数理解析研究所講究録 1380 巻 2004 年 98-105

Testing hypothesis for quantum systems

津田美幸 [1], 林正人 [1], 松本啓史 [1,2]

[1] 独立行政法人科学技術振興機構今井量子計算機構プロジェクト
[2] 国立情報学研究所

1 Introduction

A quantum information protocol usually makes use of maximally entangled states as resources ([6]). In many cases, the protocol is based on a device which repeatedly produce entangled resources with a constant level of quality. Hence it is important to test the performance of the device. Since the testing is based on a physical measurement, we should carefully analyze the noise and error. For such a testing problem, hypothesis testing theory of mathematical statistics serves an orthodox framework. In quantum theory of hypothesis testing, a lower bound of error is given by the quantum Neyman-Pearson theorem for non-asymptotic cases ([3], [5]), and by the quantum Stein's lemma for asymptotic cases ([4], [7], [2]). In these general arguments, an optimal test require measurements which can not be realized by Local Operations and Classical Communications (LOCC). On this point, these arguments are not enough as testing theory for quantum information. On the other hand, the LOCC requirement has been considered in a context of entanglement witness though their framework is not so orthodox as that of hypothesis testing.

In order to argue this requirement in hypothesis testing, it is necessary to characterize the set of LOCC tests. Recently, Virmani and Plenio [8] pointed out that some invariance restrictions measurements make the characterization more convenient. They showed some examples in a case where the number of samples is one. Such a method had been considered by Acin, Tarrach and Vidal [1] for statistical estimation of quantum states.

In this article, we consider a case where the number n of samples is large, and a case where n is just two. In the former case, we derive optimal tests

(i) with a LOCC condition between two parties,
(ii) without a LOCC condition between two parties.

In the latter case, we derive optimal tests without a LOCC condition between two parties and

(iii) with a LOCC condition between two samples,
(iv) without a LOCC condition between two samples.

As is naturally expected, the performance of the best test for (ii) is worse than that of (i). However, if the state is close to the maximally entangled state, the difference of performance asymptotically goes to zero with respect to the exponent of error. Similarly, the best test T_3 for (iii) is better than the best one T_4 for (iv). Indeed, T_3 require non-local measurement between two samples. The condition of (iv) is most restrictive so that it is the nearest to the real situation among (i)-(iv) though the mathematical story for the optimization is not so simple as those of (i)-(iii).

In Section 2, we introduce a general formulation of hypothesis testing. In Section 3, we consider (i), (ii), (iii) and (iv) in order.

2 A general setting

Let $\mathcal{H}$ be a finite-dimensional Hilbert space which describes a physical system of interest. For such $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ be the set of linear operators (matrices) on $\mathcal{H}$ and let $\mathcal{S}(\mathcal{H})(\subset \mathcal{L}(\mathcal{H}))$ be the set of density matrices on $\mathcal{H}$. Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be mutually disjoint and non-empty subsets of $\mathcal{S}(\mathcal{H})$.

Suppose that the current state $\rho(\in \mathcal{S}(\mathcal{H}))$ of the system is unknown. Suppose also that $\rho \in \mathcal{S}_0$ or $\rho \in \mathcal{S}_1$. In such a case, we would like to test

$$H_0 : \rho \in \mathcal{S}_0 \text{ versus } H_1 : \rho \in \mathcal{S}_1 \tag{1}$$

based on an appropriate measurement on $\mathcal{H}$. H_0 is called *a null hypothesis*, and H_1 is called *an alternative hypothesis*.

A test for the hypothesis (1) is given by a Positive Operator Valued Measure (POVM) on $\mathcal{H}$. Let $\mathcal{T}(\mathcal{H})$ be the set of POVM's T of the form

$$T : \{0,1\} \to \mathcal{L}(\mathcal{H}) \ (i \mapsto T_i)$$

where $T_0 + T_1 = I_{\mathcal{H}}$ (identity on $\mathcal{H}$). We often write $T(\in \mathcal{T}(\mathcal{H}))$ as $T = \{T_0, T_1\}$ to specify the form of T. Any test can be described by $T \in \mathcal{T}(\mathcal{H})$, that is, H_0 is accepted (=H_1 is rejected) if 0 is observed and H_1 is accepted (=H_0 is rejected) if 1 is observed.

