Projective Differential Geometry of Nondevelopable Surfaces. I.

BY

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CHAPTER I.

INVARIANTS.

1. Let $\mathfrak{x}_1$, $\mathfrak{x}_2$, $\mathfrak{x}_3$, $\mathfrak{x}_4$ be homogeneous coordinates of the point $\mathfrak{x}$ in the projective space of the three dimensions and suppose that a surface is given by the equations

$$\mathfrak{x}_i = \mathfrak{x}_i(u, v), \qquad (i = 1, 2, 3. 4)$$

where $\mathfrak{x}_i(u, v)$ are analytic functions of u, v in a domain R.

Consider the determinants of the matrix

$$\left\| \begin{array}{cccccc} \mathfrak{x}_1 & \dfrac{\partial \mathfrak{x}_1}{\partial u} & \dfrac{\partial \mathfrak{x}_1}{\partial v} & \dfrac{\partial^2 \mathfrak{x}_1}{\partial u^2} & \dfrac{\partial^2 \mathfrak{x}_1}{\partial u \partial v} & \dfrac{\partial^2 \mathfrak{x}_1}{\partial v^2} \\ \mathfrak{x}_2 & \dfrac{\partial \mathfrak{x}_2}{\partial u} & \dfrac{\partial \mathfrak{x}_2}{\partial v} & \dfrac{\partial^2 \mathfrak{x}_2}{\partial u^2} & \dfrac{\partial^2 \mathfrak{x}_2}{\partial u \partial v} & \dfrac{\partial^2 \mathfrak{x}_2}{\partial v^2} \\ \mathfrak{x}_3 & \dfrac{\partial \mathfrak{x}_3}{\partial u} & \dfrac{\partial \mathfrak{x}_3}{\partial v} & \dfrac{\partial^2 \mathfrak{x}_3}{\partial u^2} & \dfrac{\partial^3 \mathfrak{x}_3}{\partial u \partial v} & \dfrac{\partial^2 \mathfrak{x}_3}{\partial v^2} \\ \mathfrak{x}_4 & \dfrac{\partial \mathfrak{x}_4}{\partial u} & \dfrac{\partial \mathfrak{x}_4}{\partial v} & \dfrac{\partial^2 \mathfrak{x}_4}{\partial u^2} & \dfrac{\partial^2 \mathfrak{x}_4}{\partial u \partial v} & \dfrac{\partial^2 \mathfrak{x}_4}{\partial v^2} \end{array} \right\|$$

and put

$$\left.\begin{array}{l} a = \quad | \mathfrak{x}_u \quad \mathfrak{x}_v \quad \mathfrak{x}_{uu} \quad \mathfrak{x}_{uv} |, \\ a' = \dfrac{1}{2} | \mathfrak{x}_u \quad \mathfrak{x}_v \quad \mathfrak{x}_{uu} \quad \mathfrak{x}_{vv} |, \end{array}\right\}$$

$$
\left.\begin{aligned}
a'' &= \quad |\mathfrak{X} \quad \mathfrak{X}_{uv} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{vv}|,\\
b &= \quad |\mathfrak{X} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{uv}|,\\
b' &= \frac{1}{2}|\mathfrak{X} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{vv}|,\\
b'' &= \quad |\mathfrak{X} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uv} \quad \mathfrak{X}_{vv}|,\\
c &= \quad |\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{uv}|,\\
c' &= \frac{1}{2}|\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{vv}|,\\
c'' &= \quad |\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{uv} \quad \mathfrak{X}_{vv}|,\\
d &= \quad |\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uu}|,\\
d' &= \quad |\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uv}|,\\
d'' &= \quad |\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{vv}|,
\end{aligned}\right\}\quad\text{.........(1)}
$$

where $|\mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uu} \ \mathfrak{X}_{uv}|$ denote the determinants the i^{th} row of which consists of $\frac{\partial \mathfrak{X}_i}{\partial u}, \frac{\partial \mathfrak{X}_i}{\partial v}, \frac{\partial^2 \mathfrak{X}_i}{\partial u^2}, \frac{\partial^2 \mathfrak{X}_i}{\partial v^2}$.

We shall call these determinants the *fuudamental determinants* of the given surface.

From (1) we have

$$\frac{\partial d}{\partial v} = b - 2c' + |\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uuv}|,$$

$$\frac{\partial d'}{\partial u} = -b + |\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uuv}|,$$

$$\frac{\partial d'}{\partial v} = -c'' + |\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uvv}|,$$

$$\frac{\partial d''}{\partial u} = -b' + c'' + |\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uvv}|.$$

Therefore,

$$
\left.\begin{aligned}
\frac{\partial d}{\partial v} - \frac{\partial d'}{\partial u} &= 2(b - c'),\\
\frac{\partial d'}{\partial v} - \frac{\partial d''}{\partial u} &= 2(b' - c'').
\end{aligned}\right\}\quad\text{........................(2)}
$$

Consider the determinant

$$\begin{vmatrix}
\mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial u} & \frac{\partial \mathfrak{X}_1}{\partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial v^2} & 0 & 0 \\
\mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial u} & \frac{\partial \mathfrak{X}_2}{\partial u} & \frac{\partial^2 \mathfrak{X}_2}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_2}{\partial v^2} & 0 & 0 \\
\mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial u} & \frac{\partial \mathfrak{X}_3}{\partial v} & \frac{\partial \mathfrak{X}_3}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_3}{\partial v^2} & 0 & 0 \\
\mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial u} & \frac{\partial \mathfrak{X}_4}{\partial v} & \frac{\partial^2 \mathfrak{X}_4}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_4}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_4}{\partial v^2} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_1}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial v^2} & \mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_2}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_2}{\partial v^2} & \mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_3}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_3}{\partial v^2} & \mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_4}{\partial u^2} & \frac{\partial^3 \mathfrak{X}_4}{\partial u \partial v} & \frac{\partial^3 \mathfrak{X}_4}{\partial v^2} & \mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial v}
\end{vmatrix} \cdots (3)$$

The determinent (3) vanishes identically. For (3) is equal to

$$\begin{vmatrix}
\mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial u} & \frac{\partial \mathfrak{X}_1}{\partial v} & 0 & 0 & 0 & -\mathfrak{X}_1 & -\frac{\partial \mathfrak{X}_1}{\partial v} \\
\mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial u} & \frac{\partial \mathfrak{X}_2}{\partial v} & 0 & 0 & 0 & -\mathfrak{X}_2 & -\frac{\partial \mathfrak{X}_2}{\partial v} \\
\mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial u} & \frac{\partial \mathfrak{X}_3}{\partial v} & 0 & 0 & 0 & -\mathfrak{X}_3 & -\frac{\partial \mathfrak{X}_3}{\partial v} \\
\mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial u} & \frac{\partial \mathfrak{X}_4}{\partial v} & 0 & 0 & 0 & -\mathfrak{X}_4 & -\frac{\partial \mathfrak{X}_4}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_1}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial v^2} & \mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_2}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_2}{\partial v^2} & \mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_3}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_3}{\partial v^2} & \mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_4}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_4}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial v^2} & \mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial v}
\end{vmatrix}$$

$$
=\begin{vmatrix}
\mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial u} & \frac{\partial \mathfrak{X}_1}{\partial v} & 0 & 0 & 0 & 0 & 0 \\
\mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial u} & \frac{\partial \mathfrak{X}_2}{\partial v} & 0 & 0 & 0 & 0 & 0 \\
\mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial u} & \frac{\partial \mathfrak{X}_3}{\partial v} & 0 & 0 & 0 & 0 & 0 \\
\mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial u} & \frac{\partial \mathfrak{X}_4}{\partial v} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_1}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial v^2} & \mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_2}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_2}{\partial v^2} & \mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_3}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_3}{\partial v^2} & \mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_4}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_4}{\partial u \partial v} & \frac{\partial^3 \mathfrak{X}_4}{\partial v^2} & \mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial v}
\end{vmatrix}
$$

Expanding the last determinant by the method of Laplace we can see that it vanishes identically.

While if we expand (3) by the method of Laplace it becomes

$$db'' - 2d'b' + d''b.$$

Therefore

$$
\left.\begin{aligned}
&db'' - 2d'b' + d''b = 0. \\
&\text{Similarly, we have} \\
&dc'' - 2d'c' + d''c = 0, \\
&da'' - 2d'a' + d''a = 0.
\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (4)
$$

2. Consider the transformation

$$\overline{\mathfrak{X}} = \lambda \mathfrak{X} \qquad \cdots\cdots\cdots\cdots (5)$$

and denote by $\bar{a}$, $\bar{b}$, etc. the determinants obtained by substituting $\overline{\mathfrak{X}}$ instead of $\mathfrak{X}$ in (1).

