Testing hypothesis for quantum systems

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1 Introduction

A quantum information protocol usually makes use of maximally entangled states as resources ([6]). In many cases, the protocol is based on a device which repeatedly produce entangled resources with a constant level of quality. Hence it is important to test the performance of the device. Since the testing is based on a physical measurement, we should carefully analyze the noise and error. For such a testing problem, hypothesis testing theory of mathematical statistics serves an orthodox framework. In quantum theory of hypothesis testing, a lower bound of error is given by the quantum Neyman-Pearson theorem for non-asymptotic cases ([3], [5]), and by the quantum Stein's lemma for asymptotic cases ([4], [7], [2]). In these general arguments, an optimal test require measurements which can not be realized by Local Operations and Classical Communications (LOCC). On this point, these arguments are not enough as testing theory for quantum information. On the other hand, the LOCC requirement has been considered in a context of entanglement witness though their framework is not so orthodox as that of hypothesis testing.

In order to argue this requirement in hypothesis testing, it is necessary to characterize the set of LOCC tests. Recently, Virmani and Plenio [8] pointed out that some invariance restrictions measurements make the characterization more convenient. They showed some examples in a case where the number of samples is one. Such a method had been considered by Acin, Tarrach and Vidal [1] for statistical estimation of quantum states.

In this article, we consider a case where the number $n$ of samples is large, and a case where $n$ is just two. In the former case, we derive optimal tests
(i) with a LOCC condition between two parties,
(ii) without a LOCC condition between two parties.
In the latter case, we derive optimal tests without a LOCC condition between two parties and
(iii) with a LOCC condition between two samples,
(iv) without a LOCC condition between two samples.

As is naturally expected, the performance of the best test for (ii) is worse than that of (i). However, if the state is close to the maximally entangled state, the difference of performance asymptotically goes to zero with respect to the exponent of error. Similarly,
the best test $T_3$ for (iii) is better than the best one $T_4$ for (iv). Indeed, $T_3$ require non-local measurement between two samples. The condition of (iv) is most restrictive so that it is the nearest to the real situation among (i)-(iv) though the mathematical story for the optimization is not so simple as those of (i)-(iii).

In Section 2, we introduce a general formulation of hypothesis testing. In Section 3, we consider (i), (ii), (iii) and (iv) in order.

2 A general setting

Let $\mathcal{H}$ be a finite-dimensional Hilbert space which describes a physical system of interest. For such $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ be the set of linear operators (matrices) on $\mathcal{H}$ and let $S(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ be the set of density matrices on $\mathcal{H}$. Let $S_0$ and $S_1$ be mutually disjoint and non-empty subsets of $S(\mathcal{H})$.

Suppose that the current state $\rho \in S(\mathcal{H})$ of the system is unknown. Suppose also that $\rho \in S_0$ or $\rho \in S_1$. In such a case, we would like to test

$$\begin{align*}
H_0 & : \rho \in S_0 \text{ versus } H_1 : \rho \in S_1
\end{align*}$$

(1)

based on an appropriate measurement on $\mathcal{H}$. $H_0$ is called a null hypothesis, and $H_1$ is called an alternative hypothesis.

A test for the hypothesis (1) is given by a Positive Operator Valued Measure (POVM) on $\mathcal{H}$. Let $T(\mathcal{H})$ be the set of POVM's $T$ of the form

$$T : \{0, 1\} \rightarrow \mathcal{L}(\mathcal{H}) (i \mapsto T_i)$$

where $T_0 + T_1 = I_{\mathcal{H}}$ (identity on $\mathcal{H}$). We often write $T(\in T(\mathcal{H}))$ as $T = \{T_0, T_1\}$ to specify the form of $T$. Any test can be described by $T \in T(\mathcal{H})$, that is, $H_0$ is accepted (= $H_0$ is rejected) if 0 is observed and $H_1$ is accepted (= $H_0$ is rejected) if 1 is observed.