A type 1 error is an event such that H_1 is accepted though H_0 is true. A type 2 error is an event such that H_0 is accepted though H_1 is true. Hence the type 1 error probability $\alpha(T,\rho)$ and the type 2 error probability $\beta(T,\rho)$ are given by

$$\alpha(T,\rho) = \mathrm{Tr}(\rho T_1) \ (\rho \in \mathcal{S}_0), \ \beta(T,\rho) = \mathrm{Tr}(\rho T_0) \ (\rho \in \mathcal{S}_1).$$

A quantity $1 - \beta(T,\rho)$ is called *power*.

A test T is said to be *level-α* if $\alpha(T,\rho) \le \alpha$ for any $\rho \in \mathcal{S}_0$. Let $\mathcal{T}_C(\subset \mathcal{T}(\mathcal{H}))$ be the set of tests which satisfy a condition C. A test $T \in \mathcal{T}_C$ is called *a Most Powerful-C (MP-C) test at $\rho(\in \mathcal{S}_1)$* if $\beta(T,\rho) \le \beta(T',\rho)$ for any $T' \in \mathcal{T}_C$. A test $T \in \mathcal{T}_\alpha$ is called *a Uniformly Most Powerful-C (UMP-C) test* if T is MP-C for any $\rho \in \mathcal{S}_1$. In case that $\mathcal{S}_0$ is a closed set with respect to the natural topology, a test $T \in \mathcal{T}_\alpha$ is called *a Locally Most Powerful-C (LMP-C) test* if there is an open set S_0' such that $S_0 \subset S_0'$ and T is MP-C for any $\rho \in \mathcal{S}_0'$.

3 Problems, Answers, Proofs

3.1 Problem 1

We study how to test hypotheses for entanglement between two d-dimensional parties based on n-samples. For mathematical convenience in the optimization of tests, we require an invariance of tests. (This is the simplest case.)

For $i = 1, 2, ..., n$, let $\mathcal{H}_{A_i}$ and $\mathcal{H}_{B_i}$ be d-dimensional Hilbert spaces spanned by $|0\rangle_{A_i}, |1\rangle_{A_i}, ..., |d-1\rangle_{A_i}$ and $|0\rangle_{B_i}, |1\rangle_{B_i}, ..., |d-1\rangle_{B_i}$, respectively. Let $\mathcal{H}_i = \mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$, $\mathcal{H}_A = \bigotimes_{i=1}^n \mathcal{H}_{A_i}$, $\mathcal{H}_B = \bigotimes_{i=1}^n \mathcal{H}_{B_i}$ and $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the composite spaces.

Let

$$|\Phi^+\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |i\rangle_{A_1} \otimes |i\rangle_{B_1}$$

be an element of $\mathcal{H}_1$. Note that a state $\sigma_0 = |\Phi^+\rangle\langle\Phi^+| \in \mathcal{S}(\mathcal{H}_1)$ is a maximally entangled state between $\mathcal{H}_{A_1}$ and $\mathcal{H}_{B_1}$.

Suppose that the state ρ on $\mathcal{H}_{AB}$ is given in the form $\rho = \sigma^{\otimes n}$ for an unknown $d \times d$ matrix σ with respect to an appropriate correspondence of bases. Let

$$\mathcal{S}_0 = \{\sigma_0^{\otimes n}\},\ \mathcal{S}_1 = \{\sigma^{\otimes n} \mid \sigma_0 \neq \sigma \in \mathcal{S}(\mathcal{H}_1)\}.$$

We study how to test hypotheses

$$H_0 : \rho \in \mathcal{S}_0 \text{ versus } H_1 : \rho \in \mathcal{S}_1. \tag{2}$$

For an invariance condition, we consider an action on $\mathcal{T}(\mathcal{H}_{AB})$ given by the following linear representation U_{AB} of a d^n-dimensional special unitary group $G \cong SU(d^n)$:

$$U_{AB} : g\,(\in G) \mapsto U_A(g) \otimes \bar{U}_B(g),$$

where $U_A(g)$ and $U_B(g)$ be natural representations of $g \in G$, and where $\bar{X}$ means the contragradient of X. That is, $g \in G$ maps $T = \{T_0, T_1\} (\in \mathcal{T}(\mathcal{H}_{AB}))$ as

$$T_i \mapsto U_{AB}^\dagger(g) T_i U_{AB}(g). \tag{3}$$

Define a condition C_1 on $\mathcal{T}(\mathcal{H}_{AB})$ as

(level)	$T(\in \mathcal{T}(\mathcal{H}_{AB}))$ is level-zero,
(invariance)	$T(\in \mathcal{T}(\mathcal{H}_{AB}))$ is fixed by the action of G.