Then we have

$$\left.\begin{aligned}
\bar{d} &= \lambda^4 d, \quad \bar{d}' = \lambda^4 d', \quad \bar{d}'' = \lambda^4 d'', \\
\bar{b} &= \lambda^4\left[b + \frac{\partial \log \lambda}{\partial v} d - 2\frac{\partial \log \lambda}{\partial u} d'\right], \\
\bar{b}' &= \lambda^4\left[b' - \frac{\partial \log \lambda}{\partial u} d''\right], \\
\bar{b}'' &= \lambda^4\left[b'' - \frac{\partial \log \lambda}{\partial u} d''\right], \\
\bar{c} &= \lambda^4\left[c - \frac{\partial \log \lambda}{\partial u} d\right], \\
\bar{c}' &= \lambda_4\left[c' - \frac{\partial \log \lambda}{\partial u} d\right], \\
\bar{c}'' &= \lambda^4\left[c'' + \frac{\partial \log \lambda}{\partial u} d'' - 2\frac{\partial \log \lambda}{\partial v} d'\right], \\
\bar{a} &= \lambda^4\left[a + \frac{\partial \log \lambda}{\partial u} b - \frac{\partial \log \lambda}{\partial v} c + \begin{Bmatrix}\lambda\\ uu\end{Bmatrix} d' - \begin{Bmatrix}\lambda\\ uv\end{Bmatrix} d\right], \\
\bar{a}' &= \lambda^4\left[a' + \frac{\partial \log \lambda}{\partial u} b' - \frac{\partial \log \lambda}{\partial v} c' + \frac{1}{2}\begin{Bmatrix}\lambda\\ uu\end{Bmatrix} d'' - \frac{1}{2}\begin{Bmatrix}\lambda\\ vv\end{Bmatrix} d\right], \\
\bar{a}'' &= \lambda^4\left[a'' + \frac{\partial \log \lambda}{\partial u} b'' - \frac{\partial \log \lambda}{\partial v} c'' + \begin{Bmatrix}\lambda\\ uv\end{Bmatrix} d'' - \begin{Bmatrix}\lambda\\ vv\end{Bmatrix} d'\right],
\end{aligned}\right\} \cdots(6)$$

where

$$\begin{Bmatrix}\lambda\\ uu\end{Bmatrix} = \frac{\partial^2 \log \lambda}{\partial u^2} - \left(\frac{\partial \log \lambda}{\partial u}\right)^2,$$

$$\begin{Bmatrix}\lambda\\ uv\end{Bmatrix} = \frac{\partial^2 \log \lambda}{\partial u \partial v} - \frac{\partial \log \lambda}{\partial u} \cdot \frac{\partial \log \lambda}{\partial v},$$

$$\begin{Bmatrix}\lambda\\ vv\end{Bmatrix} = \frac{\partial^2 \log \lambda}{\partial v^2} - \left(\frac{\partial \log \lambda}{\partial v}\right)^2.$$

Let us assume that the given surface is not developable and has no porabolic point in the portion which correspond to the domain R of u, v, i. e. $d'^2 - dd'' \neq 0$ in the domain R.

Let $r (r > 0)$ and $\theta (0 \leqq \theta < 2\pi)$ be the modules and the arguments of the functions $d'^2 - dd''$ respectively and put

$$\Delta = \sqrt{r}\, e^{\frac{i\theta}{2}}. \quad \cdots\cdots\cdots\cdots(7)$$

Then, by the transformation (5) we have

$$\bar{\Delta} = \varepsilon \lambda^4 \Delta, \cdots\cdots\cdots\cdots(8)$$

where

$$\varepsilon=1, \text{ or } -1.$$

Let us call x_1, x_2, x_3, x_4 which are defined by the equations

$$x_i=\frac{\mathfrak{X}_i}{\sqrt[4]{\varDelta}}, \quad (i=1, 2, 3, 4) \quad \dots\dots\dots\dots\dots\dots(9)$$

the *normal coordinates* of the point on the given surface, and the fundamental determinants formed with the normal coordinates the *normal fundamental determinants*, and denote them by A, B,......, etc.

Then

$$\left.\begin{aligned}
D &=\frac{d}{\varDelta}, \quad D'=\frac{d'}{\varDelta}, \quad D''=\frac{d''}{\varDelta},\\
B &=\frac{b}{\varDelta}+\frac{1}{2}\frac{\partial \log \varDelta}{\partial u}D'-\frac{1}{4}\frac{\partial \log \varDelta}{\partial v}D,\\
B' &=\frac{b'}{\varDelta}+\frac{1}{4}\frac{\partial \log \varDelta}{\partial u}D'',\\
B'' &=\frac{b''}{\varDelta}+\frac{1}{4}\frac{\partial \log \varDelta}{\partial v}D'',\\
C &=\frac{c}{\varDelta}+\frac{1}{4}\frac{\partial \log \varDelta}{\partial u}D,\\
C' &=\frac{c'}{\varDelta}+\frac{1}{4}\frac{\partial \log \varDelta}{\partial v}D,\\
C'' &=\frac{c''}{\varDelta}+\frac{1}{2}\frac{\partial \log \varDelta}{\partial v}D'-\frac{1}{4}\frac{\partial \log \varDelta}{\partial u}D'',\\
A &=\frac{a}{\varDelta}-\frac{1}{4}\frac{\partial \log \varDelta}{\partial u}B+\frac{1}{4}\frac{\partial \log \varDelta}{\partial v}C+\begin{Bmatrix}\frac{1}{4}\varDelta\\ uv\end{Bmatrix}D-\begin{Bmatrix}\frac{1}{4}\varDelta\\ uu\end{Bmatrix}D',\\
A' &=\frac{a'}{\varDelta}-\frac{1}{4}\frac{\partial \log \varDelta}{\partial u}B'+\frac{1}{4}\frac{\partial \log \varDelta}{\partial v}C'+\frac{1}{2}\begin{Bmatrix}\frac{1}{4}\varDelta\\ vv\end{Bmatrix}D-\frac{1}{2}\begin{Bmatrix}\frac{1}{4}\varDelta\\ uu\end{Bmatrix}D'',\\
A'' &=\frac{a''}{\varDelta}-\frac{1}{4}\frac{\partial \log \varDelta}{\partial u}B''+\frac{1}{4}\frac{\partial \log \varDelta}{\partial v}C''+\begin{Bmatrix}\frac{1}{4}\varDelta\\ vv\end{Bmatrix}D'-\begin{Bmatrix}\frac{1}{4}\varDelta\\ uv\end{Bmatrix}D''.
\end{aligned}\right\} \quad (10)$$

We can easily see that the normal fundamental determinants remain unchanged by the transformation of the form (5), or the projective transformation, except for a sign.

Evidently, we have

$$\left.\begin{aligned}
&\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}=2(B-C'),\\
&\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}=2(B'-C''),\\
&DB''-2D'B'+D''B=0,\\
&DC''-2D'C'+D''C=0,\\
&DA''-2D'A'+D''A=0
\end{aligned}\right\}\qquad\text{..................(11)}$$

and, moreover, we have

$$D'^2-DD''=1.$$

Differentiating this equation by u and by v, we have

$$\left.\begin{aligned}
&D\frac{\partial D''}{\partial u}-2D'\frac{\partial D'}{\partial u}+D''\frac{\partial D}{\partial u}=0,\\
&D\frac{\partial D''}{\partial v}-2D'\frac{\partial D'}{\partial v}+D''\frac{\partial D}{\partial v}=0.
\end{aligned}\right\}\qquad\text{.........(12)}$$

3. Next, let us consider the transformation

$$\left.\begin{aligned}
&u=u(\xi,\ \eta),\\
&v=v(\xi,\ \eta),
\end{aligned}\right\}\qquad\text{..................(13)}$$

where

$$W=\frac{\partial(u,\ v)}{\partial(\xi,\ \eta)}\neq 0.$$

Put

$$\frac{\partial u}{\partial \xi}=p,\quad \frac{\partial u}{\partial \eta}=q,\quad \frac{\partial^2 u}{\partial \xi^2}=r,\quad \frac{\partial^2 u}{\partial \xi\partial \eta}=s,\quad \frac{\partial^2 u}{\partial \eta^2}=t,$$

$$\frac{\partial v}{\partial \xi}=p_1,\quad \frac{\partial v}{\partial \eta}=q_1,\quad \frac{\partial^2 v}{\partial \xi^2}=r_1,\quad \frac{\partial^2 v}{\partial \xi\partial \eta}=s_1,\quad \frac{\partial^2 v}{\partial \eta^2}=t_1.$$

Then

$$\frac{\partial x}{\partial \xi}=\frac{\partial x}{\partial u}p+\frac{\partial x}{\partial v}p_1,$$

$$\left.\begin{aligned}
\frac{\partial x}{\partial \eta} &= \frac{\partial x}{\partial u} q + \frac{\partial x}{\partial v} q_1, \\
\frac{\partial^2 x}{\partial \xi^2} &= \frac{\partial^2 x}{\partial u^2} p^2 + 2\frac{\partial^2 x}{\partial u \partial v} p p_1 + \frac{\partial^2 x}{\partial v^2} p_1^2 + \frac{\partial x}{\partial u} r + \frac{\partial x}{\partial v} r_1, \\
\frac{\partial^2 x}{\partial \xi \partial \eta} &= \frac{\partial^2 x}{\partial u^2} pq + \frac{\partial^2 x}{\partial u \partial v}(pq_1 + p_1 q) + \frac{\partial^2 x}{\partial v^2} p_1 q_1 + \frac{\partial x}{\partial u} s + \frac{\partial x}{\partial v} s_1, \\
\frac{\partial^2 x}{\partial \eta^2} &= \frac{\partial^2 x}{\partial u^2} q^2 + 2\frac{\partial^2 x}{\partial u \partial v} q q_1 + \frac{\partial^2 x}{\partial v^2} q_1^2 + \frac{\partial x}{\partial u} t + \frac{\partial x}{\partial v} t_1.
\end{aligned}\right\} \quad (14)$$

Denote by $\bar{a}$, $\bar{b}$, etc. the fundamental determinants referred to the parameters ξ, η, i. e. the determinants obtained by substituting in (1) $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial^2 x}{\partial \xi^2}$, $\frac{\partial^2 x}{\partial \xi \partial \eta}$, $\frac{\partial^2 x}{\partial \eta^2}$ instead of $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial^2 x}{\partial u^2}$, $\frac{\partial^2 x}{\partial u \partial v}$, $\frac{\partial^2 x}{\partial v}$ respectively.

Then we have from (1) and (14)

$$\bar{d} = W\Sigma p^2 d, \quad \bar{d'} = W\Sigma pqd, \quad \bar{d''} = W\Sigma q^2 d, \quad \ldots\ldots(15)$$

where

$$\Sigma p^2 d = p^2 d + 2pp_1 d' + p_1^2 d'',$$

$$\Sigma pqd = pqd + (pq_1 + p_1 q)d' + p_1 q_1 d'',$$

$$\Sigma q^2 d = q^2 d + 2qq_1 d' + q_1^2 d''.$$

From (15) we have

$$\bar{\Delta} = \varepsilon W^2 \Delta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

where $\varepsilon = +1$, or -1.

