On the Curvature of a Curve in n Dimensional Non-Euclidean Space.

By

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(Received Dec. 13, 1922).

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1. The curvature of a curve in n dimensional Non-Euclidean space has already been discussed by many professors—G. Kowalewski,[1] E. Stransky[2] and T. Nishiuchi[3] from different points of view. And also Prof. T. Kubota[4] has treated the problem following the method given by E. Rath[5] in Euclidean space. In this paper, I will give the generalization of Prof. Nishiuchi's discussion according to the idea of Meyer[6] which he applied in the generalization of Serret-Frenet's formula in the Euclidean case.

In n Dimentional Space.

2. Let $x_0, x_1, x_2, \ldots\ldots, x_n$ be thc Weirstrass coordinates of point in n dimensional space whose measure of curvature is $\frac{1}{k^2}$ and

$$(xx)=x_0^2+x_1^2+x_2^2+\ldots\ldots+x_n^2=0$$

be the equation to the absolute.

Suppose

$$x_c=\varphi_0(t),\ x_1=\varphi_1(t),\ x_2=\varphi_2(t), \ldots\ldots,\ x_n=\varphi_n(t)$$

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1 Wiener Berichte, **120**, (1911).
2 Wiener Berichte, **121**, (1912).
3 Mem. Coll. Sci., Kyoto, **5**. No. 3. (1921).
4 Kubota, in Sendai, Tôhoku Math. J., **21**. No. 3, 4. (1922).
5, 6 Jahresber. D. M. ver., **19**, (1910).

represent the coordinates of a point on a curve Γ, where t is a parameter and

$$(\varphi_0(t))^2+(\varphi_1(t))^2+(\varphi_2(t))^2+\ldots\ldots+(\varphi_n(t))^2=k^2.$$

Next, we consider any other curve Γ' defined by the equations

$$a_i=\psi_i(t)=x_i \sin\frac{r}{k}+y_i \cos\frac{r}{k}$$

where r is a function of t and

$$(xy)=0, \quad (aa)=k^2$$

$$i=0,\ 1,\ 2,\ \ldots\ldots,\ n.$$

Take a point $P_0[x(t)]$ on the curve Γ and take $r+1$ points $P_0'[a(t)]$, $P_1'[a(t+\Delta t)]$, $P_2'[a(t+2\Delta t)]$, ..., $P_r'[a(t+r\Delta t)]$ on the curve Γ'. Then the coordinates of a point ${}_rQ({}_r\eta)$ which is on the plane manifold passing through $r+1$ points $(P_0, P_0', P_1', \ldots\ldots, P_{r-1}')$ and orthogonol to r points $(P_0, P_0', P_1', \ldots\ldots, P_{r-2}')$ will be

$$(1)\quad {}_r\eta_i=\frac{k\begin{vmatrix}(xx) & (xa) & \ldots\ldots(xa^{r-2}) & x_i\\ (ax) & (aa) & \ldots\ldots(aa^{r-2}) & a_i\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-1}x) & (a^{r-1}a) & \ldots\ldots(a^{r-1}a^{r-2}) & a_i^{r-1}\end{vmatrix}}{\sqrt{\begin{vmatrix}(xx) & (xa) & \ldots(xa^{r-2})\\ (ax) & (aa) & \ldots(aa^{r-2})\\ \vdots & \vdots & \vdots\\ (a^{r-2}x) & (a^{r-2}a) & \ldots(a^{r-2}a^{r-2})\end{vmatrix}\cdot\begin{vmatrix}(xx) & (xa) & \ldots(xa^{r-1})\\ (ax) & (aa) & \ldots(aa^{r-1})\\ \vdots & \vdots & \vdots\\ (a^{r-1}x) & (a^{r-1}a) & \ldots(a^{r-1}a^{r-1})\end{vmatrix}}},$$

$$(i=0,\ 1,\ 2,\ \ldots\ldots,\ n)$$

where

$$a^i=\frac{d^ia}{dt^i}.$$

And it is evident that

$$({}_r\eta_p\eta)=0, \quad (r\neq p),$$

$$({}_r\eta_p\eta)=k^2, \quad (r=p).$$

Next, let ${}_r\dot{Q}({}_r\dot{\eta})$ be a point on the plane manifold passing through $r+1$ points $(P_0, P_0', P_1', \ldots P_{r-2}', P_r')$ and orthogonal to r points $(P_0, P_0', P_1', \ldots, P_{r-2}')$, then

$$
{}_r\dot{\eta}_i = \left[\begin{vmatrix} (xx) & (xa) & \ldots (xa^{r-2}) & x_i \\ (ax) & (aa) & \ldots (aa^{r-2}) & a_i \\ \vdots & \vdots & \vdots & \vdots \\ (a^{r-1}x) & (a^{r-1}a) & \ldots (a^{r-1}a^{r-2}) & a_i^{r-1} \end{vmatrix} + \begin{vmatrix} (xx) & (xa) & \ldots (xa^{r-2}) & x_i \\ \vdots & \vdots & \vdots & \vdots \\ (a^{r-2}x) & (a^{r-2}a) & \ldots (a^{r-2}a^{r-2}) & a_i^{r-2} \\ (a^{r}x) & (a^{r}a) & \ldots (a^{r}a^{r-2}) & a_i^{r} \end{vmatrix} \Delta t \right] \times
$$

(2)

$$
\times \begin{vmatrix} (xx) & \ldots (xa^{r-2}) \\ (ax) & \ldots (aa^{r-2}) \\ \vdots & \vdots \\ (a^{r-2}x) & \ldots (a^{r-2}a^{r-2}) \end{vmatrix}^{\frac{1}{2}} \left[\begin{vmatrix} (xx) & \ldots (xa^{r-1}) \\ (ax) & \ldots (aa^{r-1}) \\ \vdots & \vdots \\ (a^{r-1}x) & \ldots (a^{r-1}a^{r-1}) \end{vmatrix} \right.
$$

$$
\left. + 2 \begin{vmatrix} (xx) & \ldots (xa^{r-2}) & (xa^{r}) \\ \vdots & \vdots & \vdots \\ (a^{r-1}x) & \ldots (a^{r-1}a^{r-2}) & (a^{r-1}a^{r}) \end{vmatrix} \Delta t + \begin{vmatrix} (xx) & \ldots (xa^{r-2}) & (xa^{r}) \\ \vdots & \vdots & \vdots \\ (a^{r-2}x) & \ldots (a^{r-2}a^{r-2}) & (a^{r-2}a^{r}) \\ (a^{r}x) & \ldots (a^{r}a^{r-2}) & (a^{r}a^{r}) \end{vmatrix} \overline{\Delta t^2} \right]^{-\frac{1}{2}}.
$$

The limiting value of $\dfrac{\overline{{}_rQ\,{}_r\dot{Q}}}{\text{arc } P_0P_1} \times \dfrac{1}{k}$ *when* $P_1[x(t+\Delta t)]$ *approaches to* P_0 *shall be called the rth curvature of the curve* Γ *at the point* P_0 *with respect to the curve* Γ', in analogy to Prof. Nishiuchi's definition.[1]