Then, a UMP-C_1 test for (2) is given as follows.

Theorem 1 *A UMP-C_1 is given by $T_0 = \sigma_0^{\otimes n}$.*

Proof From the invariance, the form of a test $T = \{T_0, T_1\} \in \mathcal{T}_{C_1}$ is given by

$$T_0 = w_1 \rho_0 + w_2 (I_{\mathcal{H}_{AB}} - \rho_0) \tag{4}$$

for real numbers $w_1, w_2 (\in [0,1])$. The level-zero condition implies $w_1 = 1$. The power is maximized when $w_2 = 0$. □

Corollary 1 *The type 2 error of the UMP-C_1 test is given by F^{2n} where F is the fidelity $F = \sqrt{\mathrm{Tr}(\sigma\sigma_0)}$ between σ and σ_0.*

3.2 Problem 2

We consider a case where we can use only LOCC between the two parties. Under the invariance condition introduced in the previous section, we will see that this LOCC restriction is no longer restrictive in asymptotic sense.

Define a condition C_2 on $\mathcal{T}(\mathcal{H}_{AB})$ as

$$\begin{array}{lll} \text{(level)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is level-zero,} & (5) \\ \text{(locality)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is realized by LOCC between } \mathcal{H}_A \text{ and } \mathcal{H}_B, & (6) \\ \text{(invariance)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is fixed by the action of } G. & (7) \end{array}$$

Then, a UMP-C_2 test for (2) is given as follows.

Theorem 2 *A UMP-C_2 test is given by*

$$T_0 = \sigma_0 + \frac{d^n - 1}{d^{2n} - 1}(I_{\mathcal{H}_{AB}} - \sigma_0).$$

Proof From the invariance, $T = \{T_0, T_1\} \in \mathcal{T}_{C_1}$ is written as (4). The level-zero condition implies $w_1 = 1$. The locality condition implies that T_0 is Positive Partial Transpose (PPT) (see [8]). T_0 is PPT if and only if $w_2 \geq (d^n - 1)/(d^{2n} - 1)$. Power of PPT tests gives the maximum at $w_2 = (d^n - 1)/(d^{2n} - 1)$. In fact, this PPT test is realized by LOCC as follows. First, twirl the system as

$$\rho \mapsto \int_{g \in G} U_{AB}(g)\rho(U_{AB}(g))^\dagger \mu(dg)$$

where $\mu(\cdot)$ is the normalized uniform (Haar) measure on G. Next, measure the system by a POVM $T' = \{T'_0, T'_1\} \in \mathcal{T}(\mathcal{H}_{AB})$ where

$$T'_0 = \sum_{i_1,\ldots,i_n \in \{0,1,\ldots,d-1\}} \bigotimes_{j=1,n} |i_j\rangle_{A_j}\langle i_j|_{A_j} \otimes |i_j\rangle_{B_j}\langle i_j|_{B_j}.$$

□

Corollary 2 *The type 2 error of the UMP-C_2 test is*

$$\begin{aligned} \beta(T,\rho) &= \frac{1 + d^n F^{2n}}{d^n + 1} \\ &= \begin{cases} O(F^{2n}) & \text{if } F^2 \geq d^{-1}, \\ O(d^{-n}) & \text{if } F^2 < d^{-1} \end{cases} \end{aligned}$$

as $n \to \infty$.

This Corollary means that the LOCC condition in C_2 asymptotically gives no restriction from the viewpoint of hypothesis testing.

3.3 Problem 3

We set $d = 2$ and $n = 2$ in the setting mentioned above. We let the invariance condition be less restrictive. We will see that the best test under such a condition needs an non local POVM beyond the two samples.