Therefore we have

$$\bar{D} = \varepsilon \Sigma \frac{p^2}{W} D, \quad \bar{D}' = \varepsilon \Sigma \frac{pq}{W} D', \quad \bar{D}'' = \varepsilon \Sigma \frac{q^2}{W} D. \quad \ldots.(17)$$

Let $\mathfrak{X}$, $\mathfrak{X}'$, $\mathfrak{X}''$ be three independent variables. Then we have the following identities which are very useful in the subsequent calculations.

$$\left.\begin{aligned}
&\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{T}=\left(\begin{vmatrix} r & q \\ r_1 & q_1\end{vmatrix}-\begin{vmatrix} p & s \\ p_1 & s_1\end{vmatrix}\right)\Sigma\frac{p^2}{W^2}\mathfrak{T}+2\begin{vmatrix} p & r \\ p_1 & r_1\end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)\mathfrak{T}=\begin{vmatrix} s & q \\ s_1 & q_1\end{vmatrix}\Sigma\frac{p^2}{W^2}\mathfrak{T}+\begin{vmatrix} p & r \\ p_1 & r_1\end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)\mathfrak{T}=2\begin{vmatrix} s & q \\ s_1 & q_1\end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & s \\ p_1 & s_1\end{vmatrix}-\begin{vmatrix} r & q \\ r_1 & q_1\end{vmatrix}\right)\Sigma\frac{q^2}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)\mathfrak{T}=\left(\begin{vmatrix} s & q \\ s_1 & q_1\end{vmatrix}-\begin{vmatrix} p & t \\ p_1 & t_1\end{vmatrix}\right)\Sigma\frac{p^2}{W^2}\mathfrak{T}+2\begin{vmatrix} p & s \\ p_1 & s_1\end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\eta}\left(\frac{pq}{W}\right)\mathfrak{T}=\begin{vmatrix} t & q \\ t_1 & q_1\end{vmatrix}\Sigma\frac{p^2}{W^2}\mathfrak{T}+\begin{vmatrix} p & s \\ p_1 & s_1\end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\eta}\left(\frac{q^2}{W}\right)\mathfrak{T}=2\begin{vmatrix} t & q \\ t_1 & q_1\end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & t \\ p_1 & t_1\end{vmatrix}-\begin{vmatrix} s & q \\ s_1 & q_1\end{vmatrix}\right)\Sigma\frac{q}{W^2}\mathfrak{T},\\
&\text{where}\\
&\Sigma\frac{p^2}{W}\mathfrak{T}=\frac{p^2}{W}\mathfrak{T}+\frac{2pp_1}{W}\mathfrak{T}'+\frac{p_1^2}{W}\mathfrak{T}'',\\
&\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{T}=\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{T}+2\frac{\partial}{\partial\xi}\left(\frac{pp_1}{W}\right)\mathfrak{T}'+\frac{\partial}{\partial\xi}\left(\frac{p_1^2}{W}\right)\mathfrak{T}'',\\
&\qquad\qquad\qquad\text{etc.}
\end{aligned}\right\}(18)$$

From (1) and (14) we have

$$\bar{b}=qW\Sigma p^2c+q_1W\Sigma p^2b+\begin{vmatrix} q & r \\ q_1 & r_1\end{vmatrix}\Sigma pqd+\begin{vmatrix} s & q \\ s_1 & q_1\end{vmatrix}\Sigma p^2d.$$

Therefore, we have from (9) and (18)

$$\begin{aligned}
\bar{B}&=\varepsilon\Bigg[q\Sigma\left(\frac{p^2}{W}C\right)+q_1\left(\Sigma\frac{p^2}{W}B\right)\\
&\qquad+\left(\begin{vmatrix} s & q \\ s_1 & q_1\end{vmatrix}-\begin{vmatrix} p & t \\ p_1 & t_1\end{vmatrix}\right)\Sigma\frac{p^2}{W^2}D+\begin{vmatrix} p & s \\ p_1 & s_1\end{vmatrix}\Sigma\frac{pq}{W^2}D\Bigg]\\
&=\varepsilon\left[q\left(\Sigma\frac{p^2}{W}C\right)+q_1\left(\Sigma\frac{p^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)D\right].
\end{aligned}$$

Similarly we have

$$\bar{B}'=\varepsilon\left[q\left(\Sigma\frac{pq}{W}C\right)+q_1\left(\Sigma\frac{pq}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{pq}{W}\right)D\right],$$

$$\overline{B}''=\varepsilon\left[q\left(\Sigma\frac{q^2}{W}C\right)+q_1\left(\Sigma\frac{q^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{q^2}{W}\right)D\right],$$

$$\overline{C}=\varepsilon\left[p\left(\Sigma\frac{p^2}{W}C\right)+p_1\left(\Sigma\frac{p^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D\right],$$

$$\overline{C}'=\varepsilon\left[p\left(\Sigma\frac{pq}{W}C\right)+p_1\left(\Sigma\frac{pq}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)D\right],$$

$$\overline{C}''=\varepsilon\left[p\left(\Sigma\frac{q^2}{W}C\right)+p_1\left(\Sigma\frac{q^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)D\right].$$

Therefore, if we put

$$\left.\begin{aligned}\mathfrak{B}&=B-\frac{1}{2}\frac{\partial D}{\partial v}, & \mathfrak{C}&=C-\frac{1}{2}\frac{\partial D}{\partial u},\\ \mathfrak{B}'&=B'-\frac{1}{2}\frac{\partial D'}{\partial v}, & \mathfrak{C}'&=C'-\frac{1}{2}\frac{\partial D'}{\partial u},\\ \mathfrak{B}''&=B''-\frac{1}{2}\frac{\partial D''}{\partial v}, & \mathfrak{C}''&=C''-\frac{1}{2}\frac{\partial D''}{\partial u},\end{aligned}\right\}\quad\text{.........(19)}$$

we have

$$\left.\begin{aligned}\overline{\mathfrak{B}}&=\varepsilon\left[q\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right],\\ \overline{\mathfrak{B}}'&=\varepsilon\left[q\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right],\\ \overline{\mathfrak{B}}''&=\varepsilon\left[q\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right],\\ \overline{\mathfrak{C}}&=\varepsilon\left[p\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right],\\ \overline{\mathfrak{C}}'&=\varepsilon\left[p\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right],\\ \overline{\mathfrak{C}}''&=\varepsilon\left[p\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right].\end{aligned}\right\}\quad\text{...............(20)}$$

From (11) and (12) we have

$$\left.\begin{aligned}D\mathfrak{B}''-2\mathfrak{B}'D'+D''\mathfrak{B}=0,\\ D\mathfrak{C}''-2\mathfrak{C}'D'+D''\mathfrak{C}=0.\end{aligned}\right\}\quad\text{..............(21)}$$

Therefore we have

$$\frac{2\begin{vmatrix}\mathfrak{C} & \mathfrak{B}\\ \mathfrak{C}' & \mathfrak{B}'\end{vmatrix}}{D}=\frac{\begin{vmatrix}\mathfrak{C} & \mathfrak{B}\\ \mathfrak{C}'' & \mathfrak{B}''\end{vmatrix}}{D'}=\frac{2\begin{vmatrix}\mathfrak{C}' & \mathfrak{B}'\\ \mathfrak{C}'' & \mathfrak{B}''\end{vmatrix}}{D''}=|\mathfrak{B}\,\mathfrak{C}\,D|, \quad \ldots(22)$$

where

$$|\mathfrak{B}\,\mathfrak{C}\,D|=\begin{vmatrix}\mathfrak{B} & \mathfrak{C} & D\\ \mathfrak{B}' & \mathfrak{C}' & D'\\ \mathfrak{B}'' & \mathfrak{C}'' & D''\end{vmatrix},$$

for $$D'^2-DD''=1.$$

We shall denote by θ the last determinant.

We can see from (17) and (20) that θ is a relative invariant which satisfies the equation

$$\bar{\theta}=\varepsilon W\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots(23)$$

We shall prove, afterwards, that θ does not vanish identically for a curved surface.

From (11) and (19) we have

$$\mathfrak{B}=\mathfrak{C}', \quad \mathfrak{B}'=\mathfrak{C}''.$$

From these equations and (20) we know that

$$\mathfrak{C}du^3+3\mathfrak{B}du^2dv+3\mathfrak{C}''dudv^2+\mathfrak{B}''dv^3=0,$$

is an invariant equation. We shall show, afterwards, that this is the equation of the Darboux's curves of quadric osculation.

From (1) and (13) we have

$$\bar{a}=W^2(\Sigma p^2 a).$$

Therefore, we have from (11)

$$\begin{aligned}\bar{A}=\varepsilon\Big[&\Sigma p^2 A-\frac{1}{2}\frac{\partial \log W}{\partial \xi}\Big\{q\Big(\Sigma\frac{p^2}{W}C\Big)+q_1\Big(\Sigma\frac{p^2}{W}B\Big)+\frac{1}{2}\Sigma\frac{\partial}{\partial \eta}\Big(\frac{p^2}{W}\Big)D\Big\}\\ &+\frac{1}{2}\frac{\partial \log W}{\partial \eta}\Big\{p\Big(\Sigma\frac{p^2}{W}C\Big)+p_1\Big(\Sigma\frac{p^2}{W}B\Big)+\frac{1}{2}\Sigma\frac{\partial}{\partial \xi}\Big(\frac{p^2}{W}\Big)D\Big\}\\ &+\begin{Bmatrix}\frac{1}{2}W\\ \xi\eta\end{Bmatrix}\Sigma\frac{p^2}{W}D-\begin{Bmatrix}\frac{1}{2}W\\ \xi\xi\end{Bmatrix}\Sigma\frac{pq}{W}D\Big].\end{aligned}$$