But

$$
\sin^2 \frac{\overline{{}_rQ\,{}_r\dot{Q}}}{k} = \frac{\left\| \begin{matrix} {}_r\eta_0 & {}_r\eta_1 & \cdots\cdots & {}_r\eta_n \\ {}_r\dot{\eta}_0 & {}_r\dot{\eta}_1 & \cdots\cdots & {}_r\dot{\eta}_n \end{matrix} \right\|^2}{k^4}
$$

$$
(3) \quad = \frac{\left\| \begin{matrix} \begin{vmatrix} (xx) & \ldots (xa^{r-2}) & (xa^{r-1}) \\ \vdots & \vdots & \vdots \\ (a^{r-2}x) & \ldots (a^{r-2}a^{r-2}) & (a^{r-2}a^{r-1}) \\ (a^{r-1}x) & \ldots (a^{r-1}a^{r-2}) & (a^{r-1}a^{r-1}) \end{vmatrix} & \begin{vmatrix} (xx) & \ldots (xa^{r-2}) & (xa^{r}) \\ \vdots & \vdots & \vdots \\ (a^{r-2}x) & \ldots (a^{r-2}a^{r-2}) & (a^{r-2}a^{r}) \\ (a^{r-1}x) & \ldots (a^{r-1}a^{r-2}) & (a^{r-1}a^{r}) \end{vmatrix} \\ \begin{vmatrix} (xx) & \ldots (xa^{r-2}) & (xa^{r}) \\ \vdots & \vdots & \vdots \\ (a^{r-2}x) & \ldots (a^{r-2}a^{r-2}) & (a^{r-2}a^{r}) \\ (a^{r-1}x) & \ldots (a^{r-1}a^{r-2}) & (a^{r-1}a^{r}) \end{vmatrix} & \begin{vmatrix} (xx) & \ldots (xa^{r-2}) & (xa^{r}) \\ \vdots & \vdots & \vdots \\ (a^{r-2}x) & \ldots (a^{r-2}a^{r-2}) & (a^{r-2}a^{r}) \\ (a^{r}x) & \ldots (a^{r}a^{r-2}) & (a^{r}a^{r}) \end{vmatrix} \end{matrix} \right\|}{\begin{vmatrix} (xx) & \ldots (xa^{r-2}) & (xa^{r-1}) \\ \vdots & \vdots & \vdots \\ (a^{r-2}x) & \ldots (a^{r-2}a^{r-2}) & (a^{r-2}a^{r-1}) \\ (a^{r-1}x) & \ldots (a^{r-1}a^{r-2}) & (a^{r-1}a^{r-1}) \end{vmatrix}^2} \overline{\Delta t^2}
$$

$$
+ \theta \overline{\Delta t^2}
$$

[1] Loc. cit.

where

$$\lim_{\Delta t \to 0.} \theta = 0.$$

Applying the Sylvester theorem, we have

$$\text{(4)} \qquad \sin^2 \frac{\overline{{}_rQ_r\dot{Q}}}{k} = \frac{\Delta_{r-1}\Delta_{r+1}}{\Delta_r^2} \overline{\Delta t^2} + \theta \overline{\Delta t^2}$$

where

$$\text{(5)} \qquad \Delta_i = \begin{vmatrix} (xx) & (xa) & \dots (xa^{i-1}) \\ \vdots & \vdots & \vdots \\ (a^{i-1}x) & (a^{i-1}a) & \dots (a^{i-1}a^{i-1}) \end{vmatrix} = \left\| \begin{matrix} x_0 & x_1 & \dots x_n \\ a_0 & a_1 & \dots a_n \\ a_0' & a_1' & \dots a_n' \\ \dots & \dots & \dots \\ a_0^{i-1} & a_1^{i-1} & \dots a_n^{i-1} \end{matrix} \right\|^2.$$

On the other hand

$$\sin \frac{\overline{P_0P_1}}{k} = \sqrt{\frac{\begin{vmatrix} (xx) & (xx') \\ (x'x) & (x'x') \end{vmatrix}}{(xx)^2}} \Delta t + W\Delta t$$

$$= \frac{\sqrt{\Delta}}{k^2} \Delta t + W\Delta t$$

where

$$\lim_{\Delta t \to 0} W = 0.$$

So the r^{th} curvature $\frac{1}{R_r}$ of the curve Γ at the point P_0 with respect to the curve Γ' will be

$$\frac{1}{R_r} = \lim_{\Delta t \to 0} \frac{\overline{{}_rQ_r\dot{Q}}}{k \operatorname{arc} P_0P_1} = \lim_{\Delta t \to 0} \frac{k \sin \frac{\overline{{}_rQ_r\dot{Q}}}{k}}{k \sin \frac{\overline{P_0P_1}}{k}} \times \frac{1}{k}$$

$$\text{(6)} \qquad = k \frac{\sqrt{\Delta_{r-1}\Delta_{r+1}}}{\sqrt{\Delta}\,\Delta_r}$$

$$(r = 1, 2, \dots\dots, n)$$

$$\frac{1}{R_0}=\frac{1}{k}.$$

Also, if ${}_r\theta$ be the angle between two plane manifolds whose absolute poles are ${}_rQ$ and ${}_r\dot{Q}$ respectively, then we have

$$\frac{1}{R_r}=\lim_{\Delta t\to 0}\frac{{}_r\theta}{\text{arc } P_0P_1}.$$

From this equation we can also define $\frac{1}{R_r}$, but I think the former definition is natural.

3. Differentiating ${}_r\eta_i$ with respect to t, we have

$$\frac{d_r\eta_i}{dt}=\frac{k}{\Delta_{r-1}^{\frac{3}{2}}\Delta_r^{\frac{3}{2}}}\Bigg[\Delta_{r-1}\left\{\begin{vmatrix}(xx) & \ldots(x\alpha^{r-1})\\ \vdots & \vdots\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-1})\\ (\alpha^{r-1}x) & \ldots(\alpha^{r-1}\alpha^{r-1})\end{vmatrix}\cdot\begin{vmatrix}(xx) & \ldots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-2}) & \alpha_i^{r-2}\\ (\alpha^{r}x) & \ldots(\alpha^{r}\alpha^{r-2}) & \alpha_i^{r}\end{vmatrix}\right.$$

$$\left.-\begin{vmatrix}(xx) & \ldots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-2}) & \alpha_i^{r-2}\\ (\alpha^{r-1}x) & \ldots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx) & \ldots(x\alpha^{r-2}) & (x\alpha^{r})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-2}) & (\alpha^{r-2}\alpha^{r})\\ (\alpha^{r-1}x) & \ldots(\alpha^{r-1}\alpha^{r-2}) & (\alpha^{r-1}\alpha^{r})\end{vmatrix}\right\}$$

$$-\Delta_r\left\{\begin{vmatrix}(xx) & \ldots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-2}) & \alpha_i^{r-2}\\ (\alpha^{r-1}x) & \ldots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx) & \ldots(x\alpha^{r-3}) & (x\alpha^{r-1})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-3}x) & \ldots(\alpha^{r-3}\alpha^{r-3}) & (\alpha^{r-3}\alpha^{r-1})\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-3}) & (\alpha^{r-2}\alpha^{r-1})\end{vmatrix}\right.$$

$$\left.-\begin{vmatrix}(xx) & \ldots(x\alpha^{r-2})\\ \vdots & \vdots\\ (\alpha^{r-3}x) & \ldots(\alpha^{r-3}\alpha^{r-2})\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\cdot\begin{vmatrix}(xx) & \ldots(x\alpha^{r-3}) & (x\alpha^{r-1}) & x_i\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-3}) & (\alpha^{r-2}\alpha^{r-1}) & \alpha_i^{r-2}\\ (\alpha^{r-1}x) & \ldots(\alpha^{r-1}\alpha^{r-3}) & (\alpha^{r-1}\alpha^{r-1}) & \alpha_i^{r-1}\end{vmatrix}\right\}$$