A type 1 error is an event such that $H_1$ is accepted though $H_0$ is true. A type 2 error is an event such that $H_0$ is accepted though $H_1$ is true. Hence the type 1 error probability $\alpha(T, \rho)$ and the type 2 error probability $\beta(T, \rho)$ are given by

$$\alpha(T, \rho) = \text{Tr}(\rho T_1) \quad (\rho \in S_0), \quad \beta(T, \rho) = \text{Tr}(\rho T_0) \quad (\rho \in S_1).$$

A quantity $1 - \beta(T, \rho)$ is called power.

A test $T$ is said to be level-$\alpha$ if $\alpha(T, \rho) \leq \alpha$ for any $\rho \in S_0$. Let $T(\subset T(\mathcal{H}))$ be the set of tests which satisfy a condition $C$. A test $T \in T_C$ is called a Most Powerful-C (MP-C) test at $\rho \in S_1$ if $\beta(T, \rho) \leq \beta(T', \rho)$ for any $T' \in T_C$. A test $T \in T_0$ is called a Uniformly Most Powerful-C (UMP-C) test if $T$ is MP-C for any $\rho \in S_1$. In case that $S_0$ is a closed set with respect to the natural topology, a test $T \in T_0$ is called a Locally Most Powerful-C (LMP-C) test if there is an open set $S_0'$ such that $S_0 \subset S_0'$ and $T$ is MP-C for any $\rho \in S_0'$. 

3 Problems, Answers, Proofs

3.1 Problem 1

We study how to test hypotheses for entanglement between two $d$-dimensional parties based on $n$-samples. For mathematical convenience in the optimization of tests, we
require an invariance of tests. (This is the simplest case.)

For \( i = 1, 2, \ldots, n \), let \( \mathcal{H}_{A_i} \) and \( \mathcal{H}_{B_i} \) be \( d \)-dimensional Hilbert spaces spanned by \( |0\rangle_{A_i}, |1\rangle_{A_i}, \ldots, |d-1\rangle_{A_i} \) and \( |0\rangle_{B_i}, |1\rangle_{B_i}, \ldots, |d-1\rangle_{B_i} \), respectively. Let \( \mathcal{H}_i = \mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i} \), \( \mathcal{H}_A = \bigotimes_{i=1}^{n} \mathcal{H}_{A_i} \), \( \mathcal{H}_B = \bigotimes_{i=1}^{n} \mathcal{H}_{B_i} \), and \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \) be the composite spaces.

Let

\[
|\Phi^+\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |i\rangle_{A_i} \otimes |i\rangle_{B_i}
\]

be an element of \( \mathcal{H}_1 \). Note that a state \( \sigma_0 = |\Phi^+\rangle \langle \Phi^+| \in \mathcal{S}(\mathcal{H}_1) \) is a maximally entangled state between \( \mathcal{H}_{A_1} \) and \( \mathcal{H}_{B_1} \).

Suppose that the state \( \rho \) on \( \mathcal{H}_{AB} \) is given in the form \( \rho = \sigma^\otimes n \) for an unknown \( d \times d \) matrix \( \sigma \) with respect to an appropriate correspondence of bases. Let

\[
\mathcal{S}_0 = \{ \sigma_0^\otimes n \}, \quad \mathcal{S}_1 = \{ \sigma^\otimes n \mid \sigma_0 \neq \sigma \in \mathcal{S}(\mathcal{H}_1) \}.
\]

We study how to test hypotheses

\[
H_0 : \rho \in \mathcal{S}_0 \text{ versus } H_1 : \rho \in \mathcal{S}_1.
\]

For an invariance condition, we consider an action on \( \mathcal{T}(\mathcal{H}_{AB}) \) given by the following linear representation \( U_{AB} \) of a \( d^n \)-dimensional special unitary group \( G \cong SU(d^n) \):

\[
U_{AB} : g (\in G) \mapsto U_A(g) \otimes U_B(g),
\]

where \( U_A(g) \) and \( U_B(g) \) be natural representations of \( g \in G \), and where \( \overline{X} \) means the contragradient of \( X \). That is, \( g \in G \) maps \( T = \{ T_0, T_1 \} \in \mathcal{T}(\mathcal{H}_{AB}) \) as

\[
T_i \mapsto U_{AB}^\dagger(g) T_i U_{AB}(g).
\]

Define a condition \( C_1 \) on \( \mathcal{T}(\mathcal{H}_{AB}) \) as

\[
\begin{align*}
\text{(level)} \quad & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is level-zero,} \\
\text{(invariance)} \quad & T(\in \mathcal{T}(\mathcal{H}_{AB})) \text{ is fixed by the action of } G.
\end{align*}
\]

Then, a UMP-\( C_1 \) test for (2) is given as follows.