Let $G = G_1 \times G_2 \times G_3$ be the product group of $G_1 \cong G_2 \cong SU(2)$ and a symmetric group $G_3 \cong {}_2$ of order two. Let U_{A_1} and U_{B_1} the natural representations of G_1 on $\mathcal{H}_{A_1}$ and $\mathcal{H}_{B_1}$, respectively. Let U_{A_2} and U_{B_2} the natural representations of G_2 on $\mathcal{H}_{A_2}$ and $\mathcal{H}_{B_2}$, respectively. Let U_3 be a representation of G_3 which permutes $\mathcal{H}_1$ and $\mathcal{H}_2$ by $|x\rangle_{y_1} \leftrightarrow |x\rangle_{y_2}$ for $x = 0, 1$ and $y = A, B$. Let

$$U_{AB} = U_{A_1} \otimes \bar{U}_{B_1} \otimes U_{A_2} \otimes \bar{U}_{B_2} \otimes U_3$$

be the tensor product. We consider an action of G on $\mathcal{T}(\mathcal{H}_{AB})$ of the form (3). For such a G-action, define a condition C_3 on $\mathcal{T}(\mathcal{H}_{AB})$ as (5) to (7). Then, a LMP-C_3 test for (2) is given as follows.

Theorem 3 *A LMP-C_3 test is given in the form*

$$T_0 = \sigma_0^{\otimes 2} + \frac{1}{3}(I_4 - \sigma_0)^{\otimes 2}$$

where I_4 is a 4×4 identity matrix.

Proof From the invariance, the form of T_0 is

$$T_0 = w_0 \sigma_0^{\otimes 2} + w_1(\sigma_0 \otimes (I_4 - \sigma_0) + (I_4 - \sigma_0) \otimes \sigma_0) + w_2(I_4 - \sigma_0)^{\otimes 2}.$$

The level-zero condition implies $w_0 = 1$. The locality condition implies that T_0 is PPT. T_0 is PPT if and only if

$$2w_1 + 3w_2 - 1 \geq 0 \text{ and } -6w_1 + 9w_2 + 1 \geq 0.$$

Let $w_1 = 0$ and $w_3 = 1/3$. Since $\mathrm{Tr}(\sigma_0(I_4 - \sigma_0)) = 0$, there is an open set S_0' such that $S_0 \subset S_0'$ and such that power of tests in $\mathcal{T}_{C_3}$ is maximized in S_0'. □

Corollary 3 *The type 2 error of the LMP-C_3 test is $F^4 + (1 - F^2)^2/3$.*

3.4 Problem 4

Let $d = 2$ and $n = 2$ again. We consider a case where we can use only LOCC between the two samples as well as the two parties. Moreover, we let the invariance be further less restrictive. However, it is difficult to characterize a class of such tests for optimization. Hence, we introduce another class of tests which is convenient for optimization. We will see that the optimal solution in the latter class belongs to the former class.

Let

$$|\Psi^-\rangle = \frac{|0\rangle_{A_1}|1\rangle_{B_1} - |1\rangle_{A_1}|0\rangle_{B_1}}{\sqrt{2}}.$$

Note that $\sigma_0 = |\Psi^-\rangle\langle\Psi^-|$ is maximally entangled. Let

$$\mathcal{S}_0 = \{\sigma_0^{\otimes n}\}, \ \mathcal{S}_1 = \{\sigma^{\otimes n} \mid \sigma_0 \neq \sigma \in \mathcal{S}(\mathcal{H}_1)\}.$$

We study how to test hypotheses

$$H_0 : \rho \in \mathcal{S}_0 \text{ versus } H_1 : \rho \in \mathcal{S}_1. \tag{8}$$

Let $G(\cong SU(2))$. Let U_{A_1}, U_{B_1}, U_{A_2} and U_{B_2} be the natural representations of G on $\mathcal{H}_{A_1}$, $\mathcal{H}_{B_1}$, $\mathcal{H}_{A_2}$ and $\mathcal{H}_{B_2}$, respectively. Let

$$U_{AB} = U_{A_1} \otimes U_{B_1} \otimes U_{A_2} \otimes U_{B_2}$$

be the tensor product. We consider an action of G on $\mathcal{T}(\mathcal{H}_{AB})$ of the form (3). Define a condition C_4 on $\mathcal{T}(\mathcal{H}_{AB})$ as

$$\begin{array}{rl}
\text{(level)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is level-zero,} \\
\text{(locality A : B)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is realized by LOCC between } \mathcal{H}_A \text{ and } \mathcal{H}_B, \\
\text{(locality 1 : 2)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is realized by LOCC between } \mathcal{H}_1 \text{ and } \mathcal{H}_2, \\
\text{(invariance)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is fixed by the action of } G.
\end{array}$$

However, it is difficult to characterize the set $\mathcal{T}_{C_4}$ of tests satisfying C_4 for optimization. Hence, we introduce another conditions C_4' which is convenient for optimization.