$$+\Delta_r\left\{\begin{vmatrix}(x'x) & (x'\alpha) & \ldots(x'\alpha^{r-2}) & x'_i\\ (\alpha x) & (\alpha\alpha) & \ldots(\alpha\alpha^{r-2}) & \alpha_i\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha^{r-1}x) & (\alpha^{r-1}\alpha) & \ldots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx) & \ldots(x\alpha^{r-2})\\ \vdots & \vdots\\ (\alpha^{r-2}x) & \ldots(\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\right.$$

$$\left.-\begin{vmatrix}(xx) & \ldots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-1}x) & \ldots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx') & (x\alpha) & \ldots(x\alpha^{r-2})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x') & (\alpha^{r-2}\alpha) & \ldots(\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\right\}$$

$$-\triangle_{r-1}\left\{\begin{vmatrix}(xx) & \ldots(xa^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (a^{r-1}x) & \ldots(a^{r-1}a^{r-2}) & a_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx') & (xa) & \ldots(xa^{r-1})\\ \vdots & \vdots & \vdots\\ (a^{r-1}x') & (a^{r-1}a) & \ldots(a^{r-1}a^{r-1})\end{vmatrix}\right.$$

$$\left.-\begin{vmatrix}(xx') & (xa) & \ldots(xa^{r-2}) & x_i\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-1}x') & (a^{r-1}a) & \ldots(a^{r-1}a^{r-2}) & a_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx) & \ldots(xa^{r-1})\\ \vdots & \vdots\\ (a^{r-1}x) & \ldots(a^{r-1}a^{r-1})\end{vmatrix}\right\}\Bigg].$$

By the Sylvester theorem, we have

$$\frac{d_r\eta_i}{dt}=\frac{k}{\triangle_{r-1}^{\frac{3}{2}}\triangle_r^{\frac{3}{2}}}\Big[(\triangle_{r-1}^2N_{r+1}-\triangle_r^2N_{r-1})$$

$$+\triangle_r\begin{vmatrix}(ax) & (aa) & \ldots(aa^{r-2})\\ (a'x) & (a'a) & \ldots(a'a^{r-2})\\ \vdots & \vdots & \vdots\\ (a^{r-1}x) & (a^{r-1}a) & \ldots(a^{r-1}a^{r-2})\end{vmatrix}\cdot\begin{vmatrix}(x'x) & (x'a) & \ldots(x'a^{r-2}) & x'_i\\ (xx) & (xa) & \ldots(xa^{r-2}) & x_i\\ (ax) & (aa) & \ldots(aa^{r-2}) & a_i\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-2}x) & (a^{r-2}a) & \ldots(a^{r-2}a^{r-2}) & a^{r-2}\end{vmatrix}$$

(7)

$$-\triangle_{r-1}\begin{vmatrix}(xx') & (xx) & (xa) & \ldots(xa^{r-2})\\ (ax') & (ax) & (aa) & \ldots(aa^{r-2})\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-1}x') & (a^{r-1}x) & (a^{r-1}a) & \ldots(a^{r-1}a^{r-2})\end{vmatrix}\cdot\begin{vmatrix}(xa) & (xa') & \ldots(xa^{r-1}) & x_i\\ (aa) & (aa') & \ldots(aa^{r-1}) & a_i\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-1}a) & (a^{r-1}a') & \ldots(a^{r-1}a^{r-1}) & a_i^{r-1}\end{vmatrix}$$

where

$$N_r=\begin{vmatrix}(xx) & (xa) & \ldots(xa^{r-2}) & x_i\\ (ax) & (aa) & \ldots(aa^{r-2}) & a_i\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-1}x) & (a^{r-1}a) & \ldots(a^{r-1}a^{r-2}) & a_i^{r-1}\end{vmatrix}$$

and

$$i=0,\ 1,\ 2,\ldots,\ n$$

$$r=2,\ 3,\ldots,\ n.$$

Hence, we have

$$\frac{d_r\eta_i}{ds}=\frac{_{r+1}\eta_i}{R_r}-\frac{_{r-1}\eta_i}{R_{r-1}}+\frac{_r\xi_i}{P_r}-\frac{_r\zeta_i}{Q_r},$$

$(r=2,\ 3,\ \ldots,\ n)\ (i=0,\ 1,\ 2,\ \ldots,\ n)$.

$$(8)\qquad \frac{d_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0},\quad \frac{d_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}+\frac{{}_1\xi_i}{P_1}-\frac{\zeta_i}{Q_1},$$

$$\frac{d_{n+1}\eta_i}{ds}=-\frac{{}_n\eta_i}{R_n}+\frac{{}_{n+1}\xi_i}{P_{n+1}}-\frac{{}_{n+1}\zeta_i}{Q_{n+1}}$$

$$(i=0,\ 1,\ 2,\ \ldots,\ n)$$

These are the generalized Serret-Frenet formula, where

$${}_r\xi_i=\frac{k\begin{vmatrix}(x'x) & (x'a) & \ldots(x'a^{r-2}) & x'_i\\ (xx) & (xa) & \ldots(xa^{r-2}) & x_i\\ (ax) & (aa) & \ldots(aa^{r-2}) & a_i\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-2}x) & (a^{r-2}a) & \ldots(a^{r-2}a^{r-2}) & a_i^{r-2}\end{vmatrix}}{\sqrt{\begin{vmatrix}(xx) & (xa) & \ldots(xa^{r-2})\\ (ax) & (aa) & \ldots(aa^{r-2})\\ \vdots & \vdots & \vdots\\ (a^{r-2}x) & (a^{r-2}a) & \ldots(a^{r-2}a^{r-2})\end{vmatrix}\begin{vmatrix}(x'x') & (x'x) & (x'a) & \ldots(x'a^{r-2})\\ (xx') & (xx) & (xa) & \ldots(xa^{r-2})\\ (ax') & (ax) & (aa) & \ldots(aa^{r-2})\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-2}x') & (a^{r-2}x) & (a^{r-2}a) & \ldots(a^{r-2}a^{r-2})\end{vmatrix}}},$$

$$(9)\qquad {}_r\zeta_i=\frac{k\begin{vmatrix}(xa) & (xa') & \ldots(xa^{r-1}) & x_i\\ (aa) & (aa') & \ldots(aa^{r-1}) & a_i\\ \vdots & \vdots & \vdots & \vdots\\ (a^{r-1}a) & (a^{r-1}a') & \ldots(a^{r-1}a^{r-1}) & a_i^{r-1}\end{vmatrix}}{\sqrt{\begin{vmatrix}(aa) & \ldots(aa^{r-1})\\ \vdots & \vdots\\ (a^{r-1}a) & \ldots(a^{r-1}a^{r-1})\end{vmatrix}\begin{vmatrix}(xx) & (xa) & \ldots(xa^{r-1})\\ (ax) & (aa) & \ldots(aa^{r-1})\\ \vdots & \vdots & \vdots\\ (a^{r-1}x) & (a^{r-1}a) & \ldots(a^{r-1}a^{r-1})\end{vmatrix}}},$$