\textbf{Theorem 1} A UMP-\( C_1 \) is given by \( T_0 = \sigma_0^\otimes n \).

\textbf{Proof} From the invariance, the form of a test \( T = \{ T_0, T_1 \} \in \mathcal{T}_{C_1} \) is given by

\[
T_0 = w_1 \rho_0 + w_2 (I_{\mathcal{H}_{AB}} - \rho_0)
\]

for real numbers \( w_1, w_2 (\in [0, 1]) \). The level-zero condition implies \( w_1 = 1 \). The power is maximized when \( w_2 = 0 \). \( \square \)

\textbf{Corollary 1} The type 2 error of the UMP-\( C_1 \) test is given by \( F^2 n \) where \( F \) is the fidelity \( F = \sqrt{\text{Tr}(\sigma \sigma_0)} \) between \( \sigma \) and \( \sigma_0 \).
### 3.2 Problem 2

We consider a case where we can use only LOCC between the two parties. Under the invariance condition introduced in the previous section, we will see that this LOCC restriction is no longer restrictive in asymptotic sense.

Define a condition $C_2$ on $T(\mathcal{H}_{AB})$ as

- **(level)** $T(\in T(\mathcal{H}_{AB}))$ is level-zero, \( (5) \)
- **(locality)** $T(\in T(\mathcal{H}_{AB}))$ is realized by LOCC between $\mathcal{H}_A$ and $\mathcal{H}_B$, \( (6) \)
- **(invariance)** $T(\in T(\mathcal{H}_{AB}))$ is fixed by the action of $G$. \( (7) \)

Then, a UMP-$C_2$ test for (2) is given as follows.

**Theorem 2** A UMP-$C_2$ test is given by

$$T_0 = \sigma_0 + \frac{d^n - 1}{d^{2n} - 1}(I_{\mathcal{H}_{AB}} - \sigma_0).$$

**Proof** From the invariance, $T = \{T_0, T_1\} \in \mathcal{T}_{C_1}$ is written as (4). The level-zero condition implies $w_1 = 1$. The locality condition implies that $T_0$ is Positive Partial Transpose (PPT) (see [8]). $T_0$ is PPT if and only if $w_2 \geq (d^n - 1)/(d^{2n} - 1)$. Power of PPT tests gives the maximum at $w_2 = (d^n - 1)/(d^{2n} - 1)$. In fact, this PPT test is realized by LOCC as follows. First, twirl the system as

$$\rho \mapsto \int_{g \in G} U_{AB}(g) \rho (U_{AB}(g))^\dagger \mu(dg)$$

where $\mu(\cdot)$ is the normalized uniform (Haar) measure on $G$. Next, measure the system by a POVM $T' = \{T'_0, T'_1\} \in \mathcal{T}(\mathcal{H}_{AB})$ where

$$T'_0 = \sum_{i_0, \ldots, i_n \in \{0,1,\ldots, d-1\}} \bigotimes_{j=1}^n |i_j\rangle_{A_j} \langle i_j|_{A_j} \otimes |i_j\rangle_{B_j} \langle i_j|_{B_j}.$$ 

**Corollary 2** The type 2 error of the UMP-$C_2$ test is

$$\beta(T, \rho) = \frac{1 + d^n F^{2n}}{d^n + 1}$$

$$= \begin{cases} O(F^{2n}) & \text{if } F^2 \geq d^{-1}, \\ O(d^{-n}) & \text{if } F^2 < d^{-1} \end{cases}$$

as $n \to \infty$.

