On Characteristic Equations of Analytic Functions of Many Independent Variables.

By

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In my previous paper,[1] I investigated the property of an analytic function $f(z_1, z_2, \dots\dots\dots, z_n)$ of n independent variables $z_1, z_2, \dots\dots\dots, z_n$ which has an algebraic addition-theorem:

$$(1)\quad F(f_{11}, f_{12}, f_{21}, \dots\dots\dots\dots, f_{n2}, f_0)=0,$$

where F is an irreducible polynomial of $f_{11}, f_{12}, \dots\dots\dots, f_{n2}, f_0$, containing all of them explicitly, and

$$f_{ij}\equiv f(z_{11}+z_{12}, \dots\dots\dots, z_{i-11}+z_{i-12}, z_{ij}, z_{i+11}+z_{i+12}, \dots\dots\dots, z_{n1}+z_{n2}),$$
$$(i=1, 2, \dots\dots\dots, n;\ j=1, 2)$$
$$f_0\equiv f(z_{11}+z_{12}, \dots\dots\dots, z_{n1}+z_{n2}).$$

We now define a characteristic equation of an analytic function $f(z_1, z_2, \dots\dots\dots\dots, z_n)$ of n independent variables $z_1, z_2, \dots\dots\dots\dots, z_n$ as follows:—

$$(2)\quad G(f_{11}, f_{12}, f_{21}, \dots\dots\dots\dots, f_{n2}, f_0)=0,$$

1 These memoirs, vol. VI, p. 251.

where G is an irreducible polynomial of $f_{11}, f_{12}, \dots\dots\dots f_{n2}, f_0$, containing, all of them explicitly, and

$$f_{ij} \equiv f\left(\frac{z_{11}+z_{12}}{2}, \dots\dots\dots, \frac{z_{i-11}+z_{i-12}}{2}, z_{ij}, \frac{z_{i+11}+z_{i+12}}{2}, \dots\dots\dots, \frac{z_{n1}+z_{n2}}{2}\right),$$

$$(i=1, 2, \dots\dots\dots, n\,;\, j=1, 2)$$

$$f_0 \equiv f\left(\frac{z_{11}+z_{12}}{2}, \dots\dots\dots\dots\dots, \frac{z_{n1}+z_{n2}}{2}\right).$$

Then, in the case of *single* independent variable, *any analytic function which has an algebraic addition-theorem has also a characteristic equation, and conversely.*[1] The following is an outline of the proof given by M. Falk.

I. An analytic function which has an addition-theorem has also a characteristic equation.

Let

$$(3) \quad F(f(u), f(v), f(u+v)) = 0$$

be an algebraic addition-theorem of an analytic function $f(z)$. Then the equation (3) holds for certain branches of $f(z)$, i.e.

$$(4) \quad F(f_i(u), f_j(v), f_k(u+v)) = 0,$$

where f_i, f_j and f_k represent certain branches of f. By analytic continuations along suitably chosen paths in the u- and v- planes, we may obtain, from (4),

$$F(f_l(u'), f_l(v'), f_k(u'+v')) = 0,$$

where f_l is a branch of f. Put $u'=v'=\frac{u+v}{2}$. Then

$$(5) \quad F\left(f_l\left(\frac{u+v}{2}\right), f_l\left(\frac{u+v}{2}\right), f_k(u+v)\right) = 0.$$

[1] M. Falk: Ueber die Haupteigenschaften der-jenigen analytischen Funktionen eines Arguments, welche Additionstheoreme besitzen. (Nova Acta Regloe Societatis Scientiarum Upsaliensis. Ser. IV., Vol. 1, N. 8).

Eliminating $f_k(u+v)$ from (4) and (5), we have

$$(6)\quad G\left(f_i(u),\ f_j(v),\ f_l\left(\frac{u+v}{2}\right)\right)=0,$$

where $G(f_i, f_j, f_l)$ is an irreducible polynomial of f_i, f_j and f_l. Thus we have a characteristic equation

$$G\left(f(u),\ f(v),\ f\left(\frac{u+v}{2}\right)\right)=0. \qquad \text{Q.E.D.}$$

II. An analytic function which has a characteristic equation has also an addition-theorem.

Let

$$(7)\quad G\left(f(u),\ f(v),\ f\left(\frac{u+v}{2}\right)\right)=0$$

be a characteristic equation of an analytic function $f(z)$.