$$\frac{1}{P_r}=\frac{k}{\Delta}\,\frac{\sqrt{\begin{vmatrix}(x'x') & (x'x) & (x'a) & \ldots(x'a^{r-2})\\ (xx') & (xx) & (xa) & \ldots(xa^{r-2})\\ (ax') & (ax) & (aa) & \ldots(aa^{r-2})\\ \vdots & \vdots & \vdots & \\ (a^{r-2}x') & (a^{r-2}x) & (a^{r-2}a) & \ldots(a^{r-2}a^{r-2})\end{vmatrix}}}{\sqrt{\Delta_r}}\,\frac{\begin{vmatrix}(ax) & (aa) & \ldots(aa^{r-2})\\ (a'x) & (a'a) & \ldots(a'a^{r-2})\\ \vdots & \vdots & \vdots\\ (a^{r-1}x) & (a^{r-1}a) & \ldots(a^{r-1}a^{r-2})\end{vmatrix}}{\Delta_{r-1}}$$

$$\frac{1}{Q_r}=\frac{k}{\Delta}\frac{\sqrt{\begin{vmatrix}(aa) & \ldots(aa^{r-1}) \\ \vdots & \vdots \\ (aa^{r-1}) & \ldots(a^{r-1}a^{r-1})\end{vmatrix}}}{\sqrt{\Delta_{r-1}}}\frac{\begin{vmatrix}(xx') & (xx) & (xa) & \ldots(xa^{r-2}) \\ (ax') & (ax) & (aa) & \ldots(aa^{r-2}) \\ \vdots & \vdots & \vdots & \vdots \\ (a^{r-1}x') & (a^{r-1}x) & (a^{r-1}a) & \ldots(a^{r-1}a^{r-2})\end{vmatrix}}{\Delta_r}.$$

It is easily seen that $({}_r\xi)$ will be the coordinates of a point ${}_rX$ which is on the plane manifold passing through $r+1$ points $(P_0, P_1, P_0', P_1' \ldots, P'_{r-2})$ and orthogonal to r points $(P_0, P_0', P_1' \ldots, P_{r-2}')$. And $({}_r\zeta)$ be that of a point ${}_rZ$ which is on the plane manifold passing through $r+1$ points $(P_0, P_0', P_1', \ldots, P_{r-1}')$ and orthogonal to r points $(P_0', P_1', \ldots, P'_{r-1})$.

Next, let Δr be the difference of the distance between the points $P_0[x(t)]$, ${}_rQ'[{}_r\eta(t+\Delta t)]$ and $P_0[x(t)]$, ${}_rQ[{}_r\eta(t)]$, then

$$\frac{1}{Q_r}=\frac{1}{\cos\frac{\overline{P_0\,{}_rZ}}{k}}\left[\lim_{\Delta t\to 0}\frac{\Delta r}{\text{arc}\,P_0P_1}\right],$$

$$\frac{1}{P_r}=\frac{\cos\frac{\overline{{}_rQ_rZ}}{k}}{\cos\frac{\overline{{}_rQ_rX}}{k}}\frac{1}{Q_r}.$$

We shall call $\frac{1}{P_r}\left(\frac{1}{Q_r}\right)$ the r^{th} *left* (*right*) *curvature of the curve* Γ at the point $P_0(x)$ with respect to the curve Γ'.

4. Specially, if Γ' be a curve traced by a point on the tangent to the curve Γ at the point P_0, then we have

$$a_i=x_i\sin\frac{r}{k}+\rho k x_i'\cos\frac{r}{k},$$

where

$$\rho=\frac{dt}{ds},$$

$$(i=0,\ 1,\ 2,\ 3, \ldots,\ n).$$

And we have

$$(10) \qquad {}_r\eta_i = \frac{k \begin{vmatrix} (xx) & (xx') & \dots & (xx^{r-1}) & x_i \\ (x'x) & (x'x') & \dots & (x'x^{r-1}) & x'_i \\ \vdots & \vdots & & \vdots & \vdots \\ (x^r x) & (x^r x') & \dots & (x^r x^{r-1}) & x^r_i \end{vmatrix}}{\sqrt{\begin{vmatrix} (xx) & \dots & (xx^{r-1}) \\ \vdots & & \vdots \\ (x^{r-1}x) & \dots & (x^{r-1}x^{r-1}) \end{vmatrix} \begin{vmatrix} (xx) & \dots & (xx^r) \\ \vdots & & \vdots \\ (x^r x) & \dots & (x^r x^r) \end{vmatrix}}},$$

$$(11) \qquad \frac{1}{R_r^2} = k^2 \frac{\triangle_{r-1}\triangle_{r+1}}{\triangle_r^2 \triangle_1},$$

where

$$\triangle_i = \begin{vmatrix} (xx) & \dots & (xx^i) \\ \vdots & & \vdots \\ (x^i x) & \dots & (x^i x^i) \end{vmatrix}.$$

And the Serret-Frenet formula will be

$$\frac{d\,{}_r\eta_i}{ds} = \frac{{}_{r+1}\eta_i}{R_r} - \frac{{}_{r-1}\eta_i}{R_{r-1}}, \quad (r = 1, 2, 3, \dots, n-1)$$

$$(12) \qquad \frac{d\,{}_0\eta_i}{ds} = \frac{{}_1\eta_i}{k}, \quad \frac{d\,{}_n\eta_i}{ds} = -\frac{{}_{n-1}\eta_i}{R_{n-1}},$$

$$(i = 0, 1, 2, \dots, n).$$

There are the formula given by Prof. Nishiuchi.[1]

In Euclidean Space.

5. If we put

$$x_0 = k\dot{x}_0, \; x_1 = \dot{x}_1, \; x_2 = \dot{x}_2, \; \dots, \; x_n = \dot{x}_n,$$

$$y_0 = k\dot{y}_0, \; y_1 = \dot{y}_1, \; y_2 = \dot{y}_2, \; \dots, \; y_n = \dot{y}_n$$

and

$$\frac{\dot{x}_i}{\dot{x}_0} = \mathfrak{x}_i,$$

$$\frac{\dot{y}_i}{\dot{y}_0} = \mathfrak{y}_i, \quad (i = 1, 2, \dots, n)$$

1 loc. cit.

and take the limiting value when $\frac{1}{k^2}$ tends to zero, we have

$$\frac{1}{R_r^2}=\frac{D_{r-1}D_{r+1}}{\triangle D_r^2}, \tag{13}$$

where

$$D_i=\left\|\begin{matrix} y_1-x_1 & y_2-x_2 & \cdots\cdots & y_n-x_n \\ \bar{a}_1' & \bar{a}_2' & \cdots\cdots & \bar{a}_n' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \bar{a}_1^{i-1} & \bar{a}_2^{i-1} & & \bar{a}_n^{i-1} \end{matrix}\right\|^2$$

$$\bar{a}_i=x_i+r(y_i-x_i)$$

$$\triangle=(x'x').$$

6. If we consider the case when r tends to infinity in the formula above, then

$$\frac{1}{R_r^2}=\frac{D_{r-1}D_{r+1}}{\triangle D_r^2} \tag{14}$$

where

$$D_i=\left\|\begin{matrix} a_1 & a_2 & \ldots a_n \\ a_1' & a_2' & \ldots a_n' \\ \cdots & \cdots & \cdots \\ a_1^{i-1} & a_2^{i-1} & \ldots a_n^{i-1} \end{matrix}\right\|^2 \tag{14$_1$}$$

