta \mu_1 n,$$

$$\theta \frac{\partial \mu_1}{\partial u} + \mu_1 \frac{\partial \theta}{\partial v} = \theta \mu m.$$

Substituting the above values λ, λ_1 in (11) and (12), we know that $\bar{x}$ is given by the equations

$$\theta \frac{\partial \bar{x}}{\partial u} + \frac{\partial \theta}{\partial u} \bar{x} = \theta \mu y,$$

$$\theta \frac{\partial \bar{x}}{\partial v} + \frac{\partial \theta}{\partial v} \bar{x} = \theta \mu_1 z.$$

Put

$$x = \theta \bar{x}, \qquad p = \mu \theta, \qquad q = \mu_1 \theta.$$

Then we know that x is given by the equations

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= py, \\ \frac{\partial x}{\partial v} &= qz. \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (13)$$

where the functions p, q are a pair of solutions of the system of the partial differential equations

$$\left.\begin{aligned} \frac{\partial p}{\partial v} &= nq, \\ \frac{\partial q}{\partial n} &= mp. \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (14)$$

As by assumption, both S_y and S_z are not degenerate, the four points (y), (z), $\left(\frac{\partial y}{\partial u}\right)$, $\left(\frac{\partial z}{\partial v}\right)$ are not coplanar.

Therefore, x may be expressed in the form

$$x = w_1 y + w_2 z + w_3 \frac{\partial y}{\partial u} + w_4 \frac{\partial z}{\partial v}.$$

By (9) and (13) the above functions w_1, w_2, w_3, w_4 must satisfy the system of the equations

$$\left.\begin{aligned}
&\frac{\partial w_1}{\partial u} + n w_2 + a w_3 + n_v w_4 = p, \\
&\frac{\partial w_2}{\partial u} + b w_3 + m n w_4 = 0, \\
&\frac{\partial w_3}{\partial u} + c w_3 + w_1 = 0, \\
&\frac{\partial w_4}{\partial u} + d w_3 = 0, \\
&\frac{\partial w_1}{\partial v} + w_3 m n + a' w_4 = 0, \\
&\frac{\partial w_2}{\partial v} + m w_1 + m_u w_3 + b' w_4 = q, \\
&\frac{\partial w_3}{\partial v} + c' w_4 = 0, \\
&\frac{\partial w_4}{\partial v} + w_2 + d' w_4 = 0.
\end{aligned}\right\} \qquad (15)$$

And x must satisfy the system of the equations

$$\frac{w_3}{p}\frac{\partial^2 x}{\partial u^2} + \frac{w_4}{q}\frac{\partial^2 x}{\partial v^2} + \left(\frac{w_1}{p} - \frac{w_3 p_u}{p^2}\right)\frac{\partial x}{\partial u} + \left(\frac{w_2}{q} - \frac{w_4 q_u}{q^2}\right)\frac{\partial x}{\partial v} = x,$$

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{p_v}{p}\frac{\partial x}{\partial u} + \frac{q_u}{q}\frac{\partial x}{\partial v}.$$

The asymptotic lines S_y, S_z and S are given by the equations

$$\frac{d}{m} du^2 + dv^2 = 0,$$

$$du^2 + \frac{c'}{n} dv^2 = 0,$$

$$\frac{pw_4}{qw_3}du^2-dv^2=0.$$

If the given congruence is a W-congruence and the asymptotic lines of the focal surfaces and S-correspond, we must have

$$\frac{d}{m}=\frac{n}{c'}=\frac{-pw_4}{qw_3}, \qquad (16)$$

or

$$\frac{dw_3}{w_4}=-\frac{mp}{q},$$

$$\frac{c'w_4}{w_3}=-\frac{nq}{p}.$$

These are equivalent to

$$\frac{1}{w_4}\frac{\partial w_4}{\partial u}=\frac{1}{q}\frac{\partial q}{\partial u},$$

$$\frac{1}{w_1}\frac{\partial w_3}{\partial v}=\frac{1}{p}\frac{\partial p}{\partial v}.$$

Integrating, we have

$$w_3=p\,U(u),$$

$$w_4=q\,U(v).$$

where $U(u)$ and $V(v)$ are certain functions of u and v only respectively.

