On the Principle of the Light-Velocity

By

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In the special theory of relativity, we assume that the velocity of the light is absolutely constant for any observer; whence it follows that the velocity of the light is maximum. In the general theory of relativity, the assumption on the light becomes $ds^2=0$. If we intend to treat the theories deductively, we must at first assume that $ds^2=0$ for the light. Thus in the deductive treatment, the principle on the light is not banal for understanding. In this note its meaning is considered.

The statical gravitation-field of Einstein is, after Schwalzschild,[1]

$$ds^2=-\frac{dr^2}{\gamma}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+\gamma dt^2, \qquad (1)$$

$$\gamma=1-\frac{2m}{r}.$$

The motion of a particle in the plane $\varphi=$const., without any *masses*, is given by the equations

$$\left.\begin{aligned} r^2\frac{d\theta}{ds}&=h, \\ \frac{dt}{ds}&=\frac{k}{\gamma}, \\ \left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\theta}{ds}\right)^2&=k^2-1+\frac{2m}{r}-\frac{2mh^2}{r^3}, \end{aligned}\right\} \qquad (2)$$

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1 Eddington, Report on the relativity theory of gravitation.

where h and k are integration-constants.

In newtonion dynamics of the planetary motion, we have

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\frac{2m}{r}-\frac{m}{a}, \tag{3}$$

where m is the gravitational mass of the sun, a, the semimajor axis of the orbit. By the principle of equivalence, we have

$$k^2-1=-\frac{m}{a}, \quad \text{or} \quad k=\sqrt{1-\frac{m}{a}}. \tag{4}$$

Therefore we must limit the variation of the constant k. If a be small or m be great k would be imaginary and the principle of equivalence would be fatal. Now if we restrict the principle in the limit that (4) should be true, we have nothing to say. But we *extend* to apply the principle for any central planetary motion, as newtonian dynamics does, though actually by observations, we can not yet ascertain any posslble planetary motian given by the dynamics.

When k becomes great, (4) would be impossible. The reason is that (3) belongs to the elliptic motion. For the parabolic and hyperbolic motions, in newtonian dynamics, we have

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\frac{2m}{r}, \tag{5}$$

respectively

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\frac{2m}{r}+\frac{m}{a}. \tag{6}$$

Comparing (2) and (5), we have by the principle of equivalence

$$k^2-1=0, \quad \text{or} \quad k=1. \tag{7}$$

From (2) and (6),

$$k^2-1=\frac{m}{a}, \quad \text{or} \quad k=\sqrt{1+\frac{m}{a}}. \tag{8}$$

We observe that even when (4) be an imaginary equation, yet (8) is real. (6) is their limiting case.

Taking the light for our signal, assume that the velocity of the light-signal is maximum. By analogy with newtonian case, it is natural to take it ∞. By the last equation of (2), we have $k=\infty$; hence by the second,

$$ds=0. \tag{9}$$

To obtain the equation of motion of the signal, eliminating ds from (2), we have

$$\frac{r^2}{\gamma}\frac{d\theta}{dt}=\frac{h}{k}.$$

If the signal really exist, we must have

$$\lim_{k\to\infty}\frac{h}{k}=l, \ (\textit{finite, determinate}).$$

Or else, by the last equation of (2), the velocity could not be ∞. We have therefore,

$$\left.\begin{aligned}\frac{r^2}{\gamma}\frac{d\theta}{dt}&=l,\\ \frac{1}{\gamma^2}\left(\frac{dr}{dt}\right)^2+\frac{r^2}{\gamma}\left(\frac{d\theta}{dt}\right)^2&=1,\end{aligned}\right\} \tag{10}$$

which can easily be deduced from Fermat's principle

$$\delta\int dt=0,$$

where $ds^2=0$.

Eliminating dt between the equations (10), the path of the light is

$$\left(\frac{du}{d\theta}\right)^2=\frac{1}{l^2}-u^2+2mu^3, \quad u=\frac{1}{r}, \tag{11}$$

and the light does not propagate in a straight line. It seems verified by the famous astronomical observation.

For the special theory of relativity, we put $m=0$, and (10) becomes

$$\left.\begin{aligned}r^2\frac{d\theta}{dt}&=l\\ \left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2&=1\end{aligned}\right\} \tag{12}$$

Hence the velocity of the light is unity and it propagates in a straight line. The *Massbestimmung* (1) becomes

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + dt^2.$$

Changing into the rectangular cartesian coordinates (x, y, z) and the unit of the time, we have

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2, \tag{13}$$

where c is the velocity of the light in vacuum. For the light-signal, we have by (9),

$$dt^2 = \frac{dx^2 + dy^2 + dz^2}{c^2}.$$

We arrived at this result under the assumption that the light-signal has the maximum velocity. In newtonian dynamics, we consider the signal of ∞ —velocity but not of the light. For this case, tending $c \rightarrow \infty$, we have

$$dt = 0. \tag{14}$$

Since in newtonian dynamics, there is only one kind of the time, this equation just corresponds to (9) in Einstein's theory. From these considerations, it seems unnecessary to reject ∞ —*velocity* (with respect to the proper time) in Einstein's theory.