$$a_i=y_i-x_i.$$

And if we take

$$\lim_{k\to\infty}\frac{k_r\eta_i}{_r\eta_0}={_r}a_i,$$

$$({_r}a_r a)=1,$$

then we have

$$(15)\qquad {}_r\alpha_i = \frac{\begin{vmatrix} (\alpha\alpha) & (\alpha\alpha') & \ldots(\alpha\alpha^{r-2}) & \alpha_i \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^{r-1}\alpha) & (\alpha^{r-1}\alpha') & \ldots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1} \end{vmatrix}}{\sqrt{\begin{vmatrix} (\alpha\alpha) & \ldots(\alpha\alpha^{r-2}) \\ \vdots & \vdots \\ (\alpha^{r-2}\alpha) & \ldots(\alpha^{r-2}\alpha^{r-2}) \end{vmatrix} \cdot \begin{vmatrix} (\alpha\alpha) & \ldots(\alpha\alpha^{r-1}) \\ \vdots & \vdots \\ (\alpha^{r-1}\alpha) & \ldots(\alpha^{r-1}\alpha^{r-1}) \end{vmatrix}}}$$

$$(r = 1,\ 2,\ \ldots,\ n+1).$$

$${}_1\alpha_i = \alpha_i,$$

$$({}_r\alpha_p\alpha) = 0, \quad (\alpha \neq p)$$

$$({}_r\alpha_p\alpha) = 1 \quad (\alpha = p).$$

Now, instead of the curve Γ', we consider a straight line g through the point P_0 on the curve Γ whose direction cosines are $({}_1\alpha)$. If we take n consecutive lines g', g'', ... , g^n through the points P_1, P_2, ... , P_n respectively and let the lines p_1, p_2, ... , p_n be parallel to the lines g', g'' ...... , g^n through the point P_0, then the equation (15) gives the direction cosines of the line $({}_r\alpha)$ which is in the plane manifold possing through the lines g, p_1, p_2, ... , p_{r-1} and orthogonal to the one passing through the lines g, p_1, p_2, ... , p_{r-2}.

And $\frac{1}{R_r}$ which is given by the equation (14) shall be called the r^{th} *curvature of the curve* Γ at the point P_0 with respect to the line g.

Let ${}_r\theta$ be the angle between the line $({}_r\alpha)$ and the line which is in the plane manifold passing g, p_1, p_2, ... , p_{r-2}, p_r and orthogonal to the one passing through g, p_1, p_2, ... , p_{r-2}, then it is evident that

$$\lim_{\Delta t \to 0} \frac{{}_r\theta}{\text{arc}\, P_0P_1} = \frac{1}{R_r}.$$

Differentiating ${}_r\alpha_i$ with respect to t, we have.

$$(16)\qquad \frac{d\,{}_r\alpha_i}{dt} = \frac{1}{D_{r-1}^{\frac{3}{2}}\, D_{r+1}^{\frac{3}{2}}} \{D_{r-1}^2 N_{r+1} - D_r^2 N_{r-1}\}$$

where

$$N_r = \begin{vmatrix} (\alpha\alpha) & \ldots(\alpha\alpha^{r-2}) & \alpha_i \\ \vdots & \vdots & \vdots \\ (\alpha^{r-1}\alpha) & \ldots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{i-1} \end{vmatrix}.$$

Hence, we have

$$\frac{d_r a_i}{ds} = \frac{{}_{r+1}a_i}{R_r} - \frac{{}_{r-1}a_i}{R_{r-1}}, \quad (r=2,\ 3,\ \ldots\ldots,\ n).$$

(17)
$$\frac{d_1 a_i}{ds} = \frac{{}_2 a_i}{R_1},$$

$$\frac{d_{n+1} a_i}{ds} = -\frac{{}_n a_i}{R_n}$$

$$(i=1,\ 2,\ \ldots,\ n).$$

These are generalized Brunel[1] and Landsbergs[1] formula.

7. From the equations (17), we have

$$\left(\frac{d_r a}{ds}\ \frac{d_r a}{ds}\right) = \frac{1}{R_r^2} + \frac{1}{R_{r-1}^2} \equiv \frac{1}{r_r^2} (\equiv K_r^2),$$

$$\left(\frac{d_1 a}{ds}\ \frac{d_1 a}{ds}\right) = \frac{1}{R_1^2} \equiv \frac{1}{r_1^2} (\equiv K_1^2),$$

$$\left(\frac{d_{n+1} a}{ds}\ \frac{d_{n+1} a}{ds}\right) = \frac{1}{R_n^2} \equiv \frac{1}{r_{n+1}^2} (\equiv K_{n+1}^2).$$

So we get

$$\frac{1}{R_2^2} = \frac{1}{r_2^2} - \frac{1}{r_1^2},$$

$$\frac{1}{R_3^2} = \frac{1}{r_3^2} - \frac{1}{R_2^2} = \frac{1}{r_3^2} - \frac{1}{r_2^2} + \frac{1}{r_1^2}$$

(18) ....

$$\frac{1}{R_i^2} = \frac{1}{r_i^2} - \frac{1}{R_{i-1}^2} = \frac{1}{r_i^2} - \frac{1}{r_{i-1}^2} + \frac{1}{r_{i-2}^2} \pm \ldots + (-1)^{i-1} \frac{1}{r_1^2},$$

....

$$\frac{1}{R_n^2} = \frac{1}{r_n^2} - \frac{1}{R_{n-1}^2} = \frac{1}{r_n^2} - \frac{1}{r_{n-1}^2} + \ldots\ldots + (-1)^{n-1} \frac{1}{r_1^2}$$

$$= \frac{1}{r_{n+1}^2}.$$

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1 Math. Ann. **19**.

2 Crelle J. **114**.

Putting

$$\frac{1}{\rho_i}=+\sqrt{\frac{1}{r_i^2}-\frac{1}{r_{i-1}^2}+\ldots\ldots+(-1)^{i-1}\frac{1}{r_1^2}}=\frac{1}{R_i},$$

$$(i=2,\ 3,\ \ldots\ldots,\ n-1)$$

$$\frac{1}{\rho_n}=\frac{1}{r_{n+1}}=-\sqrt{\frac{1}{r_n^2}-\frac{1}{r_{n-1}^2}+\ldots\ldots+(-1)^{n-1}\frac{1}{r_1^2}}$$

We have

$$\frac{da_i}{ds}=\frac{{}_2a_i}{r_1},$$

$$\frac{d_2a_i}{ds}=\frac{{}_3a_i}{\rho_2}-\frac{{}_1a_i}{r_1}$$

(19) $$\frac{d_3n_i}{ds}=\frac{{}_4a_i}{\rho_3}-\frac{{}_2a_i}{\rho_2}$$

....

$$\frac{d_na_i}{ds}=\frac{{}_{n+1}a_i}{\rho_n}-\frac{{}_{n-1}a_i}{\rho_{n-1}}$$

$$\frac{d_{n+1}a_i}{ds}=-\frac{{}_na_i}{\rho_n}$$

The formulas (18) and (19) are those which were given by Meyer[1] who defines K_r as the r^{th} curvature of the curve Γ with respect to the line g.