Substituting the above values of w_3, w_4 in (16) we have

$$\frac{d}{m}=\frac{n}{c'}=-\frac{V(v)}{U(u)}.$$

Therefore by the transformation

$$\bar{u}=\int\frac{du}{\sqrt{U(u)}},$$

$$\bar{v}=\int\frac{dv}{\sqrt{V(v)}}.$$

we can reduce c, d, m and n so as to the satisfy the relations

$$d = -m,$$
$$c' = -n.$$

Let us suppose that we have performed this transformation, Then we have

$$\frac{pw_4}{qw_3} = 1,$$

from which we can easily see that

$$\left.\begin{aligned} kw_3 &= p, \\ kw_4 &= q, \end{aligned}\right\} \qquad (17)$$

where k is a constant.

From these relations and (15) we have

$$\left.\begin{aligned} kw_1 &= -(p_u + cp), \\ kw_2 &= -(q_v + d'q). \end{aligned}\right\} \qquad (18)$$

By (14), (15), (17) and (18), p and q must satisfy the syetem of equations

$$\left.\begin{aligned} &\frac{\partial p}{\partial v} = nq, \qquad \frac{\partial q}{\partial u} = mp, \\ &\frac{\partial^2 p}{\partial u^2} = (a - c_u - k)p + (n_v - nd')q - c\frac{\partial p}{\partial u} - n\frac{\partial q}{\partial v}, \\ &\frac{\partial^2 q}{\partial v^2} = (m_u - mc)p + (b' - d'_u - k)q - m\frac{\partial p}{\partial u} - d'\frac{\partial q}{\partial v}. \end{aligned}\right\} \qquad (19)$$

When

$$d = -m, \qquad c' = -n.$$

The conditions (10) become

$$\left.\begin{aligned} &c: \quad \text{a function of } u \text{ only}, \\ &d: \quad \text{a function of } v \text{ only}, \\ &b = m_v + md', \qquad a' = n_u + nc, \\ &m_{uu} - m_{vv} - md'_v - m_v d' - m_u c = m(a - b'), \\ &n_{vv} - n_{uu} - mc'_u - n_u c - u_v d' = n(a - b'), \\ &2\frac{\partial}{\partial u}(mn) = \frac{\partial a}{\partial v}, \\ &2\frac{\partial}{\partial v}(mn) = \frac{\partial b'}{\partial n}. \end{aligned}\right\} \qquad (10')$$

We can easily see that the system (19) is completely integrable by (10′). Therefore, we have four linearly independent pairs of solutions for (19).

The surface (S) which is in the relation above described with the given congruence, is given by the system

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial n^2}+\frac{\partial^2 x}{\partial v^2}&=\left(c+\frac{2p_u}{p}\right)\frac{\partial x}{\partial u}+\left(d'+\frac{2q_v}{q}\right)\frac{\partial x}{dv}+kx,\\ \frac{\partial^2 x}{\partial u\partial v}&=\frac{p_v}{p}\frac{\partial x}{\partial u}+\frac{q_u}{q}\frac{\partial x}{\partial v}. \end{aligned}\right\} \qquad (20)$$

Therefore, we know that *if a surface (S) has W-congruence (D) as its congruence Γ, and the asymptotic lines of (S) correspond to those of the focal surfaces of (D), the surface (S) and the focal surfaces admit the projeceive deformation and the edges of regression of the developables of (D) form the conjugate net of projective deformation on the focal surface on which they lie with their conjugate lines.*

Reciprocally, if W-congruence (D) is such that the focal surfaces of it admit projective deformation and edges of regression of the developables of it are the lines of the conjugate net of projective deformation of the focal surface on which they lie, there are surfaces which have the congruence (D) as a congruence Γ of them and of which the asymptotic lines correspond to those of the focal surfaces of (D).