This Corollary means that the LOCC condition in $C_2$ asymptotically gives no restriction from the viewpoint of hypothesis testing.
3.3 Problem 3

We set $d = 2$ and $n = 2$ in the setting mentioned above. We let the invariance condition be less restrictive. We will see that the best test under such a condition needs an non local POVM beyond the two samples.

Let $G = G_1 \times G_2 \times G_3$ be the product group of $G_1 \cong G_2 \cong SU(2)$ and a symmetric group $G_3 \cong S_3$ of order two. Let $U_{A_1}$ and $U_{B_1}$ the natural representations of $G_1$ on $\mathcal{H}_{A_1}$ and $\mathcal{H}_{B_1}$, respectively. Let $U_{A_2}$ and $U_{B_2}$ the natural representations of $G_2$ on $\mathcal{H}_{A_2}$ and $\mathcal{H}_{B_2}$, respectively. Let $U_3$ be a representation of $G_3$ which permutes $H_1$ and $H_2$ by $|x\rangle_{y_1} \leftrightarrow |x\rangle_{y_2}$ for $x = 0, 1$ and $y = A, B$. Let

$$U_{AB} = U_{A_1} \otimes U_{B_1} \otimes U_{A_2} \otimes U_{B_2} \otimes U_3$$

be the tensor product. We consider an action of $G$ on $T(\mathcal{H}_{AB})$ of the form $(3)$. For such a $G$-action, define a condition $C_3$ on $T(\mathcal{H}_{AB})$ as $(5)$ to $(7)$. Then, a LMP-$C_3$ test for $(2)$ is given as follows.

**Theorem 3** A LMP-$C_3$ test is given in the form

$$T_0 = \sigma_0^{\otimes 2} + \frac{1}{3}(I_4 - \sigma_0)^{\otimes 2}$$

where $I_4$ is a $4 \times 4$ identity matrix.

**Proof** From the invariance, the form of $T_0$ is

$$T_0 = w_0 \sigma_0^{\otimes 2} + w_1 (\sigma_0 \otimes (I_4 - \sigma_0) + (I_4 - \sigma_0) \otimes \sigma_0) + w_2 (I_4 - \sigma_0)^{\otimes 2}.$$  

The level-zero condition implies $w_0 = 1$. The locality condition implies that $T_0$ is PPT. $T_0$ is PPT if and only if

$$2w_1 + 3w_2 - 1 \geq 0 \quad \text{and} \quad -6w_1 + 9w_2 + 1 \geq 0.$$  

Let $w_1 = 0$ and $w_3 = 1/3$. Since $\text{Tr}(\sigma_0(I_4 - \sigma_0)) = 0$, there is an open set $S_0'$ such that $S_0 \subset S_0'$ and such that power of tests in $T_{C_3}$ is maximized in $S_0'$. \hfill \Box

**Corollary 3** The type 2 error of the LMP-$C_3$ test is $F^4 + (1 - F^2)^2/3$.

3.4 Problem 4

Let $d = 2$ and $n = 2$ again. We consider a case where we can use only LOCC between the two samples as well as the two parties. Moreover, we let the invariance be further less restrictive. However, it is difficult to characterize a class of such tests for optimization. Hence, we introduce another class of tests which is convenient for optimization. We will see that the optimal solution in the latter class belongs to the former class.

Let

$$|\Psi^-\rangle = \frac{|0\rangle_{A_1}|1\rangle_{B_1} - |1\rangle_{A_1}|0\rangle_{B_1}}{\sqrt{2}}.$$  

Note that $\sigma_0 = |\Psi^-\rangle\langle \Psi^-|$ is maximally entangled. Let

$$S_0 = \{\sigma_0^{\otimes n}\}, \quad S_1 = \{\sigma^{\otimes n} \mid \sigma_0 \neq \sigma \in S(\mathcal{H}_2)\}.$$
We study how to test hypotheses

\[ H_0 : \rho \in S_0 \text{ versus } H_1 : \rho \in S_1. \]  