3 Dimensional Space.

8. Let $(\mathfrak{X})$ be a point orthogonal to (x) and on the tangent, $(\mathfrak{Y})$ orthogonal to (x) on the principal normal, $(\mathfrak{Z})$ orthogonal to (x) on the binormal, then

$$(x\mathfrak{X})=(x\mathfrak{Y})=(x\mathfrak{Z})=(\mathfrak{X}\mathfrak{Y})=(\mathfrak{X}\mathfrak{Z})=(\mathfrak{Y}\mathfrak{Z})=0.$$

If we, hereafter, take as our parameter on the given curves the length of arc, then

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1 loc. cit.

$$x'_i=\frac{dx_i}{ds}, \quad (x'x')=1, \quad (i=0, 1, 2, 3)$$

and

$$\mathfrak{X}_i=kx'_i,$$

$$\mathfrak{Y}_i=\frac{\rho}{k}(x_i+kx''_i),$$

$$\mathfrak{Z}_i=\rho\frac{\partial}{\partial t_i}\begin{vmatrix} t\, x\, x'\, x'' \end{vmatrix},$$

$$(\mathfrak{X}\mathfrak{X})=k^2, \ (\mathfrak{Y}\mathfrak{Y})=k^2, \ (\mathfrak{Z}\mathfrak{Z})=k^2$$

and

$$\frac{d\mathfrak{X}_i}{ds}=\frac{\mathfrak{Y}_i}{\rho}-\frac{x_i}{k},$$

$$\frac{d\mathfrak{Y}_i}{ds}=-\frac{\mathfrak{X}_i}{\rho}-\frac{\mathfrak{Z}_i}{\tau},$$

$$\frac{d\mathfrak{Z}_i}{ds}=\frac{\mathfrak{Y}_i}{\tau}, \qquad (i=0, 1, 2, 3)$$

where ρ and τ are the curvature and torsion respectively.

9. If the curve Γ' be the evolute of the curve Γ or the curve traced by a point on the tangent of the curve P, then

$${}_0\eta_i=x_i,$$

$${}_1\eta_i=\mathfrak{X}_i,$$

$${}_2\eta_i=\mathfrak{Y}_i,$$

$${}_3\eta_i=-\mathfrak{Z}_i,$$

$$R_0=k, \ R_1=\rho, \ R_2=\tau, \ \frac{1}{P_r}=0, \ \frac{1}{Q_r}=0.$$

and we have

$$\frac{d\,{}_0\eta_i}{ds}=\frac{{}_1\eta_i}{R_0}, \quad \frac{d\,{}_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}-\frac{{}_0\eta_i}{R_0}$$

$$\frac{d\,{}_2\eta_i}{ds}=\frac{{}_3\eta_i}{R_2}-\frac{{}_1\eta_i}{R_1}, \quad \frac{d\,{}_3\eta_i}{ds}=-\frac{{}_2\eta_i}{R_2} \quad (i=0, 1, 2, 3)$$

which are the formulas given by Bianchi.[1]

10. Suppose Γ' be a curve traced by the point $(\mathcal{Y})$ which is orthogonal to (x) and on the principal normal. We get

$$a_i=\mathcal{Y}_i,$$

$$ {}_0\eta_i=x_i,$$

$$ {}_1\eta_i=\mathcal{Y}_i,$$

$$ {}_2\eta_i=k\frac{\mathcal{Y}'_i}{\sqrt{(\mathcal{Y}'\mathcal{Y}')}}=\frac{-\frac{\mathcal{X}_i}{\rho}-\frac{\mathcal{Z}_i}{\tau}}{\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}},$$

$$ {}_3\eta_i=\frac{\frac{\mathcal{X}_i}{\tau}-\frac{\mathcal{Z}_i}{\rho}}{\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}},$$

$$(i=0,\ 1,\ 2,\ 3).$$

and

$$\frac{1}{R_1}=\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}},$$

$$\frac{1}{R_2}=\frac{\tau\frac{d\rho}{ds}-\rho\frac{d\tau}{ds}}{\rho^2+\tau^2}=\frac{\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\tau}\right)-\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)}{\frac{1}{\rho^2}+\frac{1}{\tau^2}}.$$

$$\frac{1}{P_0}=\frac{1}{k},\quad \frac{1}{P_1}=0,\quad \frac{1}{P_2}=0,$$

$$\frac{1}{P_3}=\frac{1}{k^2\tau\rho}\,\frac{1}{\left(\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\tau}\right)-\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right)},$$

$$\frac{1}{Q_0}=0,\quad \frac{1}{Q_1}=0,\quad \frac{1}{Q_2}=-\frac{1}{k\rho\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}}$$

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[1] Lezioni di Geometria Differenziale, p. 457. A. Razzaboni, Le formale del Frenet in geometria iperbolica e laro applicazioni. (Bologna, Gamberini 1897). Also see Nischiuchi, loc. cit.

$$\frac{1}{Q_3}=-\frac{\sqrt{\dfrac{1}{\rho^2k^2}\left(\dfrac{1}{\rho^2}+\dfrac{1}{\tau^2}\right)+E^2}}{k\tau E\sqrt{\dfrac{1}{\rho^2}+\dfrac{1}{\tau^2}}},$$

$${}_0\xi_i=\mathscr{X}_i,\quad {}_3\xi_i=-\frac{\dfrac{\mathscr{X}_i}{\tau}-\dfrac{\mathscr{Y}_i}{\rho}}{\sqrt{\dfrac{1}{\rho^2}+\dfrac{1}{\tau^2}}}=-{}_3\eta_i,$$

$${}_2\zeta_i=x_i,\quad {}_3\zeta_i=\frac{1}{k}\,\frac{k\,E\,x_i+\dfrac{1}{\rho}\left(\dfrac{\mathscr{Y}_i}{\rho}-\dfrac{\mathscr{X}_4}{\tau}\right)}{\sqrt{\dfrac{1}{k^2\rho^2}\left(\dfrac{1}{\rho^2}+\dfrac{1}{\tau^2}\right)+E^2}}$$

$$(i=0,1,2,3)$$

where

$$E=\frac{1}{\rho}\,\frac{d}{ds}\left(\frac{1}{\tau}\right)-\frac{1}{\tau}\,\frac{d}{ds}\left(\frac{1}{\rho}\right).$$

Hence, we have

$$\frac{1}{R_1^2}\left(\frac{1}{Q_3^2}-\frac{1}{k^2P_3^2}\right)=\frac{1}{k^2\tau^2},$$

$$\frac{1}{P_3R_2}=\frac{1}{\rho\tau}\cdot\frac{1}{R_1^2},$$

$$\frac{1}{R_1Q_2}+\frac{1}{k\rho}=0.$$

and

$$\frac{d\,{}_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0},\quad \frac{d\,{}_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1},$$

$$\frac{d\,{}_2\eta_i}{ds}=\frac{{}_3\eta_i}{R_2}-\frac{{}_1\eta_i}{R_1}-\frac{{}_2\zeta_i}{Q_2},$$

$$\frac{d\,{}_3\eta_i}{ds}=-\frac{{}_2\eta_i}{R_2}+\frac{{}_3\xi_2}{P_3}-\frac{{}_3\zeta_i}{Q_3}.$$

$$(i=0,1,2,3)$$

11. If Γ and Γ' have the same principal normal, then the equations of the curve Γ' will be

$$\alpha_i = x_i \sin\frac{r}{k} + \mathfrak{Y}_i \cos\frac{r}{k},$$

$$(i=0, 1, 2, 3),$$

and the coordinates of a points on the principal normal to Γ' and orthogonal to (α) will be

$$\overline{\mathfrak{Y}}_i = x_i \cos\frac{r}{k} - \mathfrak{Y}_i \sin\frac{r}{k},$$

$$(i=0, 1, 2, 3).$$

And we have

$$(\overline{\mathfrak{Y}}\, d\alpha)=0$$

or

$$k^2\, dr=0$$

i. e.

$$r=\text{const.},$$

and we can find such quantities λ, μ, ν that

$$\overline{\mathfrak{Y}}_i = \lambda\, \alpha_i + \mu\, \alpha'_i + \nu\, \alpha''_i .$$

Multiplying through α'_i, $\mathfrak{z}_i$ and adding, we get

$$\mu(\alpha'\, \alpha') + \nu(\alpha''\, \alpha')=0,$$

$$\mu(\alpha'\, \mathfrak{z}) + \nu(\alpha''\, \mathfrak{z})=0.$$

Eliminating μ, ν,

$$\begin{vmatrix} (\alpha'\, \alpha') & (\alpha''\, \alpha') \\ (\alpha'\, \mathfrak{z}) & (\alpha''\, \mathfrak{z}) \end{vmatrix} = 0$$

or

$$k\left\{\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\tau}\right) - \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\} = \tan\frac{r}{k}\frac{d}{ds}\left(\frac{1}{\tau}\right).$$

Hence, we have

$$\frac{k}{\tau}+\frac{Ck}{\rho}=C\tan\frac{r}{k} \quad (C\text{: integration const.})$$

This is the corresponding relation given by Bertrand in the Euclidean case.