We have elsewhere[1] proved that if there is an isothermal conjugate net on a surface (S) such that the asymptotic lines of (S) and its minus-first or first Laplace-Dardoux transform correspond, (S) admits the projective deformation and the conjugate net is that of projective deformation,

Therefore, we may say as follows:—

If the edges of regression of the developables of a W-congruence form an isothermal conjugate net with their conjugate lines on the focal surface on which they lie, there are surfaces which have that congruence as a congruence Γ of them and of which the asymptotic lines correspond to those of the focal surfaces of that congruence.

Consider a congruence D_λ which is given by the system

[1] J. Kanitani, Mem. Coll. Sci., **6**, 29–43 (1922).

$$\left.\begin{aligned}
&\frac{\partial y}{\partial v}=mz, \qquad\qquad \frac{\partial z}{\partial u}=ny,\\
&\frac{\partial^2 y}{\partial u^2}=(a+\lambda)y+bz+c\frac{\partial y}{\partial u}-m\frac{\partial z}{\partial v},\\
&\frac{\partial^2 y}{\partial v^2}=a'y+(b'+\lambda)z-n'\frac{\partial y}{\partial u}+d'\frac{\partial z}{\partial v},\\
&\qquad (\lambda:\text{ const.})
\end{aligned}\right\}\quad(21)$$

which is completely integrable by (10′).

Denote by D_0 the congruence given by the system which is obtained by putting $\lambda=0$ in (21) and by $S_{y\lambda}$, $S_{z\lambda}$ the focal surfaces which are described by the points (y), (z) respectively, which are defined by (21).

Evidentiy $S_{y\lambda}$, and $S_{z\lambda}$ are projectively deformable to S_y and S_z respectivly.

The surface S_λ which is given by

$$\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}=\left(c+\frac{2p_u}{p}\right)\frac{\partial x}{\partial n}+\left(d'+\frac{2q_v}{q}\right)\frac{\partial x}{\partial v}+(k+\lambda)x,$$

$$\frac{\partial^2 x}{\partial u\partial v}=\frac{p_v}{p}\frac{\partial x}{\partial u}+\frac{q_u}{q}\frac{\partial x}{\partial v}.$$

where p, q are the same function as those which appear in (20), *i.e.*, a pair of solutions of (19), has the same relation with D_λ as those which S has with D_0 and is projectively deformable to S.

Therefore we have the following theorem.

Let D be a congruence which is formed by tangent lines of a family of curves of a conjugate net of projective deformation of a surface S of the first kind which affords only ∞^1 surfaces which are projectively deformable to it, and Σ be a surface which has D as its congruence Γ and of which the asymptotic lines correspond to those of the focal surfaces of D, that is, those of S.

If a surface S_λ be projectively deformable to S, there are surfaces projectively deformable to Σ among those which have the same relations with S_λ as Σ has with S.

§ 3.

In this paragraph we shall consider a surface (S) which are in the following relation with the congruence which is given by system (9).

1° There is a one to one correspondence between the points on (S) and the lines of the given congruence.

2° A point on (S) lies on the corresponding line of the congruence.

3° Developables of the congruence cut (S) in a conjugate net.

Coordinates (x) of such surface are given by[1]

$$mx = p\frac{\partial y}{\partial v} - \frac{\partial p}{\partial v}y,$$

where p is a solution of the equation

$$\frac{\partial^2 p}{\partial u \partial v} = \frac{m_u}{m}p + mnp,$$

which is satisfied by the coordinates (y) of a point on the focal surface S_y.