Similarly we have

$$\begin{aligned}\bar{A}' = \varepsilon\Bigg[& \Sigma pq A - \frac{1}{2}\frac{\partial \log W}{\partial \xi}\left\{q\left(\Sigma\frac{pq}{W}C\right)+q_1\left(\Sigma\frac{pq}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{pq}{W}\right)D\right\} \\ & +\frac{1}{2}\frac{\partial \log W}{\partial \eta}\left\{p\left(\Sigma\frac{pq}{W}C\right)+p_1\left(\Sigma\frac{pq}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)D\right\} \\ & +\frac{1}{2}\left\{\begin{matrix}\frac{1}{2}W\\ \eta\eta\end{matrix}\right\}\Sigma\frac{p^2}{W}D-\frac{1}{2}\left\{\begin{matrix}\frac{1}{2}W\\ \xi\xi\end{matrix}\right\}\Sigma\frac{q^2}{W}D\Bigg],\end{aligned}$$

$$\begin{aligned}\bar{A}'' = \varepsilon\Bigg[& \Sigma q^2 A - \frac{1}{2}\frac{\partial \log W}{\partial \xi}\left\{q\left(\Sigma\frac{q^2}{W}C\right)+q_1\left(\Sigma\frac{q^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{q^2}{W}\right)D\right\} \\ & +\frac{1}{2}\frac{\partial \log W}{\partial \eta}\left\{p\left(\Sigma\frac{q^2}{W}C\right)+q_1\left(\Sigma\frac{q^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)D\right\} \\ & +\left\{\begin{matrix}\frac{1}{2}W\\ \eta\eta\end{matrix}\right\}\Sigma\frac{pq}{W}D-\left\{\begin{matrix}\frac{1}{2}W\\ \xi\xi\end{matrix}\right\}\Sigma\frac{q^2}{W}D\Bigg].\end{aligned}$$

Therefore, if we put

$$\left.\begin{aligned}\mathfrak{A} &= A + \frac{1}{2}\frac{\partial \log\theta}{\partial u}B - \frac{1}{2}\frac{\partial\log\theta}{\partial v}C + \left\{\begin{matrix}\frac{1}{2}\theta\\ uu\end{matrix}\right\}D' - \left\{\begin{matrix}\frac{1}{2}\theta\\ uv\end{matrix}\right\}D, \\ \mathfrak{A}' &= A' + \frac{1}{2}\frac{\partial \log\theta}{\partial u}B' - \frac{1}{2}\frac{\partial\log\theta}{\partial v}C' + \frac{1}{2}\left\{\begin{matrix}\frac{1}{2}\theta\\ uu\end{matrix}\right\}D'' - \frac{1}{2}\left\{\begin{matrix}\frac{1}{2}\theta\\ vv\end{matrix}\right\}D, \\ \mathfrak{A}'' &= A'' + \frac{1}{2}\frac{\partial \log\theta}{\partial u}B'' - \frac{1}{2}\frac{\partial\log\theta}{\partial v}C'' + \left\{\begin{matrix}\frac{1}{2}\theta\\ uv\end{matrix}\right\}D'' - \left\{\begin{matrix}\frac{1}{2}\theta\\ vv\end{matrix}\right\}D',\end{aligned}\right\}\quad(24)$$

we have

$$\left.\begin{aligned}\bar{\mathfrak{A}} &= \varepsilon[\Sigma p^2\mathfrak{A}], \\ \bar{\mathfrak{A}}' &= \varepsilon[\Sigma pq\mathfrak{A}], \\ \bar{\mathfrak{A}}'' &= \varepsilon[\Sigma q^2\mathfrak{A}],\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

From (25) we know that $\mathfrak{A}\mathfrak{A}''-\mathfrak{A}'^2$ is a relative invariant and

$$\mathfrak{A}du^2+2\mathfrak{A}'dudv+\mathfrak{A}''dv^2=0,$$

is an invariant equation. Afterwards, we shall interpretate them geometrically.

From (11) and (12) we have

$$D\mathfrak{A}''-2D'\mathfrak{A}'+D''\mathfrak{A}=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

If the number ε in the equation (16) is equal to -1, then let us carry out the transformation

$$\overline{\mathfrak{X}}=e^{\frac{i\pi}{4}}\mathfrak{X}.$$

again. By this transformation $\bar{\varDelta}$ remains unchanged by its definition, but the signs of $\bar{D}$, $\bar{D}$, $\bar{D}''$; $\overline{\mathfrak{B}}$, $\overline{\mathfrak{B}}$, $\overline{\mathfrak{B}}''$; $\overline{\mathfrak{C}}$, $\overline{\mathfrak{C}}'$, $\overline{\mathfrak{C}}''$; $\overline{\mathfrak{A}}$, $\overline{\mathfrak{A}}'$, $\overline{\mathfrak{A}}''$ are changed.

Therefore, we may suppose the number ε in the equations (17), (20), (23) and (25) to be equal to $+1$.

4. From (20) we have

$$\frac{\partial\overline{\mathfrak{B}}}{\partial\xi}-\frac{\partial\overline{\mathfrak{C}}}{\partial\eta}=q\left\{\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\}+q_1\left\{\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\}$$

$$-p\left\{\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\}-p_1\left\{\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\}$$

$$+\Sigma p^2\left(\frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{B}}{\partial v}\right).$$

And from the relation

$$\mathfrak{B}=\mathfrak{C}',\quad \mathfrak{B}'=\mathfrak{C}'',$$

we have

$$q\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)=p\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right),$$

$$q\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)=p\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right).$$

In virtue of the last two equations and (19), we have

$$q\left\{\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\}+q_1\left\{\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\}-p\left\{\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\}-p_1\left\{\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\}$$

$$=-\frac{\partial\log W}{\partial\xi}\left\{q\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right\}$$

$$+\frac{\partial\log W}{\partial\eta}\left\{p\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)-2p\varphi(\mathfrak{C})-2p_1\varphi(\mathfrak{B}),\right.$$

where

$$\varphi(\mathfrak{T})=\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix} \Sigma\frac{p^2}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & s \\ p_1 & s_1 \end{vmatrix}-\begin{vmatrix} r & q \\ r_1 & q_1 \end{vmatrix}\right)\Sigma\frac{pq}{W^2}\mathfrak{T}+\begin{vmatrix} r & p \\ r_1 & p_1 \end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T}.$$

Therefore, we have

$$\left.\begin{aligned}
\frac{\partial\overline{\mathfrak{B}}}{\partial\xi}-\frac{\partial\overline{\mathfrak{C}}}{\partial\eta} &= -\frac{\partial\log W}{\partial\xi}\left\{q\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right\} \\
&\quad+\frac{\partial\log W}{\partial\eta}\left\{p\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right\} \\
&\quad-2p\varphi(\mathfrak{C})-2p_1\varphi(\mathfrak{B})+\Sigma p^2\left(\frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{C}}{\partial v}\right).
\end{aligned}\right\}$$

Similarly, we have

$$\left.\begin{aligned}
\frac{\partial\overline{\mathfrak{B}}'}{\partial\xi}-\frac{\partial\overline{\mathfrak{C}}'}{\partial\eta} &= -\frac{\partial\log W}{\partial\xi}\left\{q\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right\} \\
&\quad+\frac{\partial\log W}{\partial\eta}\left\{p\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right\} \\
&\quad-q\varphi(\mathfrak{C})-q_1\varphi(\mathfrak{B})-p\psi(\mathfrak{C})-p_1\psi(\mathfrak{B})+\Sigma pq\left(\frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{C}}{\partial v}\right), \\
\frac{\partial\overline{\mathfrak{B}}''}{\partial\xi}-\frac{\partial\overline{\mathfrak{C}}''}{\partial\eta} &= -\frac{\partial\log W}{\partial\xi}\left\{q\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right\} \\
&\quad+\frac{\partial\log W}{\partial\eta}\left\{p\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right\} \\
&\quad-2q\psi(\mathfrak{C})-2q_1\psi(\mathfrak{B})+\Sigma q^2\left(\frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{C}}{\partial v}\right),
\end{aligned}\right\} \quad (27)$$

where

$$\psi(\mathfrak{T})=\begin{vmatrix} t & q \\ t_1 & q_1 \end{vmatrix} \Sigma\frac{p^2}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & t \\ p_1 & t_1 \end{vmatrix}-\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}\right)\Sigma\frac{pq}{W^2}\mathfrak{T}+\begin{vmatrix} s & p \\ s_1 & p_1 \end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T}$$

From (12) and (20) we have

$$\frac{2\begin{vmatrix} D_u & \mathfrak{C} \\ D_u' & \mathfrak{C}' \end{vmatrix}}{D}=\frac{\begin{vmatrix} D_u & \mathfrak{C} \\ D_u' & \mathfrak{C}'' \end{vmatrix}}{D'}=\frac{2\begin{vmatrix} D_u' & \mathfrak{C}' \\ D_u'' & \mathfrak{C}'' \end{vmatrix}}{D''}=|\mathfrak{C}\ D_u\ D|,$$

$$\left.\begin{aligned}
&\frac{2\begin{vmatrix} D_u & \mathfrak{B} \\ D'_u & \mathfrak{B}' \end{vmatrix}}{D}=\frac{\begin{vmatrix} D_u & \mathfrak{B} \\ D''_u & \mathfrak{B}'' \end{vmatrix}}{D'}=\frac{2\begin{vmatrix} D'_u & \mathfrak{B}' \\ D''_u & \mathfrak{B}'' \end{vmatrix}}{D''}=|\mathfrak{B}\ D_u\ D|,\\
&\frac{2\begin{vmatrix} D_v & \mathfrak{C} \\ D'_v & \mathfrak{C}' \end{vmatrix}}{D}=\frac{\begin{vmatrix} D_v & \mathfrak{C} \\ D''_v & \mathfrak{C}'' \end{vmatrix}}{D'}=\frac{2\begin{vmatrix} D_v' & \mathfrak{C}' \\ D_v'' & \mathfrak{C}'' \end{vmatrix}}{D''}=|\mathfrak{C}\ D_v\ D|,\\
&\frac{2\begin{vmatrix} D_v & \mathfrak{B} \\ D'_v & \mathfrak{B}' \end{vmatrix}}{D}=\frac{\begin{vmatrix} D_v & \mathfrak{B} \\ D_v'' & \mathfrak{B}'' \end{vmatrix}}{D'}=\frac{2\begin{vmatrix} D_v' & \mathfrak{B}' \\ D_v'' & \mathfrak{B}'' \end{vmatrix}}{D''}=|\mathfrak{B}\ D_v\ D|.
\end{aligned}\right\}\quad(28)$$