And

$$\frac{1}{R_1}=\frac{\sqrt{(a'\,a')}}{k\cos\frac{r}{k}}=\frac{1}{\tau}\sqrt{\frac{1}{C^2}+1}\,, \quad \frac{1}{R_2}=0$$

$$\frac{1}{P_0}=\frac{1}{k}\,, \quad \frac{1}{P_1}=\frac{1}{k}\tan\frac{r}{k}\,, \quad \frac{1}{P_2}=0$$

$$\frac{1}{Q_0}=0, \quad \frac{1}{Q_1}=0,$$

$$\frac{1}{Q_2}=\frac{\sin\frac{r}{k}-\frac{k}{\rho}\cos\frac{r}{k}}{k\cos\frac{r}{k}\sqrt{(a'\,a')}}=\frac{1}{k\cos\frac{r}{k}\sqrt{1+C^2}}\,,$$

$${}_0\eta_i=x_i, \quad {}_1\eta_i=\mathscr{Y}_i, \quad {}_2\eta_i=\frac{k\,a'}{\sqrt{(a'\,a')}}\,,$$

$${}_0\xi_i=\mathscr{X}_i, \quad {}_1\xi_i=-\mathscr{X}_i, \quad {}_2\xi_i=\mathscr{X}_i,$$

$${}_2\zeta_i=\cos\frac{r}{k}x_i-\sin\frac{r}{k}\mathscr{Y}_i=\overline{\mathscr{Y}}_i\,.$$

Hence, we have

$$\frac{1}{R_1}\,\frac{1}{Q_2}=\frac{1}{C\,k\cos\frac{r}{k}}\,\frac{1}{\tau}$$

$$=\frac{1}{k\cos\frac{r}{k}}\left[\frac{1}{k}\tan\frac{r}{k}-\frac{1}{\rho}\right]$$

or

$$\frac{1}{R_1}\,\frac{1}{Q_2}=\frac{1}{k\cos\frac{r}{k}}\left[\frac{1}{P_1}-\frac{1}{\rho}\right],$$

and the generalized Serret-Frenet formula will be

$$\frac{d\,{}_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0},$$

$$\frac{d\,{}_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}+\frac{{}_1\xi_i}{P_1},$$

$$\frac{d\,{}_2\eta_i}{ds}=-\frac{{}_1\eta_i}{R_1}-\frac{{}_2\zeta_i}{Q_2}, \qquad (i=0, 1, 2, 3)$$

And $({}_3\eta)$ will be on the absolute.

12. If the principal normal of the curve Γ be the binormal of the curve Γ', we have

$$a_i=x_i\sin\frac{r}{k}+\mathfrak{y}_i\cos\frac{r}{k}$$

$$(i=0, 1, 2, 3)$$

and let $(\bar{\mathfrak{z}})$ be the coordinates of the points on the binormal of Γ' and orthogonal to (a), then

$$\bar{\mathfrak{z}}_i=x_i\cos\frac{r}{k}-\mathfrak{y}_i\sin\frac{r}{k},$$

and we have

$$(a'\,\bar{\mathfrak{z}}_i)=0,$$

$$(a''\,\bar{\mathfrak{z}}_i)=0.$$

Hence we get

$$r=\text{const.},$$

$$\left(\frac{k^2}{\rho^2}+\frac{k^2}{\tau^2}-1\right)\sin\frac{r}{k}\cos\frac{r}{k}=\frac{k}{\rho}\left(\sin^2\frac{r}{k}-\cos^2\frac{r}{k}\right).$$

And

$$\frac{1}{R_1}=\frac{\sqrt{(a'\,a')}}{k\cos\frac{r}{k}},$$

$$\frac{1}{R_2}=\frac{k\cos\frac{r}{k}\sin\frac{r}{k}\left[\left(\sin\frac{r}{k}-\frac{r}{\rho}\cos\frac{r}{k}\right)\frac{d}{ds}\left(\frac{1}{\tau}\right)+\frac{k}{\tau}\frac{d}{ds}\left(\frac{1}{\tau}\right)\cos\frac{r}{k}\right]}{\sin\frac{r}{k}-\frac{k}{\rho}\cos\frac{r}{k}}$$

$$\frac{1}{P_0}=\frac{1}{k}, \quad \frac{1}{P_1}=\frac{1}{k}\tan\frac{r}{k}, \quad \frac{1}{P_2}=0, \quad \frac{1}{P_3}=0.$$

$$\frac{1}{Q_0}=0, \quad \frac{1}{Q_1}=0, \quad \frac{1}{Q_2}=\frac{1}{k}\tan\frac{r}{k}\sqrt{(a'\,a')}$$

$$\frac{1}{Q_3}=-\frac{1}{\tau\sqrt{(a'\,a')}}$$

Hence, we have

$$\frac{1}{Q_2}\frac{1}{Q_3}=-\frac{\tan\frac{r}{k}}{k\tau},$$

$$k^2\cos^2\frac{r}{k}\frac{d}{ds}\left(\frac{R_1}{Q_3}\right)+\frac{Q_2^2}{R_2}=0$$

and the generalized Serret-Frenet formula will be

$$\frac{d\,{}_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0},$$

$$\frac{d\,{}_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}+\frac{{}_1\xi_i}{P_1},$$

$$\frac{d\,{}_2\eta_i}{ds}=\frac{{}_3\eta_i}{R_2}-\frac{{}_1\eta_i}{R_1}-\frac{{}_2\zeta_i}{Q_2}.$$

$$\frac{d\,{}_3\eta_i}{ds}=-\frac{{}_2\eta_i}{R_2}-\frac{{}_3\zeta_i}{Q_3}.$$

$$(i=0,\ 1,\ 2,\ 3)$$

13. Suppose Γ' be a curve traced by the points $(\mathfrak{z})$ which is orthogonal to (x) and on the binormal, then

$$a_i=\mathfrak{z}_i$$

$$\frac{1}{R_1}=\frac{1}{\tau}, \quad \frac{1}{R_2}=\frac{1}{\rho},$$

$$\frac{1}{P_0}=\frac{1}{k}, \quad \frac{1}{P_1}=0, \quad \frac{1}{P_2}=0, \quad \frac{1}{P_3}=0.$$

$$\frac{1}{Q_1}=0, \quad \frac{1}{Q_2}=0, \quad \frac{1}{Q_3}=\frac{1}{k}.$$

$$_0\eta_i = x_i, \qquad _1\eta_i = \mathfrak{z}_i, \qquad _2\eta_i = \mathscr{Y}_i$$