If we put

$$q = \frac{1}{m}\frac{\partial p}{\partial v},$$

we have

$$x = pz - qy, \qquad \dots\dots (22)$$

$$\left.\begin{aligned} \frac{\partial p}{\partial v} &= mq, \\ \frac{\partial q}{\partial u} &= np. \end{aligned}\right\} \qquad \dots\dots (23)$$

From (22) and (23) we have

$$\frac{\partial x}{\partial u} = \frac{\partial p}{\partial u}z - q\frac{\partial y}{\partial u},$$

$$\frac{\partial x}{\partial v} = p\frac{\partial z}{\partial v} - \frac{\partial q}{\partial u}y,$$

from which follows

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial x}{\partial u} + \frac{p_u}{p}\frac{\partial x}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)x.$$

If we put

$$w = ap + bq + c\frac{\partial p}{\partial u} + d\frac{\partial q}{\partial v} - \frac{\partial^2 p}{\partial u^2},$$

1 Darboux, II., chap. X.

$$\tau = a'p + b'q + c'\frac{\partial p}{\partial u} + d'\frac{\partial q}{\partial v} - \frac{\partial^2 q}{\partial^2 v},$$

we have from (10), (22) and (23).

$$\frac{\partial^3 x}{\partial u^2} = \left(\frac{np}{q} + c\right)\frac{\partial x}{\partial u} - \frac{dq}{p}\frac{\partial x}{\partial v} + \left(a + \frac{dq_v}{p} - \frac{np_u}{q}\right)x - wz,$$

$$\frac{\partial^2 x}{\partial v^2} = -\frac{c'p}{q}\frac{\partial x}{\partial u} + \left(d' + \frac{mq}{p}\right)\frac{\partial x}{\partial v} + \left(b' + \frac{c'p_u}{q} - \frac{mq_v}{p}\right)x + \tau y.$$

from which we know that x satisfies the equation of the form

$$\frac{\partial^2 x}{\partial u^2} = A\frac{\partial^2 x}{\partial v^2} + B\frac{\partial x}{\partial u} + C\frac{\partial x}{\partial v} + Dx,$$

where

$$A = -\frac{wq}{\tau p}.$$

If the asymptotic lines of (S) and the focal surfaces S_y, S_z correspond, we must have

$$\frac{d}{m} = \frac{n}{c'} = \frac{-wq}{\tau p}. \quad \text{.. (24)}$$

But we know from (10) that w and τ satisfy the equation

$$\left.\begin{aligned} \frac{\partial w}{\partial v} &= -d\tau, \\ \frac{\partial w}{\partial u} &= -c'w. \end{aligned}\right\} \quad \text{.. (25)}$$

From (24) and (25) we can see, as in the preceding paragraph, that by the transformation of the form

$$\bar{u} = U(u), \qquad \bar{v} = V(v),$$

m, n, c, c may be reduced so as to satisfy the equation

$$d = -m,$$
$$c' = -n.$$

Let us suppose that we have performed the above transformation. Then we have

$$w = kp,$$
$$\tau = kq,$$

where k is a constant, and p and q satisfy the system of the equations

$$\frac{\partial p}{\partial v} = mq, \qquad \frac{\partial q}{\partial u} = np,$$

$$\frac{\partial^2 p}{\partial u^2} = (a-b)p + bq + c\frac{\partial p}{\partial u} + d\frac{\partial q}{\partial v},$$

$$\frac{\partial^2 q}{\partial v^2} = a'p + (b'-k)p + c'\frac{\partial p}{\partial u} + d'\frac{\partial q}{\partial v},$$

which is completely integrable by (10′).

The surface (S) which satisfies our conditions is given by the system

$$\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial x}{\partial u} + \left(d' + \frac{2p_v}{p}\right)\frac{\partial \lambda}{\partial v}$$
$$+ \left(a + b' - \frac{2mq_v}{p} - \frac{2np_u}{q} - k\right)x.$$

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial x}{\partial u} + \frac{p_u}{p}\frac{\partial x}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)x.$$

Therefore we know that the theorems in § 2 also hold, if we substitute congruence Γ' for the congruence Γ.