From (17) and (20) we have

$$\begin{aligned}
|\overline{\mathfrak{C}}\ \overline{D}_\xi\ \overline{D}| = p\,|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D\ \ \Sigma\frac{p^2}{W}D\,|\\
+p_1\,|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D\ \ \Sigma\frac{p^2}{W}D\,|\\
+p^2\,|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{p^2}{W}D_u\ \ \Sigma\frac{p^2}{W}D\,|\\
+pp_1\{\,|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{p^2}{W}D_v\ \ \Sigma\frac{p^2}{W}D| + |\Sigma\frac{p^2}{W}\mathfrak{B}\ \Sigma\frac{p^2}{W}D_u\ \Sigma\frac{p^2}{W}D|\}\\
+p_1^2|\Sigma\frac{p^2}{W}\mathfrak{B}\ \Sigma\frac{p^2}{W}D_v\ \Sigma\frac{p^2}{W}D\,|.
\end{aligned}$$

But in virtue of (19) and the equation

$$\left(\Sigma\frac{p^2}{W}D\right)\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)-2\left(\Sigma\frac{pq}{W}D\right)\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+\left(\Sigma\frac{q^2}{W}D\right)\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)=0,$$

which follows from the equation

$$D\mathfrak{C}''-2D\mathfrak{C}'+D''\mathfrak{C}=0,$$

we have

$$(\Sigma\frac{p^2}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)D)-(\Sigma\frac{pq}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D)=(\Sigma\frac{p^2}{W}D)\varphi(\mathfrak{C}),$$

$$(\Sigma\frac{p^2}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)D)-(\Sigma\frac{q^2}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\ \frac{p^2}{W}D)=2(\Sigma\frac{pq}{W}D)\varphi(\mathfrak{C}),$$

$$(\Sigma \frac{pq}{W}\mathfrak{C})(\Sigma \frac{\partial}{\partial \xi}\left(\frac{q^2}{W}\right)D)-(\Sigma \frac{q^2}{W}\mathfrak{C})(\Sigma \frac{\partial}{\partial \xi}\left(\frac{pq}{W}\right)D)=(\Sigma \frac{q^2}{W}D)\varphi(\mathfrak{C}).$$

The last three equations also hold if we substitute $\mathfrak{B}$ for $\mathfrak{C}$. Therefore, we have

$$|\overline{\mathfrak{C}}\,\overline{D}_\xi\,\overline{D}| = -2p\varphi(\mathfrak{C})-2p_1\varphi(\mathfrak{B})+p^2|\mathfrak{C}\,D_u\,D|$$
$$+pp_1\{|\mathfrak{C}\,D_v\,D|+|\mathfrak{B}\,D_u\,D|\}+p_1^2|\mathfrak{B}\,D_v\,D|.$$

Similarly, we have

$$\tfrac{1}{2}\{|\overline{\mathfrak{C}}\,\overline{D}_\eta\,\overline{D}|+|\overline{\mathfrak{B}}\,\overline{D}_\xi\,D|\}$$
$$=-p\psi(\mathfrak{C})-p_1\psi(\mathfrak{B})-q\varphi(\mathfrak{C})-q_1\varphi(\mathfrak{B})$$
$$+pq|\mathfrak{C}\,D_u\,D|+\frac{pq_1+p_1q}{2}\{|\mathfrak{C}\,D_v\,D|+|\mathfrak{B}\,D_u\,D|\}$$
$$+p_1q_1|\mathfrak{B}\,D_v\,D|,$$

$$|\overline{\mathfrak{B}}\,\overline{D}_\eta\,\overline{D}| = -2q\psi(\mathfrak{C})-2q_1\psi(\mathfrak{B})+q^2|\mathfrak{C}\,D_w\,D|$$
$$+qq_1\{|\mathfrak{C}\,D_v\,D|+|\mathfrak{B}\,D_u\,D|\}+q_1^2|\mathfrak{B}\,D_v\,D|.$$

Therefore, if we put

$$\left.\begin{aligned} L &= \frac{\partial \mathfrak{B}}{\partial u}-\frac{\partial \mathfrak{C}}{\partial v}-|\mathfrak{C}\,D_u\,D|,\\ L' &= \frac{\partial \mathfrak{B}'}{\partial u}-\frac{\partial \mathfrak{C}'}{\partial v}-\frac{1}{2}\{|\mathfrak{C}\,D_v\,D|+|\mathfrak{B}\,D_u\,D|\}\\ L'' &= \frac{\partial \mathfrak{B}''}{\partial u}-\frac{\partial \mathfrak{C}''}{\partial v}-|\mathfrak{B}\,D_v\,D|,\end{aligned}\right\} \quad (29)$$

$$\left.\begin{aligned} \mathfrak{L} &= L+\frac{\partial \log\theta}{\partial u}\mathfrak{B}-\frac{\partial \log\theta}{\partial v}\mathfrak{C},\\ \mathfrak{L}' &= L'+\frac{\partial \log\theta}{\partial u}\mathfrak{B}'-\frac{\partial \log\theta}{\partial v}\mathfrak{C}',\\ \mathfrak{L}'' &= L''+\frac{\partial \log\theta}{\partial u}\mathfrak{B}''-\frac{\partial \log\theta}{\partial v}\mathfrak{C}''\end{aligned}\right\} \quad (30)$$

we have

$$\left.\begin{aligned} \overline{\mathfrak{L}} &= \Sigma p^2 \mathfrak{L}, \\ \overline{\mathfrak{L}}' &= \Sigma pq \mathfrak{L}, \\ \overline{\mathfrak{L}}'' &= \Sigma q^2 \mathfrak{L}. \end{aligned}\right\} \quad \text{..............................(31)}$$

From (28) we have

$$D'' \,|\, \mathfrak{C}\, D_u\, D \,| = 2\left(\mathfrak{C}'' \frac{\partial D'}{\partial u} - \mathfrak{C}' \frac{\partial D''}{\partial u}\right) = 2\left(\mathfrak{B}' \frac{\partial D'}{\partial u} - \mathfrak{B} \frac{\partial D''}{\partial u}\right),$$

$$D' \{ \,|\, \mathfrak{C}\, D_v\, D \,| + |\, \mathfrak{B}\, D_u\, D \,| \,\}$$

$$= \mathfrak{B}'' \frac{\partial D}{\partial u} - \mathfrak{B} \frac{\partial D''}{\partial u} + \mathfrak{C}'' \frac{\partial D}{\partial v} - \mathfrak{C} \frac{\partial D''}{\partial v},$$

$$D \,|\, \mathfrak{B}\, D_v\, D \,| = 2\left(\mathfrak{B}' \frac{\partial D}{\partial v} - \mathfrak{B} \frac{\partial D'}{\partial v}\right) = 2\left(\mathfrak{C}'' \frac{\partial D}{\partial v} - \mathfrak{C}' \frac{\partial D'}{\partial v}\right).$$

Hence, we have

$$DL'' - 2D'L' + D''L$$

$$= \frac{\partial}{\partial u}(D\mathfrak{B}'' - 2D'\mathfrak{B}' + D''\mathfrak{B})$$

$$- \frac{\partial}{\partial v}(D\mathfrak{C}' - 2D'\mathfrak{C}' + D''\mathfrak{C}) = 0,$$

$$D\mathfrak{L}'' + 2D'\mathfrak{L}' + D''\mathfrak{L}$$

$$= DL'' - 2D'L' + D''L$$

$$+ \frac{\partial \log \theta}{\partial u}(D\mathfrak{B}'' - 2D'\mathfrak{B}' + D''\mathfrak{B})$$

$$- \frac{\partial \log \theta}{\partial v}(D\mathfrak{C}'' - 2D'\mathfrak{C}' + D''\mathfrak{C}) = 0.$$

Therefore, we have

$$\left.\begin{aligned} \frac{2\begin{vmatrix} \mathfrak{C} & \mathfrak{L} \\ \mathfrak{C}' & \mathfrak{L}' \end{vmatrix}}{D} = \frac{\begin{vmatrix} \mathfrak{C} & \mathfrak{L} \\ \mathfrak{C}'' & \mathfrak{L}'' \end{vmatrix}}{D'} = \frac{2\begin{vmatrix} \mathfrak{C}' & \mathfrak{L}' \\ \mathfrak{C}'' & \mathfrak{L}'' \end{vmatrix}}{D''} = |\, \mathfrak{L}\ \mathfrak{C}\ D \,|, \\ \frac{2\begin{vmatrix} \mathfrak{B} & \mathfrak{L} \\ \mathfrak{B}' & \mathfrak{L}' \end{vmatrix}}{D} = \frac{\begin{vmatrix} \mathfrak{B} & \mathfrak{L} \\ \mathfrak{B}'' & \mathfrak{L}'' \end{vmatrix}}{D'} = \frac{2\begin{vmatrix} \mathfrak{B}' & \mathfrak{L}' \\ \mathfrak{B}'' & \mathfrak{L}'' \end{vmatrix}}{D''} = |\, \mathfrak{L}\ \mathfrak{B}\ D \,|. \end{aligned}\right\} \text{...(32)}$$

From (17), (20) and (31) we have

$$\left.\begin{aligned} |\bar{\mathfrak{L}}\ \bar{\mathfrak{C}}\ \bar{D}| &= pW|\mathfrak{L}\,\mathfrak{C}\,D| + p_1W|\mathfrak{L}\,\mathfrak{B}\,D|, \\ |\bar{\mathfrak{L}}\ \bar{\mathfrak{B}}\ \bar{D}| &= qW|\mathfrak{L}\,\mathfrak{C}\,D| + q_1W|\mathfrak{L}\,\mathfrak{B}\,D|. \end{aligned}\right\} \cdots\cdots\cdots(33)$$