$$\begin{array}{rl}
\text{(level)} & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is level-zero,} \\
\text{(locality \& invariance)} & T = \{T_0, T_1\} \text{ is of the form}
\end{array}$$

$$T_0 = \sum_{i=1}^{q} \lambda_i \int_{g\in G} (U_{AB}(g))^\dagger M_{A_1}^i \otimes M_{B_1}^i \otimes M_{A_2}^i \otimes M_{B_2}^i U_{AB}(g) \mu(dg),$$

where

(i) q is an element of the set of natural numbers,
(ii) λ_i is a positive number,
(iii) $M_{A_1}^i \in \mathcal{L}(\mathcal{H}_{A_1})$, $M_{B_1}^i \in \mathcal{L}(\mathcal{H}_{B_1})$, $M_{A_2}^i \in \mathcal{L}(\mathcal{H}_{A_2})$ and $M_{B_2}^i \in \mathcal{L}(\mathcal{H}_{B_2})$ are projections with rank one satisfying

$$M_{A_1}^i = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \bar{M}_{B_1}^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \; M_{A_2}^i = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \bar{M}_{B_2}^i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

(iv) $\mu(\cdot)$ is the normalized Haar measure on G.
Then, a UMP-C_4' test for (8) is given as follows.

Theorem 4 *A UMP-C_4' test is given by*

$$T_0 = \frac{1}{10} Q_0 + + \frac{1}{3} Q_1 + \frac{1}{6} Q_2 + \frac{1}{3} Q_3 + \sigma_0^{\otimes 2},$$

where Q_0, Q_1, Q_2 and Q_3 are projections defined below.

(Definitions of Q_0, Q_1 and Q_2) Let $_4$ be a symmetric group of order four. Let V be the linear representation of $_4$ which is dual to U_{AB}. Let

$$\mathbf{n}_0 = \square\square\square\square, \; \mathbf{n}_1 = \begin{smallmatrix}\square\square\square \\ \square\end{smallmatrix} \text{ and } \mathbf{n}_2 = \begin{smallmatrix}\square\square \\ \square\square\end{smallmatrix}$$

be Young indexes. For $i = 0, 1, 2$, let $U_{\mathbf{n}_i}$ and $V_{\mathbf{n}_i}$ be irreducible subrepresentations of U_{AB} and V, that is,

$$U_{AB} \otimes V = U_{\mathbf{n}_0} \otimes V_{\mathbf{n}_0} \oplus U_{\mathbf{n}_1} \otimes V_{\mathbf{n}_1} \oplus U_{\mathbf{n}_2} \otimes V_{\mathbf{n}_2}.$$

Let

$$\mathcal{U}_{AB} \otimes \mathcal{V} = \mathcal{U}_{\mathbf{n}_0} \otimes \mathcal{V}_{\mathbf{n}_0} \oplus \mathcal{U}_{\mathbf{n}_1} \otimes \mathcal{V}_{\mathbf{n}_1} \oplus \mathcal{U}_{\mathbf{n}_2} \otimes \mathcal{V}_{\mathbf{n}_2}.$$

denote the corresponding vector spaces.

Definition 1 Let Q_0 be the projection on $\mathcal{U}_{\mathbf{n}_0} \otimes \mathcal{V}_{\mathbf{n}_0}$.

Let $p_{12} \in {}_4$ be an element such that $V(p_{12})$ transposes $|x\rangle_{y_1}$ and $|x\rangle_{y_2}$ for $x = 0, 1$ and $y = A, B$. Let q_1 be the projection on the subset of $\mathcal{V}_{\mathbf{n}_1}$ which is fixed by p_{12}.

Definition 2 *Let Q_1 be $Q \otimes q_1$ where Q is the projection on $\mathcal{U}_{\mathbf{n}_1}$.*

Let $p_{AB} \in {}_4$ be an element such that $V(p_{AB})$ transposes $|x\rangle_{A_y}$ and $|x\rangle_{B_y}$ for $x = 0, 1$ and $y = 1, 2$. Let q_2 be the projection on the subset of $\mathcal{V}_{\mathbf{n}_1}$ which is fixed by p_{AB}. Let q_3 be the projection on the subset of $\mathcal{V}_{\mathbf{n}_1}$ which is -1-scaled by both p_{12} and p_{AB}.