§ 4.

Consider two surfaces S_x and S_X which are in the relations explained in § 3 with the given congruence given by the system.

$$\frac{\partial y}{\partial v} = mz, \qquad \frac{\partial z}{\partial u} = ny,$$

$$\frac{\partial^2 x}{\partial u^2} = ay + bz + c\frac{\partial y}{\partial u} - m\frac{\partial z}{\partial u},$$

$$\frac{\partial^2 x}{\partial v^2} = a'y + b'z - n\frac{\partial y}{\partial u} + d'\frac{\partial z}{\partial v},$$

of which the conditions of completely integrability are (10′).

Suppose S_x and S_X are described by the points (x) and (X) respectively such that.

$$\left.\begin{aligned} x &= pz - qy, \\ X &= Pz - Qy, \end{aligned}\right\} \qquad (26)$$

where (p, q) and (P, Q) are pairs of the solutions of the system

$$\left.\begin{aligned} &\frac{\partial p}{\partial v} = mq, \qquad \frac{\partial q}{\partial u} = np, \\ &\frac{\partial^2 p}{\partial u^2} = (a-k)p + bq + c\frac{\partial p}{\partial u} - m\frac{\partial q}{\partial v}, \\ &\frac{\partial^2 q}{\partial v^2} = a'p + (b'-k)q - n\frac{\partial p}{\partial u} + d'\frac{\partial q}{\partial v}, \end{aligned}\right\} \qquad (27)$$

and the system

$$\left.\begin{aligned} &\frac{\partial P}{\partial v} = mQ, \qquad \frac{\partial Q}{\partial u} = nP, \\ &\frac{\partial^2 P}{\partial u^2} = (a-k_1)P + bQ + c\frac{\partial P}{\partial u} - m\frac{\partial Q}{\partial v}, \\ &\frac{\partial^2 Q}{\partial v^2} = a'P + (b'-K)Q - n\frac{\partial P}{\partial u} + d'\frac{\partial Q}{\partial v}, \end{aligned}\right\} \qquad (28)$$

respectively.

Then x satisfies the system of equations

$$\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial x}{\partial u} + \left(d' + \frac{2p_v}{p}\right)\frac{\partial x}{\partial v},$$

$$+\left(a + b' - \frac{2mq_v}{p} - \frac{2np_v}{q} - k\right)x,$$

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial x}{\partial u} + \frac{p_u}{p}\frac{\partial x}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)x,$$

while X satisfies the system of the equations

$$\frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial X}{\partial u} + \left(d' + \frac{2p_v}{p}\right)\frac{\partial X}{\partial v},$$

$$+\left(a + b' - \frac{2mq_v}{p} - \frac{2np_v}{q} - k_1\right)X,$$

$$\frac{\partial^2 X}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial X}{\partial u} + \frac{p_u}{p}\frac{\partial X}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)X.$$

Put

$$\theta = pQ - qP, \qquad \text{...} (29)$$

Then we have from (27) and (28).

$$\frac{\partial\theta}{\partial u} = \frac{\partial p}{\partial u}Q - q\frac{\partial P}{\partial u},$$

$$\frac{\partial\theta}{\partial v} = p\frac{\partial Q}{\partial v} - \frac{\partial q}{\partial v}Q,$$

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial p}{\partial u}\frac{\partial Q}{\partial v} - \frac{\partial P}{\partial u}\frac{\partial q}{\partial v} + mn\theta,$$

$$\frac{\partial^2\theta}{\partial u^2} = \left(c + \frac{q_u}{q}\right)\frac{\partial\theta}{\partial u} + \frac{p_v}{p}\frac{\partial\theta}{\partial v} + \left(a - \frac{mq_v}{p} - \frac{np_u}{q}\right)\theta$$

$$- kpQ + k_1 qP,$$

$$\frac{\partial^2\theta}{\partial v^2} = \frac{q_u}{q}\frac{\partial\theta}{\partial u} + \left(d' + \frac{p_v}{p}\right)\frac{\partial\theta}{\partial v} + \left(b' - \frac{np_u}{q} - \frac{mq_v}{p}\right)\theta$$

$$- k_1 pQ + kqP,$$

from which follow

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial\theta}{\partial u} + \left(d' + \frac{p_v}{p}\right)\frac{\partial\theta}{\partial v}$$

$$+ \left\{a + b' - \frac{2mq_v}{p} - \frac{2np_u}{q} - (k + k_1)\right\}\theta,$$