5. From (24) we have

$$\frac{\partial\bar{\mathfrak{A}}}{\partial\eta} - \frac{\partial\bar{\mathfrak{A}}'}{\partial\xi} = \Sigma\frac{\partial}{\partial\eta}(p^2)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(pq)\mathfrak{A}$$

$$+pW\left(\frac{\partial\mathfrak{A}}{\partial v} - \frac{\partial\mathfrak{A}'}{\partial u}\right) + p_1W\left(\frac{\partial\mathfrak{A}'}{\partial v} - \frac{\partial\mathfrak{A}''}{\partial u}\right),$$

$$\frac{\partial\bar{\mathfrak{A}}'}{\partial\eta} - \frac{\partial\bar{\mathfrak{A}}''}{\partial\xi} = \Sigma\frac{\partial}{\partial\eta}(pq)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(q^2)\mathfrak{A}$$

$$+qW\left(\frac{\partial\mathfrak{A}}{\partial v} - \frac{\partial\mathfrak{A}'}{\partial u}\right) + q_1W\left(\frac{\partial\mathfrak{A}'}{\partial v} - \frac{\partial\mathfrak{A}''}{\partial u}\right).$$

And in virtue of (18), we have

$$\Sigma\frac{\partial}{\partial\eta}(p^2)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(pq)\mathfrak{A} = W\varphi(\mathfrak{A}),$$

$$\Sigma\frac{\partial}{\partial\eta}(pq)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(q^2)\mathfrak{A} = W\psi(\mathfrak{A}).$$

But we have as in **n°4.**

$$|\bar{\mathfrak{A}}\ \bar{D}_\xi\ \bar{D}| = -2W\varphi(\mathfrak{A}) + pW|\mathfrak{A}\,D_u\,D| + p_1W|\mathfrak{A}\,D_v\,D|,$$

$$|\bar{\mathfrak{A}}\ \bar{D}_\eta\ \bar{D}| = -2W\psi(\mathfrak{A}) + qW|\mathfrak{A}\,D_u\,D| + q_1W|\mathfrak{A}\,D_v\,D|.$$

Therefore, we have

$$\left.\begin{aligned} &\frac{\partial\bar{\mathfrak{A}}}{\partial\eta} - \frac{\partial\bar{\mathfrak{A}}'}{\partial\xi} + \frac{1}{2}|\bar{\mathfrak{A}}\ \bar{D}_\xi\ \bar{D}| \\ &\quad = pW\left\{\frac{\partial\mathfrak{A}}{\partial\eta} - \frac{\partial\mathfrak{A}'}{\partial\xi} + \frac{1}{2}|\mathfrak{A}\,D_u\,D|\right\} \\ &\quad + p_1W\left\{\frac{\partial\mathfrak{A}'}{\partial\eta} - \frac{\partial\mathfrak{A}''}{\partial\xi} + \frac{1}{2}|\mathfrak{A}\,D_v\,D|\right\}, \\ &\frac{\partial\bar{\mathfrak{A}}'}{\partial\eta} - \frac{\partial\bar{\mathfrak{A}}''}{\partial\xi} + \frac{1}{2}|\bar{\mathfrak{A}}\ \bar{D}_u\ \bar{D}| \end{aligned}\right\} \cdots\cdots\cdots(34)$$

$$=qW\left\{\frac{\partial\mathfrak{A}}{\partial\eta}-\frac{\partial\mathfrak{A}'}{\partial\xi}+\frac{1}{2}|\mathfrak{A}\,D_u\,D|\right\}$$
$$+q_1W\left\{\frac{\partial\mathfrak{A}'}{\partial\eta}-\frac{\partial\mathfrak{A}''}{\partial\xi}+\frac{1}{2}|\mathfrak{A}\,D_v\,D|\right\}$$

Similarly, we have

$$\left.\begin{aligned}
&\frac{\partial\bar{\mathfrak{L}}}{\partial\eta}-\frac{\partial\bar{\mathfrak{L}}'}{\partial\xi}+\frac{1}{2}|\bar{\mathfrak{L}}\,\bar{D}_\xi\,\bar{D}| \\
&\qquad=pW\left\{\frac{\partial\mathfrak{L}}{\partial\eta}-\frac{\partial\mathfrak{L}'}{\partial\xi}+\frac{1}{2}|\mathfrak{L}\,D_u\,D|\right\} \\
&\qquad+p_1W\left\{\frac{\partial\mathfrak{L}'}{\partial\eta}-\frac{\partial\mathfrak{L}''}{\partial\xi}+\frac{1}{2}|\mathfrak{L}\,D_v\,D|\right\} \\
&\frac{\partial\bar{\mathfrak{L}}'}{\partial\eta}-\frac{\partial\bar{\mathfrak{L}}''}{\partial\xi}+\frac{1}{2}|\bar{\mathfrak{L}}\,\bar{D}_\eta\,\bar{D}| \\
&\qquad=qW\left\{\frac{\partial\mathfrak{L}}{\partial\eta}-\frac{\partial\mathfrak{L}'}{\partial\xi}+\frac{1}{2}|\mathfrak{L}\,D_u\,D|\right\} \\
&\qquad+q_1W\left\{\frac{\partial\mathfrak{L}'}{\partial\eta}-\frac{\partial\mathfrak{L}''}{\partial\xi}+\frac{1}{2}|\mathfrak{L}\,D_v\,D|\right\}.
\end{aligned}\right\}\quad\cdots\cdots\cdots(35)$$

From (17) and (18) we have

$$\frac{\partial\bar{D}}{\partial\eta}-\frac{\partial\bar{D}'}{\partial\xi}+\frac{\partial\log\theta}{\partial\eta}\bar{D}-\frac{\partial\log\theta}{\partial\xi}\bar{D}'$$
$$=\varphi(D)+p\left\{\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}+\frac{\partial\log\theta}{\partial v}D-\frac{\partial\log\theta}{\partial u}D'\right\}$$
$$+p_1\left\{\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}+\frac{\partial\log\theta}{\partial v}D'-\frac{\partial\log\theta}{\partial u}D''\right\},$$

$$\frac{\partial\bar{D}'}{\partial\eta}-\frac{\partial\bar{D}''}{\partial\xi}+\frac{\partial\log\theta}{\partial\eta}\bar{D}'-\frac{\partial\log\theta}{\partial\xi}\bar{D}''$$
$$=\psi(D)+q\left\{\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}+\frac{\partial\log\theta}{\partial v}D-\frac{\partial\log\theta}{\partial u}D'\right\}$$
$$+q_1\left\{\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}+\frac{\partial\log\theta}{\partial v}D'-\frac{\partial\log\theta}{\partial u}D''\right\}.$$

While we have as in **n°4.**

$$|\overline{D}_\xi\ \overline{D}_\eta\ \overline{D}| = 2\frac{\partial\varphi(D)}{\partial\eta} - 2\frac{\partial\psi(D)}{\partial\xi} + W|D_u\, D_v\, D|.$$

Therefore, if we put

$$\theta = \frac{1}{2}\Bigg[\frac{\partial^2 D}{\partial v^2} - 2\frac{\partial^2 D'}{\partial u\partial v} + \frac{\partial^2 D''}{\partial u^2} - \frac{1}{2}|D_u\, D_v\, D|$$

$$\frac{\partial}{\partial v}\left(\frac{\partial \log\theta}{\partial v}D - \frac{\partial\log\theta}{\partial u}D'\right) - \frac{\partial}{\partial u}\left(\frac{\partial\log\theta}{\partial v}D' - \frac{\partial\log\theta}{\partial u}D''\right)\Bigg],$$

we have

$$\overline{\Theta} = W\Theta. \quad\text{..............................(36)}$$

Accordingly, Θ is a relative invariant and $\frac{\Theta}{\theta}$ is an absolute invariant.

From (33), (34), (35) and (36) we know that if we put

$$\mathfrak{F}^{(1)} = \frac{\partial}{\partial v}(2\mathfrak{A} - \mathfrak{L}) - \frac{\partial}{\partial u}(2\mathfrak{A}' - \mathfrak{L}')$$

$$+\frac{1}{2}|2\mathfrak{A} - \mathfrak{L}\, D_u\, D| - |\mathfrak{L}\,\mathfrak{C}\, D| + \theta\frac{\partial}{\partial u}\left(\frac{\Theta}{\theta}\right),$$

$$\mathfrak{F}^{(2)} = \frac{\partial}{\partial v}(2\mathfrak{A}' - \mathfrak{L}') - \frac{\partial}{\partial u}(2\mathfrak{A}'' - \mathfrak{L}'')$$

$$+\frac{1}{2}|2\mathfrak{A} - \mathfrak{L}\, D_v\, D| - |\mathfrak{L}\,\mathfrak{B}\, D| + \theta\frac{\partial}{\partial v}\left(\frac{\Theta}{\theta}\right),$$

we have

$$\left.\begin{aligned}\overline{\mathfrak{F}}^{(1)} &= pW\mathfrak{F}^{(1)} + p_1W\mathfrak{F}^{(2)},\\ \overline{\mathfrak{F}}^{(2)} &= qW\mathfrak{F}^{(1)} + qW\mathfrak{F}^{(2)}\end{aligned}\right\} \quad\text{............(37)}$$

and we can easily see that

$$\mathfrak{F}^{(1)} = 2\left(\frac{\partial A}{\partial v} - \frac{\partial A'}{\partial u}\right) + \frac{\partial^2 C}{\partial v^2} - 2\frac{\partial^2 C'}{\partial u\partial v} + \frac{\partial^2 C''}{\partial u^2}$$

$$+\frac{\partial}{\partial v}|C D_u\, D| - \frac{\partial}{\partial u}|B - D_v\, C D|$$

$$+|A\, D_u\, D| - |L\, C D|,$$

$$\left.\begin{aligned}\mathfrak{F}^{(2)}&=2\left(\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial u}\right)+\frac{\partial^2 B}{\partial v^2}-2\frac{\partial^2 B'}{\partial u\partial v}+\frac{\partial^2 B''}{\partial u^2}\\&\frac{\partial}{\partial v}\,|\,C-D_u\,B\,D\,|\,-\frac{\partial}{\partial u}\,|\,B\,D_v\,D\,|\\&+|\,A\,D_v\,D\,|-|\,L\,B\,D\,|\,.\end{aligned}\right\}\quad(38)$$