Let \( G(\cong SU(2)) \). Let \( U_{A_1}, U_{B_1}, U_{A_2} \) and \( U_{B_2} \) be the natural representations of \( G \) on \( \mathcal{H}_{A_1}, \mathcal{H}_{B_1}, \mathcal{H}_{A_2} \) and \( \mathcal{H}_{B_2} \), respectively. Let

\[ U_{AB} = U_{A_1} \otimes U_{B_1} \otimes U_{A_2} \otimes U_{B_2} \]

be the tensor product. We consider an action of \( G \) on \( T(\mathcal{H}_{AB}) \) of the form (3). Define a condition \( C_4 \) on \( T(\mathcal{H}_{AB}) \) as

\[ T(\in T(\mathcal{H}_{AB})) \text{ is level-zero,} \]

\[ T(\in T(\mathcal{H}_{AB})) \text{ is realized by LOCC between } \mathcal{H}_{A} \text{ and } \mathcal{H}_{B}, \]

\[ T(\in T(\mathcal{H}_{AB})) \text{ is realized by LOCC between } \mathcal{H}_{1} \text{ and } \mathcal{H}_{2}, \]

\[ T(\in T(\mathcal{H}_{AB})) \text{ is fixed by the action of } G. \]

However, it is difficult to characterize the set \( T_{C_4} \) of tests satisfying \( C_4 \) for optimization. Hence, we introduce another conditions \( C'_4 \) which is convenient for optimization.

\[ T(\in T(\mathcal{H}_{AB})) \text{ is level-zero,} \]

\[ T = \{T_0,T_1\} \text{ is of the form} \]

\[ T_0 = \sum_{i=1}^{q} \lambda_i \int_{g \in G} (U_{AB}(g))^{\dagger} M_{A_1}^i \otimes M_{B_1}^i \otimes M_{A_2}^i \otimes M_{B_2}^i U_{AB}(g) \mu(dg), \]

where

(i) \( q \) is an element of the set of natural numbers,

(ii) \( \lambda_i \) is a positive number,

(iii) \( M_{A_1}^i \in \mathcal{L}(\mathcal{H}_{A_1}), M_{B_1}^i \in \mathcal{L}(\mathcal{H}_{B_1}), M_{A_2}^i \in \mathcal{L}(\mathcal{H}_{A_2}) \) and \( M_{B_2}^i \in \mathcal{L}(\mathcal{H}_{B_2}) \) are projections with rank one satisfying

\[ M_{A_1}^i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \]

\[ M_{A_2}^i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \]

(iv) \( \mu(\cdot) \) is the normalized Haar measure on \( G \).

Then, a UMP-\( C'_4 \) test for (8) is given as follows.

**Theorem 4** A UMP-\( C'_4 \) test is given by

\[ T_0 = \frac{1}{10} Q_0 + \frac{1}{3} Q_1 + \frac{1}{6} Q_2 + \frac{1}{3} Q_3 + \sigma_0^{22}, \]

where \( Q_0, Q_1, Q_2 \) and \( Q_3 \) are projections defined below.

**Definitions of \( Q_0, Q_1 \) and \( Q_2 \)** Let \( 4 \) be a symmetric group of order four. Let \( V \) be the linear representation of \( 4 \) which is dual to \( U_{AB} \). Let

\[ n_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, n_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } n_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
be Young indexes. For $i = 0, 1, 2$, let $U_{n_i}$ and $V_{n_i}$ be irreducible subrepresentations of $U_{AB}$ and $V$, that is,

$$U_{AB} \otimes V = U_{n_0} \otimes V_{n_0} \oplus U_{n_1} \otimes V_{n_1} \oplus U_{n_2} \otimes V_{n_2}.$$ 

Let

$$U_{AB} \otimes V = U_{n_0} \otimes V_{n_0} \oplus U_{n_1} \otimes V_{n_1} \oplus U_{n_2} \otimes V_{n_2}.$$ 

denote the corresponding vector spaces.

**Definition 1** Let $Q_0$ be the projection on $U_{n_0} \otimes V_{n_0}$.

Let $p_{n_1} \in 4$ be an element such that $V(p_{n_1})$ transposes $|x\rangle_{y_1}$ and $|x\rangle_{y_2}$ for $x = 0, 1$ and $y = A, B$. Let $q_1$ be the projection on the subset of $V_{n_1}$ which is fixed by $p_{n_1}$.