Definition 3 Let Q_2 be $Q \otimes q_2$. Let Q_3 be $Q \otimes q_3$.

Proof of Theorem 4 Since $\alpha(T, \sigma_0^{\otimes 2}) = 0$ (level-zero) and since the partial trace $\mathrm{Tr}_{\mathcal{H}_B}(\sigma_0^{\otimes 2})$ over $\mathcal{H}_B$ is $4^{-1} I_{\mathcal{H}_A}$, it holds that $\mathrm{Tr}_{\mathcal{H}_B}(T_0) = I_{\mathcal{H}_A}$. Since T_0 is G-invariant, it also holds that

$$\frac{1}{2} \sum_{i=1}^{q} f_i = \langle \phi | \mathrm{Tr}_B(T_B) | \phi \rangle = 1$$

where

$$\begin{aligned} f_i &= \mathrm{Tr}(M_{A_1}^i M_{A_2}^i) = \mathrm{Tr}(M_{B_1}^i M_{B_2}^i), \\ |\phi\rangle &= \frac{|0\rangle_{A_1} |0\rangle_{A_2} + |1\rangle_{A_1} |1\rangle_{A_2}}{\sqrt{2}}. \end{aligned}$$

The type 2 error for $\rho = \sigma^{\otimes 2}$ is of the form

$$\beta(T, \rho) = \sum_{i=1}^{q} \lambda_i (a(\sigma) f_i^2 + b(\sigma) f_i + c(\sigma))$$

where

$$\begin{aligned} a(\sigma) &= \frac{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2}{15} + \frac{6}{15}(3x_{23}^2 + 3y_{12}^2 + 3y_{13}^2), \\ \sigma &= \begin{pmatrix} x_1 & x_{12} & x_{13} & x_{14} \\ x_{12} & x_2 & x_{23} & x_{24} \\ x_{13} & x_{23} & x_3 & x_{34} \\ x_{14} & x_{24} & x_{34} & x_4 \end{pmatrix} + \sqrt{-1} \begin{pmatrix} & -y_{12} & -y_{13} & -y_{14} \\ y_{12} & & -y_{23} & -y_{24} \\ y_{13} & y_{23} & & -y_{34} \\ y_{14} & y_{24} & y_{34} & \end{pmatrix}, \end{aligned}$$

and $b(\sigma)$ and $c(\sigma)$ are some functions of σ. Therefore, $\beta(T, \rho)$ is minimized by $q = 4$, $\lambda_1 = \cdots = \lambda_4 = 1$ and $f_1 = \cdots = f_4 = 1/2$. For example,

$$M_{A_1} = \begin{pmatrix} 1 & \\ & \end{pmatrix}, \quad M_{B_1} = \begin{pmatrix} & \\ & 1 \end{pmatrix}, \quad M_{A_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_{B_2} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

give the UMP-C_4'. □

Corollary 4 *The UMP-C_4' test in Theorem 4 is in $\mathcal{T}_{C_4}$.*

Corollary 5 *The type 2 error of the UMP-C_4' test is given by*

$$ {}^t vRv - \frac{2}{15}(x_{23}^2 + y_{12}^2 + y_{13}^2) $$

where

$$ v = \begin{pmatrix} x_1 - 1/2 \\ x_2 - 1/2 \\ x_3 - 1/2 \end{pmatrix}, \quad R = \frac{1}{15}\begin{pmatrix} 6 & 7 & 7 \\ 7 & 6 & 7 \\ 7 & 7 & 6 \end{pmatrix}. $$

References

[1] Acin, A., Tarrach, R. and Vidal G. 2000 *Phys. Rev. A* **61**, 62307.

[2] Hayashi, M. 2002 *J. Phys. A* **35** 10759-10773.

[3] Helstrom, C. W. 1976 *Quantum detection and estimation theory* Academic Press.

[4] Hiai, F. and Petz, D. 1991 *Comm. Math. Phys.*, **143**, 99-114.

[5] Holevo, A. S. 1982 *Probabilistic and statistical aspects of quantum theory* North-Holland Publishing Co.

[6] Nielsen, M. A. and Chuang, I. L. 2000 *Quantum computation and quantum information* Cambridge University Press.

[7] Ogawa, T. and Nagaoka, H. 2000 *IEEE Trans. Inform. Theory* **46** 2428-2433.

[8] Virmani, S. and Plenio, M. B. 2003 *Phys. Rev. A* **67** 062308.