From (20), (25), (31) and (34) we know that if we put

$$\begin{aligned}\mathfrak{F}^{(3)}&=\theta\frac{\partial}{\partial v}\left[\frac{1}{\theta}\left\{\frac{\partial\mathfrak{A}}{\partial v}-\frac{\partial\mathfrak{A}'}{\partial u}+\frac{1}{2}\,|\,\mathfrak{A}\,D_u\,D\,|-|\,\mathfrak{A}\,\mathfrak{C}\,D\,|\right\}\right]\\&-\theta\frac{\partial}{\partial u}\left[\frac{1}{\theta}\left\{\frac{\partial\mathfrak{A}'}{\partial v}-\frac{\partial\mathfrak{A}''}{\partial u}+\frac{1}{2}\,|\,\mathfrak{A}\,D_v\,D\,|-|\,\mathfrak{A}\,\mathfrak{B}\,D\,|\right\}\right]\\&-|\,\mathfrak{L}\,\mathfrak{A}\,D\,|\,,\end{aligned}$$

we have

$$\bar{\mathfrak{F}}^{(3)}=W^2\mathfrak{F}^{(3)},$$

and we can easily see that

$$\left.\begin{aligned}\mathfrak{F}^{(3)}&=\frac{\partial^2 A}{\partial v^2}-2\frac{\partial^2 A'}{\partial u\partial v}+\frac{\partial A''}{\partial u^2}-\frac{\partial}{\partial v}\,|\,A\,C-D_u\,D\,|\\&+\frac{\partial}{\partial u}\,|\,A\,B-D_v\,D\,|-|\,L\,A\,D\,|\\&-\frac{1}{2}\,\frac{\partial\log\theta}{\partial v}\mathfrak{F}^{(1)}+\frac{1}{2}\,\frac{\partial\log\theta}{\partial u}\mathfrak{F}^{(2)}.\end{aligned}\right\}\quad\cdots(39)$$

We shall show, in the subsequent chapter, that $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$ and $\mathfrak{F}^{(3)}$ vanish identically.

CHAPTER II.

FUNDAMENTAL EQUATIONS OF THE PROJECTIVE DIFFERENTIAL GEOMETRY OF NONDEVELOPABLE SURFACES.

1. Put

$$y=D\frac{\partial x}{\partial v}-D'\frac{\partial x}{\partial u},$$

$$\left.\begin{aligned} z &= D'\frac{\partial x}{\partial v} - D''\frac{\partial x}{\partial u}, \\ w &= \frac{1}{2}\left(\frac{\partial y}{\partial v} - \frac{\partial z}{\partial u}\right). \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Then we have

$$|x\, y\, z\, w| = (DD'' - D'^2)^2 = 1.$$

Therefore, the four points x, y, z, w are not coplanar. From (1) we have

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= -D'y + Dz, \\ \frac{\partial x}{\partial v} &= -D''y + D'z. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

From the definition of the fundamental determinants we have

$$\left.\begin{aligned} & Ax - B\frac{\partial x}{\partial u} + C\frac{\partial x}{\partial v} - D'\frac{\partial^2 x}{\partial u^2} + D\frac{\partial^2 x}{\partial u \partial v} = 0, \\ & 2A'x - 2B'\frac{\partial x}{\partial u} + 2C'\frac{\partial x}{\partial v} - D''\frac{\partial^2 x}{\partial u^2} + D\frac{\partial^2 x}{\partial v^2} = 0, \\ & A''x - B''\frac{\partial x}{\partial u} + C''\frac{\partial x}{\partial v} - D''\frac{\partial^2 x}{\partial u \partial v} + D'\frac{\partial^2 x}{\partial v^2} = 0. \end{aligned}\right\} \quad \cdots\cdots\cdots (3)$$

From (1), (2) and (3) we have

$$\left.\begin{aligned} \frac{\partial y}{\partial u} &= [D'(B - \frac{\partial D}{\partial v}) - D(C'' - \frac{\partial D''}{\partial u})]y \\ &\quad - [D'(C - \frac{\partial D}{\partial u}) - D(B - \frac{\partial D'}{\partial u})]z - Ax, \\ \frac{\partial y}{\partial v} &= w + [D''(B - \frac{\partial D}{\partial v}) - D'(C'' - \frac{\partial D''}{\partial u})]y \\ &\quad - [D'(B - \frac{\partial D}{\partial v}) - D(C'' - \frac{\partial D''}{\partial u})]z - A'x, \\ \frac{\partial z}{\partial u} &= -w + [D''(B - \frac{\partial D}{\partial v}) - D'(C'' - \frac{\partial D''}{\partial u})]y \end{aligned}\right\} \cdots (2_2)$$

$$-[D'(B-\frac{\partial D}{\partial v})-D(C''-\frac{\partial D''}{\partial u})]z-A'x,$$

$$\frac{\partial z}{\partial v}=[D''(C''-\frac{\partial D'}{\partial v})-D'(B''-\frac{\partial D''}{\partial v})]y$$

$$-[D''(B-\frac{\partial D}{\partial v})-D'(C''-\frac{\partial D''}{\partial u})]z-A''x.$$

From (2_2) by writing

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u}\right),$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right),$$

we have

$$\left.\begin{aligned}
\frac{\partial w}{\partial u}&=[D''(A-L)-D'(A'-L')+D'(\theta-\theta_{oo})]y\\
&\quad-[D'(A-L)-D(A'-L')+D(\theta-\theta_{oo})]z\\
&\quad-[\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}-|A\,C-D_u\,D|]x,\\
\frac{\partial w}{\partial v}&=[D''(A'-L')-D'(A''-L'')+D''(\theta-\theta_{oo})]y\\
&\quad-[D'(A'-L')-D(A''-L'')+D'(\theta-\theta_{oo})]z\\
&\quad-[\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial u}-|A\,B-D_v\,D|\,x.
\end{aligned}\right\}\quad\ldots\ldots(2_3)$$

where

$$\theta_{oo}=\frac{1}{2}\frac{\partial^2 D}{\partial v^2}-\frac{\partial^2 D'}{\partial u\partial v}+\frac{1}{2}\frac{\partial^2 D''}{\partial u^2}-\frac{1}{4}|D_u\,D_v\,D|.$$

The equations (2_1) (2_2) and (2_3) form a system of simultaneous linear partial differential equations.

The conditions of the complete integrability of this system which is obtained by writing

$$\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right)$$

are

$$\left.\begin{aligned}
&2\left(\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}\right)+\frac{\partial^2 C}{\partial v^2}-2\frac{\partial^2 C'}{\partial u\partial v}+\frac{\partial^2 C''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|C D_u D|-\frac{\partial}{\partial u}|C B-D_v D|\\
&\quad+|A D_u D|-|L C D|=0,\\
&2\left(\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial u}\right)+\frac{\partial^2 B}{\partial v^2}-2\frac{\partial^2 B'}{\partial u\partial v}+\frac{\partial^2 B''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|C-D_u B D|-\frac{\partial}{\partial w}|B D_v D|\\
&\quad+|A D_v D|-|L C D|=0,\\
&\frac{\partial^2 A}{\partial v^2}-2\frac{\partial^2 A'}{\partial u\partial v}+\frac{\partial^2 A''}{\partial u^2}-\frac{\partial}{\partial v}|A C-D_u D|\\
&\quad+\frac{\partial}{\partial u}|A B-D_v D|-|L A D|=0.
\end{aligned}\right\}\quad\ldots\ldots(4)$$

From (38), (39) (**Chap. I**) and (4) we know that $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$ and $\mathfrak{F}^{(3)}$ vanish identically.

Let (X, Y, Z, W) a pair of the solutions of the system formed by the equations (2_1), (2_2), (2_3) and consider the system formed by the first equation of (2_1), the first and the third of (2_2), and the first of (2_3). If we regard v as a parameter, the latter system is that of the normal linear ordinary differential equations of which the normal coordinates x_i and y_i, z_i, w_i $(i=1, 2, 3, 4)$ defined by the equation (1) are a fundamental system of the solutions.[1]

Therefore, we have

$$X=\sum_{i=1}^{4} C_i(v)x_i,$$

$$Y=\sum_{i=1}^{4} C_i(v)y_i,$$

$$Z=\sum_{i=1}^{4} C_i(v)z_i,$$

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1 Poussin, Cours d'Analyse Infinitesimal, Vol. II. p. 256.

$$W=\sum_{i=1}^{4} C_i(v) w_i$$

where $C_i(v)$ $(i=1, 2, 3, 4)$ are functions of v only.

Substituting these values of X, Y, Z, W in the seconde quations of (2_1) the second and the fourth of (2_2), and the second of (2_3), we have

$$\sum_{i=1}^{4} \frac{dC_i}{dv} x_i = 0, \quad \sum_{i=1}^{4} \frac{dC_i}{dv} y_i = 0,$$

$$\sum_{i=1}^{4} \frac{dC_i}{dv} z_i = 0, \quad \sum_{i=1}^{4} \frac{dC_i}{dv} w_i = 0.$$

Therefore $C_i (i=1, 2, 3, 4)$ are constants.