$$_3\eta_i = -\mathscr{X}_i. \quad _3\zeta_i = -x_i \qquad (i = 0, 1, 2, 3)$$

And the generalized Serret-Frenet formula will be

$$\frac{d\,_0\eta_i}{ds} = \frac{_0\xi_i}{P_0}$$

$$\frac{d\,_1\eta_i}{ds} = \frac{_2\eta_i}{R_1}, \qquad \frac{d\,_2\eta_i}{ds} = \frac{_3\eta_i}{R_2} - \frac{_1\eta_i}{R_1}$$

$$\frac{d\,_3\eta_i}{ds} = -\frac{_2\eta_i}{R_2} - \frac{_3\zeta_i}{Q_3}$$

$$(i = 0, 1, 2, 3).$$

14. If the binormal of the curve Γ be the principal normal to the curve Γ' and let $(\overline{\mathscr{Y}})$ be the coordinates of the point on the binormal of the curve Γ' and orthogonal to (a), then

$$a_i = x_i \sin\frac{r}{k} + \mathfrak{z}_i \cos\frac{r}{k},$$

$$\overline{\mathscr{Y}}_i = x_i \cos\frac{r}{k} - \mathfrak{z}_i \sin\frac{r}{k}$$

$$(i = 0, 1, 2, 3)$$

and

$$(\overline{\mathscr{Y}}\, d\, a) = 0$$

we get

$$r = \text{const.}$$

More over we can find such quantities λ, μ, ν that

$$\overline{\mathscr{Y}}_i = \lambda\, a_i + \mu\, a_i' + \nu\, a_i''$$

Multiplying through a_i', $\mathscr{Y}_i$ and adding, we have

$$\mu(a'\, a') + \nu(a''\, a') = 0,$$

$$\mu(a'\, \mathscr{Y}) + \nu(a''\, \mathscr{Y}) = 0$$

or

$$\begin{vmatrix} (\alpha'\,'\alpha) & (\alpha'\,\alpha'') \\ (\alpha'\,\mathfrak{Y}) & (\alpha''\,\mathfrak{Y}) \end{vmatrix} = 0$$

i. e.

$$\frac{1}{\rho}\tan^2\frac{r}{k} + k\frac{d}{ds}\left(\frac{1}{\tau}\right)\tan\frac{r}{k} + \frac{k^2}{\rho\,\tau^2} = 0.$$

and

$$\frac{1}{R_1} = \frac{\sqrt{(\alpha'\,\alpha')}}{k\cos\frac{r}{k}} = \sqrt{\frac{1}{k^2}\tan^2\frac{r}{k} + \frac{1}{\tau^2}}, \quad \frac{1}{R_2} = 0$$

$$\frac{1}{P_0} = \frac{1}{k}, \quad \frac{1}{P_1} = \frac{1}{k}\tan\frac{r}{k}, \quad \frac{1}{P_2} = 0$$

$$\frac{1}{Q_0} = 0, \quad \frac{1}{Q_1} = 0, \quad \frac{1}{Q_2} = \frac{\tan\frac{r}{k}}{k\sqrt{(\alpha'\,\alpha')}}.$$

So we have

$$\frac{1}{R_1}\,\frac{1}{Q_2} = \text{const.} \quad \frac{1}{P_1} = \text{const.}$$

and

$$\frac{d\,{}_0\eta_i}{ds} = \frac{{}_0\xi_i}{P_0},$$

$$\frac{d\,{}_1\eta_i}{ds} = \frac{{}_2\eta_i}{R_1} + \frac{{}_1\xi_i}{P_1},$$

$$\frac{d\,{}_2\eta_i}{ds} = -\frac{{}_1\eta_i}{R_1} - \frac{{}_2\zeta_i}{Q_2}$$

where

$${}_0\eta_i = x_i, \quad {}_1\eta_i = \mathfrak{z}_i, \quad {}_2\eta_i = \frac{k\alpha'_i}{\sqrt{(\alpha'\,\alpha')}}$$

$${}_0\xi_i = \mathfrak{X}_i, \quad {}_1\xi_i = -\mathfrak{X}_i$$

$${}_2\zeta_i = \overline{\mathfrak{Y}}_i, \qquad (i = 0, 1, 2, 3).$$

And $({}_8\eta)$ will be on the absolute.

15. If the binormal of the curve Γ' be that of the curve Γ, and let $(\overline{\xi})$ be the coordinates of a points on the binormal to Γ' and orthogonal to (α), then

$$\alpha_i = x_i \sin\frac{r}{k} + \xi_i \cos\frac{r}{k},$$

$$\overline{\xi}_i = x_i \cos\frac{r}{k} - \xi_i \sin\frac{r}{k}$$

$$= \frac{\partial}{\partial t_i} \left| t\ \alpha\ \alpha'\ \alpha'' \right|$$

$$(i=0,\ 1,\ 2,\ 3),$$

and

$$(\alpha' \overline{\xi}) = (\alpha'' \overline{\xi}) = 0.$$

Hence, we have

$$r = \text{const.}$$

$$\frac{1}{\tau^2} = \frac{1}{k^2}.$$

In this case, it is evident that the binormals will generate a ruled surface with vanisching Gaussian curvature.[1]

And

$$\frac{1}{R_1} = \frac{1}{k \cos\frac{r}{k}}, \quad \frac{1}{R_2} = \frac{1}{\rho}.$$

$$\frac{1}{P_0} = \frac{1}{k}, \quad \frac{1}{P_1} = \frac{1}{k}\tan\frac{r}{k}, \quad \frac{1}{P_2} = 0, \quad \frac{1}{P_3} = 0.$$

$$\frac{1}{Q_0} = 0, \quad \frac{1}{Q_1} = 0, \quad \frac{1}{Q_2} = \frac{1}{k}\tan\frac{r}{k}, \quad \frac{1}{Q} = \frac{1}{k}$$

$${}_3\eta_i = \mathcal{Y}_i \sin\frac{r}{k} - \mathcal{X}_i \cos\frac{r}{k},$$

$${}_2\eta_i = \mathcal{X}_i \sin\frac{r}{k} + \mathcal{Y}_i \cos\frac{r}{k},$$

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[1] Coolidge. Non-Euclidean Geometry.

$$ {}_1\eta_i = \mathfrak{y}_i , $$

$$ {}_1\xi_1 = -\mathfrak{X}_i , $$

$$ {}_3\xi_i = \mathfrak{y}_i \sin\frac{r}{k} - x_i \cos\frac{r}{k} $$

$$ {}_2\zeta_i = x_i \cos\frac{r}{k} - \mathfrak{y}_i \sin\frac{r}{k} $$

$$ (i=0,\ 1,\ 2,\ 3) $$

So the generalized formula will be

$$ \frac{d\,{}_0\eta_i}{ds} = \frac{{}_0\xi_i}{P_0} , $$

$$ \frac{d\,{}_2\eta_i}{ds} = \frac{{}_2\eta_i}{R_1} + \frac{{}_1\xi_i}{P_1} $$

$$ \frac{d\,{}_2\eta_i}{ds} = \frac{{}_3\zeta_i}{R_2} - \frac{{}_1\eta_i}{R_1} - \frac{{}_2\zeta_i}{Q_2} $$

$$ \frac{d\,{}_3\eta_i}{ds} = -\frac{{}_2\eta_i}{R_2} - \frac{{}_3\zeta_i}{Q_3} $$

$$ (i=0,\ 1,\ 2,\ 3) $$

In conclusion, the author wishes to express his sincere thanks to Prof. Nischiuchi for his kind guidance and valuable remarks and also to Asst. Prof. T. Matsumoto for his kind remarks.