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{q_v}{q}\frac{\partial\theta}{\partial u} + \frac{p_u}{p}\frac{\partial\theta}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)\theta,$$

Similarly we have

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = \left(c + \frac{2Q_u}{Q}\right)\frac{\partial\theta}{\partial u} + \left(d' + \frac{P_v}{P}\right)\frac{\partial\theta}{\partial v},$$

$$+ \left\{a + b' - \frac{2mQ_v}{P} - \frac{2nP_u}{Q} - (k + k_1)\right\}\theta,$$

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{P_v}{P}\frac{\partial\theta}{\partial u} + \frac{P_u}{P}\frac{\partial\theta}{\partial v} + \left(mn - \frac{P_u Q_v}{PQ}\right)\theta.$$

But from (26) and (29), we easily see that

$$Q\left(\theta\frac{\partial x}{\partial u}-\frac{\partial\theta}{\partial u}x\right)\equiv q\left(\theta\frac{\partial X}{\partial u}-\frac{\partial\theta}{\partial u}X\right),$$

$$P\left(\theta\frac{\partial x}{\partial v}-\frac{\partial\theta}{\partial v}x\right)\equiv p\left(\frac{\partial X}{\partial v}-\frac{\partial\theta}{\partial v}X\right).$$

Therefore we have the following theorem:—

Let D be a W-congruence such that the edges of regression of its developable surfaces from an isothermal conjugate net on the focal surface on which they lie with their conjugate lines, and S and T be two surfaces which have D as their congruence Γ' and of which the asymptotic lines correspond to those of the focal surfaces of D. Then the lines of intersection of the tangent planes of S and T at the corresponding points from a congruence Γ conjugate to S, as well as to T such that the asymptotic lines of S, T and their focal surfaces correspond.

Let $(P_1,\ Q_1)$, $(P_2,\ Q_2)$, $(P_3,\ Q_4)$ and $(P_4,\ Q_Q)$ be four lieearly indedendent pairs of solution of the system (28).

Then $\theta_1,\ \theta_2,\ \theta_3,\ \theta_4$ such that

$$\theta_i=pQ_i-qP_i\quad(i=1,\ 2,\ 3,\ 4).$$

are also linearly independent.

For if there be a relation such that

$$c_1\theta_1+c_2\theta_2+c_3\theta_3+c_4\theta_4=0,$$

we have by (27) and (28),

$$\sum_{i=1}^{4}\frac{\partial}{\partial v}\left(\frac{c_iP_i}{p}\right)=0,$$

$$\sum_{i=1}^{4}\frac{\partial}{\partial u}\left(\frac{c_iQ_i}{q}\right)=0,$$

Integrating, we have

$$\sum_{i=1}^{4}c_iP_i=U(u)p,$$

$$\sum_{i=1}^{4}c_iQ_i=V(v)q.$$

Differentiating the former by v and comparing the result with the latter, we have

$$U(u)=V(v)=C,$$

where C is a constant.

If $k_1 \neq k$, we must have $C=0$.

This contradicts our assumption.

When $k_1=k$, we may take

$$P_1=p, \qquad Q_1=q,$$

and reduce to a similar contradiction.

Therefore, we know that *congruences Γ conjugate to S such that the asymptotic lines of their focal surfaces correspond to those of S, are those which are formed by intersections of tangent planes at corresponding points of S and a surface which has the same relation as S with D.*