Therefore, we know that if X_i, Y_i, Z_i, W_i $(i=1, 2, 3, 4)$ be four systems of the solutions which satisfy the relations

$$|X\,Y\,Z\,W| \neq 0,$$

The surface described by the point X is a projective transform of the given surface, in another words, *auy surface of which the normal fundamental determinants are equal to those of the surface S is a projective transform of S.*

But by the transformation

$$\bar{x}=e^{\frac{i\pi}{4}} x,$$

the signs of all the normal fundamental determinants are changed. Therefore, we know that *any surface of which the normal fundamental determinants are equal, except for a sign, to those of the surface S is a projective transform of S.*

2. Next, let us suppose that we are given twelve function D, D', D''; B, B', B''; C, C', C''; A, A', A'' which are holomorph in the domain R and satisfy the equations

$$\left.\begin{aligned} &D'^2-DD''=1, \\ &DB''-2D'B'+DB''=0, \\ &DC''-2D'C'+DC''=0, \\ &DA''-2D'A'+DA''=0, \end{aligned}\right|$$

$$\left.\begin{aligned}
&\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}=2(B-C')\\
&\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}=2(B'-C'')\\
&2\left(\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}\right)+\frac{\partial^2 C}{\partial v^2}-2\frac{\partial^2 C'}{\partial u\partial v}+\frac{\partial^2 C''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|C D_u D|-\frac{\partial}{\partial u}|C B-D_v D|\\
&\quad+|A D_u D|-|L C D|=0,\\
&2\left(\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial u}\right)+\frac{\partial^2 B}{\partial v^2}-2\frac{\partial^2 B'}{\partial u\partial v}+\frac{\partial^2 B''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|C-D_u B D|-\frac{\partial}{\partial u}|B D_v D|\\
&\quad+|A D_v D|-|L C D|=0,\\
&\frac{\partial^2 A}{\partial v^2}-2\frac{\partial^2 A'}{\partial u\partial v}+\frac{\partial^2 A''}{\partial u^2}-\frac{\partial}{\partial v}|A C-D_u D|\\
&\quad+\frac{\partial}{\partial u}|A B-D_v D|-|L A D|=0.
\end{aligned}\right\}\cdots\cdots(5)$$

Then the system of the total differential equations

$$\begin{aligned}
&dx=-(D'y-Dz)du-(D''y-D'z)dv,\\
&dy=[\{D'(B-\frac{\partial D}{\partial v})-D(C''-\frac{\partial D''}{\partial u})\}y\\
&\qquad-\{D'(C-\frac{\partial D}{\partial u})-D(B-\frac{\partial D'}{\partial u})\}z-A'x]\,du\\
&\quad+[w+\{D''(B-\frac{\partial D}{\partial v})-D'(C-\frac{\partial D''}{\partial w})\}y\\
&\qquad-\{D'(B-\frac{\partial D}{\partial v})-D(C''-\frac{\partial D''}{\partial w})\}z-A'x]dv,\\
&dz=[-w+\{D''(B-\frac{\partial D}{\partial v})-D'(C''-\frac{\partial D''}{\partial w})\}y
\end{aligned}$$

$$\left.\begin{aligned} &-\{D'(B-\frac{\partial D}{\partial v}(-D(C''-\frac{\partial D''}{\partial w}))\}z-A'x]du \\ &+[\{D''(C''-\frac{\partial D'}{\partial v})-D'(B''-\frac{\partial D''}{\partial v})\}y \\ &-\{D''(B-\frac{\partial D}{\partial v})-D'(C''-\frac{\partial D''}{\partial w})z-A''x]dv, \\ dw=&[\{D''(A-L)-D'(A'-L')+D'(\theta-\theta_{oo})\}y \\ &-\{D'(A-L)-D(A'-L')+D(\theta-\theta_{oo})\}z \\ &-\{\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}-|A\ C-D_u\ D|\ \}x]du \\ &+[\{D''(A'-L')-D'(A''-L'')+D''(\theta-\theta_{oo})\}y \\ &-\{D'(A'-L')-D(A''-L'')+D'(\theta-\theta_{oo})\}z \\ &-\{\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial w}-|A\ B-D_v\ D|\ \}x]dv \end{aligned}\right\}\cdots(6)$$

are completely integrable.

Therefore there are one and only one system of the functions x, y, z w which satisfy (6), and are holomorph in the vicinity of x_o, y_o, z_o, w_o, and assume respectively the values x_o, y_o, z_o, w_o for $u=u_o$, $v=v_o$, (u_o, v_o) being a point in the domain R.[1]

Take constants $(\lambda_i, \mu_i, \nu_i, \sigma_i)$ $(i=1, 2, 3, 4)$ which satisfy

$$|\lambda\,\mu\,\nu\,\sigma|=1.$$

Let (x_i, y_i, z_i, w_i) $(i=1, 2, 3, 4)$ be the integrals which assume respectively the values $(\lambda_i, \mu_i, \nu_i, \sigma_i)$ for $u=u_o$, $v=v_o$.

From (6) we have

$$\frac{\partial}{\partial u}|x\,y\,z\,w|=0,$$

$$\frac{\partial}{\partial v}|x\,y\,z\,w|=0.$$

Therefore $|x\,y\,z\,w|$ is a constant,

1 Grousat, Leçons sur l'integration des équation aux dérivées partielles du premier ordre, p. 73.

But

$$|x\, y\, z\, w| = 1$$

for $u=u_0$, $v=v_v$. Accordingly this relation holds for all values of u, v.

From (6) we have

$$\left.\begin{aligned}
\frac{\partial^2 x}{\partial u^2} &= -Dw-(2C'-B)y+Cz+(AD'-A'D)x, \\
\frac{\partial^2 x}{\partial u \partial v} &= -D'w-C''y+Bz+\frac{1}{2}(AD''-A''D)x, \\
\frac{\partial^2 x}{\partial v^2} &= -D''w-B''y+(2B'-C'')z+(A'D''-A''D')x.
\end{aligned}\right\}\cdots(7)$$

From (6) and (7) we have

$$|x \quad x_u \quad x_v \quad x_{uu}| = D,$$

$$\frac{1}{2}|x \quad x_u \quad x_v \quad x_{uv}| = D',$$

$$|x \quad x_u \quad x_v \quad x_{vv}| = D'',$$

$$|x \quad x_v \quad x_{uu} \quad x_{uv}| = B,$$

$$\frac{1}{2}|x \quad x_v \quad x_{uu} \quad x_{vv}| = B',$$

$$|x \quad x_v \quad x_{uv} \quad x_{vv}| = B'',$$

$$|x \quad x_u \quad x_{uu} \quad x_{uv}| = C,$$

$$\frac{1}{2}|x \quad x_u \quad x_{uu} \quad x_{vv}| = C',$$

$$|x \quad x_u \quad x_{uv} \quad x_{vv}| = C'',$$

$$|x_u \quad x_v \quad x_{uu} \quad x_{uv}| = A,$$

$$\frac{1}{2}|x_u \quad x_v \quad x_{uu} \quad x_{vv}| = A',$$

$$|x_u \quad x_v \quad x_{uv} \quad x_{vv}| = A''.$$

Therefore we know that *if we are given twelve function of* u, v D, D', D''; B, B', B''; C, C', C''; A, A', A'' *which satisfy the equa-*

tion (5), *we have nnique surface which has them as the normal fundamental determinants, except for a projective transformation.*

3. Let S and S_1 be two nondevelopable surfaces which are projectively deformable to each other.

Let us suppose that they are defined by the equations

$$\begin{aligned} \mathfrak{x}_i &= \mathfrak{x}_i(u, v), \\ \mathfrak{X}_i &= \mathfrak{X}_i(u, v), \end{aligned} \qquad (i=1,\ 2,\ 3,\ 4)$$

respectively and that parameters u, v are so chosen that the corresponding points on S and S_1 correspond to the same values of u, v. Denote by a, b, ... ; a_1, b_1 ...... the fundamental determinants of S and S_1 respectively.

Fubini[1] has proved that for S and S_1 to be projectively deformable to each other, it is necessary and sufficient that

$$\left.\begin{aligned} d &= \lambda d_1, \quad d' = \lambda d_1', \quad d'' = \lambda d_1'', \\ b &= \lambda[b_1 - 2\beta d_1' + \gamma d_1], \\ b' &= \lambda[b_1' - \beta d_1''], \\ b'' &= \lambda[b_1'' - \gamma d_1''], \\ c &= \lambda[c_1 - \beta d_1], \\ c' &= \lambda[c_1' - \gamma d_1], \\ c'' &= \lambda[c_1'' - 2\gamma d_1' + \beta d_1''] \end{aligned}\right\} \quad \text{..............(8)}$$

where λ, β, γ are functions of u, v.

From (8) we have

$$\varDelta = \varepsilon\lambda\varDelta_1 \qquad (\varepsilon = +1, \text{ or } -1)\text{......(9)}$$

From (18) and the equations

$$\frac{\partial d}{\partial v} - \frac{\partial d'}{\partial u} = 2(b - c'),$$

$$\frac{\partial d'}{\partial v} - \frac{\partial d''}{\partial u} = 2(b' - c''),$$

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[1] Rendiconti del circolo mathematico di Palermo. **41**, p. 135 (1916).

$$\frac{\partial d_1'}{\partial v}-\frac{\partial d_1'}{\partial u}=2(b_1-c_1'),$$

$$\frac{\partial d_1'}{\partial v}-\frac{\partial d_1''}{\partial u}=2(b_1'-c_1''),$$

we have

$$\left(\beta-\frac{1}{4}\frac{\partial \log \lambda}{\partial u}\right)d_1'-\left(\gamma-\frac{1}{4}\frac{\partial \log \lambda}{\partial v}\right)d_1=0,$$

$$\left(\beta-\frac{1}{4}\frac{\partial \log \lambda}{\partial w}\right)d_1''-\left(\gamma-\frac{1}{4}\frac{\partial \log \lambda}{\partial v}\right)d_1'=0.$$

Therefore we have

$$\left.\begin{aligned}&\beta-\frac{\partial \log \lambda}{\partial u}=0.\\&\gamma-\frac{1}{4}\frac{\partial \log \lambda}{\partial v}=0.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(10)$$

Let A, B, ... ; A_1, B_1, ... be the normal fundamental determinants of S and S_1 respectively.

Then we have from (8), (9) and (10)

$$\left.\begin{aligned}&D_1=\varepsilon D,\ \ D_1'=\varepsilon D',\ \ D_1''=\varepsilon D'',\\&B_1=\varepsilon B,\ \ B_1'=\varepsilon B',\ \ B_1''=\varepsilon B'',\\&C_1=\varepsilon C,\ \ C_1'=\varepsilon C',\ \ C_1''=\varepsilon C''.\end{aligned}\right\}\quad\ldots\ldots(11)$$

where $\varepsilon=+1$, or -1.

Therefore we know that for the two surfaces S and S_1 to be projectively deformable to each other, it is necessary and sufficient that we can choose the parameters u, v so that corresponding points on S and S_1 correspond to the same values of u, v and satisfy equation (11).

(*To be Continued.*)