**Definition 2** Let $Q_1$ be $Q \otimes q_1$ where $Q$ is the projection on $U_{n_1}$.

Let $p_{AB} \in 4$ be an element such that $V(p_{AB})$ transposes $|x\rangle_{A_1}$ and $|x\rangle_{B_2}$ for $x = 0, 1$ and $y = 1, 2$. Let $q_2$ be the projection on the subset of $V_{n_2}$ which is fixed by $p_{AB}$. Let $q_3$ be the projection on the subset of $V_{n_3}$ which is $-1$-scaled by both $p_{n_1}$ and $p_{AB}$.

**Definition 3** Let $Q_2$ be $Q \otimes q_2$. Let $Q_3$ be $Q \otimes q_3$.

**Proof of Theorem 4** Since $\alpha(T, \sigma_0^{\otimes 2}) = 0$ (level-zero) and since the partial trace $\operatorname{Tr}_{\mathcal{H}_B}(\sigma_0^{\otimes 2})$ over $\mathcal{H}_B$ is $4^{-1} I_{\mathcal{H}_A}$, it holds that $\operatorname{Tr}_{\mathcal{H}_B}(T_0) = I_{\mathcal{H}_A}$. Since $T_0$ is $G$-invariant, it also holds that

$$\frac{1}{2} \sum_{i=1}^{4} f_i = \langle \phi | \operatorname{Tr}_{\mathcal{H}_B}(T_B) | \phi \rangle = 1$$

where

$$f_i = \operatorname{Tr}(M_{A_1}^{i} M_{A_2}^{i}) = \operatorname{Tr}(M_{B_1}^{i} M_{B_2}^{i}),$$

$$|\phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_{A_1} |0\rangle_{A_2} + |1\rangle_{A_1} |1\rangle_{A_2}).$$

The type 2 error for $\rho = \sigma^{\otimes 2}$ is of the form

$$\beta(T, \rho) = \sum_{i=1}^{4} \lambda_i (a(\sigma) f_i^2 + b(\sigma) f_i + c(\sigma))$$

where

$$a(\sigma) = \frac{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2}{15} + \frac{6}{15} (3x_2^2 + 3y_1^2 + 3y_3^2),$$

$$\sigma = \frac{1}{\sqrt{-1}} \begin{pmatrix} x_1 & x_{12} & x_{13} & x_{14} \\ x_{12} & x_2 & x_{23} & x_{24} \\ x_{13} & x_{23} & x_3 & x_{34} \\ x_{14} & x_{24} & x_{34} & x_4 \end{pmatrix} + \begin{pmatrix} -y_1 & -y_{12} & -y_{13} & -y_{14} \\ y_{12} & y_1 & -y_{23} & -y_{24} \\ y_{13} & y_{23} & y_3 & -y_{34} \\ y_{14} & y_{24} & y_{34} & y_4 \end{pmatrix},$$

and $b(\sigma)$ and $c(\sigma)$ are some functions of $\sigma$. Therefore, $\beta(T, \rho)$ is minimized by $q = 4$, $\lambda_1 = \cdots = \lambda_4 = 1$ and $f_1 = \cdots = f_4 = 1/2$. For example,

$$M_{A_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad M_{B_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad M_{A_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_{B_2} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

give the UMP-C$_4$. 

$\square$
Corollary 4  The UMP-$C_4$ test in Theorem 4 is in $\mathcal{T}_{C_4}$.

Corollary 5  The type 2 error of the UMP-$C_4$ test is given by

$$^t\mathbf{v}R\mathbf{v} - \frac{2}{15}(x_{23}^2 + y_{12}^2 + y_{13}^2)$$

where

$$
\mathbf{v} = \begin{pmatrix}
x_1 - 1/2 \\
x_2 - 1/2 \\
x_3 - 1/2 
\end{pmatrix}, \quad 
R = \frac{1}{15} \begin{pmatrix} 6 & 7 & 7 \\ 7 & 6 & 7 \\ 7 & 7 & 6 \end{pmatrix}
$$

References