(i) Suppose that $z=0$ be a regular point of $f(z)$. Let u, v, u', v' be any four values of z such that $u+v=u'+v'$. Then, by eliminating $f\left(\frac{u+v}{2}\right)\equiv f\left(\frac{u'+v'}{2}\right)$ from (7) and

$$(8)\quad G\left(f(u'),\ f(v'),\ f\left(\frac{u'+v'}{2}\right)\right)=0,$$

we have

$$H(f(u),\ f(v),\ f(u'),\ f(v'))=0,$$

where H is an irreducible polynomial of its arguments f's. As $z=0$ is a regular point, $\lim f(v')=b$ is finite and determinate, so that we may write down H in the form

$$\begin{aligned} H(f(u),\ f(v),\ f(u'),\ f(v'))\equiv & F(f(u),\ f(v),\ f(u')) \\ & +F_1(f(u),\ f(v),\ f(u')),\ (f(v')-b)+\cdots\cdots\cdots \\ & +F_r(f(u),\ f(v),\ f(u')),\ (f(v')-b)^r=0, \end{aligned}$$

where F's are polynomials of $f(u)$, $f(v)$, $f(u')$ and v' is a point in the vicinity of $v'=0$. As H is irreducible, $F(f(u), f(v), f(u')\not\equiv 0$. In the limit $v'=0$, we have $u'=u+v$ and accordingly

$$F(f(u), f(v), f(u+v))=0. \qquad \text{Q.E.D.}$$

(ii) Suppose that $z=0$ be a singular point of $f(z)$. It may be easily proved that there is a value $z=a$ of z such that $f(z)$ is regular in the vicinity of $z=a$ and $z=2a$.

Put $\qquad u=u'+a,\ v=v'+a,$

and

$$f(u)\equiv f(u'+a)\equiv g(u'),\quad f(v)\equiv g(v').$$

Then the new origin is a regular point of $g(u')$, and we have, by (7),

$$G\left(g(u'),\ g(v'),\ g\left(\frac{u'+v'}{2}\right)\right)=0,$$

since $\dfrac{(u'+a)+(v'+a)}{2}=\dfrac{u'+v'}{2}+a.$ Accordingly, by (i),

$$F(g(u'),\ g(v'),\ g(u'+v'))=0,$$

or as $\qquad g(u')\equiv f(u'+a)\equiv f(u),\ g(v')\equiv f(v),$ and $g(u'+v')\equiv f(u+v-a),$

$$(8)\quad F(f(u), f(v), f(u+v-a))=0,$$

since $v=2a$ is a regular point of $f(v)$, by putting $v=2a$ and then by replacing u by $u+v-a$, we have

$$(10)\quad F(f(u+v-a), f(2a), f(u+v))=0,$$

where $f(u+v-a)$ will represent by analytic continuation as in §1, the same branch of f as that in (9) and accordingly they are identical. Eliminating $f(u+v-a)$ from (9) and (10), we have

$$F_1(f(u), f(v), f(u+v))=0. \qquad \text{Q.E.D.}$$

Now, what is the case of *many* independent variables? This is our present question, and the answer is as follows:—

Theorem. In order that an analytic function $f(z_1, z_2, \ldots\ldots\ldots, z_n)$ *of* n *independent variables* $z_1, z_2, \ldots\ldots\ldots, z_n$ *has a characteristic equation, it is necessary and sufficient that* $f(z_1, z_2, \ldots\ldots\ldots\ldots, z_n)$ *is an algebraic function of*

$$(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots\ldots\ldots\ldots\ldots(z_{r_1}+a_{r_1})^{p_{r_1}}, \quad \ldots\ldots\ldots\ldots,$$

$$(z_{r_{s-1}+1}+a_{r_{s-1}+1})^{p_{r_{s-1}+1}}\ldots\ldots\ldots\ldots\ldots\ldots(z_{r_s}+a_{r_s})^{p_{r_s}},$$

$$e^{\alpha_1(z_{r_s+1}+a_{r_s+1})\ldots\ldots\ldots\ldots(z_{r_{s+1}}+a_{r_{s+1}})} \quad \ldots\ldots\ldots\ldots\ldots,$$

$$e^{\alpha_{t-s}(z_{r_{t-1}+1}+a_{r_{t-1}+1})\ldots\ldots\ldots(z_{r_t}+a_{r_t})},$$

$$\wp((z_{r_t+1}+a_{r_t+1})\ldots\ldots\ldots\ldots(z_{r_{t+1}}+a_{r_{t+1}}), \quad \omega_1^{(1)}, \omega_2^{(1)}),$$

$\ldots\ldots\ldots\ldots\ldots$, *where the coefficients aud* $a_1, a_2, \ldots\ldots\ldots\ldots\ldots, a_{r_{t+1}}, \ldots\ldots\ldots$, *are constants (zero being iucluded),* $p_1, p_2, \ldots\ldots\ldots, p_{r_s}$ *are integers positive or negative (zero being excluded), and* $\alpha_1, \ldots\ldots\ldots, \alpha_{t-s}, \omega_1^{(1)}, \omega_2^{(1)}, \ldots\ldots\ldots$ *are constants (zero being excluded and* $\frac{\omega_2^{(1)}}{\omega_1^{(1)}}, \ldots\ldots\ldots$ *being not real).*

As $\frac{(z_{i1}+a_i)+(z_{i2}+a_i)}{2}=\frac{z_{i1}+z_{i2}}{2}+a_i$, we may assume, without loss of generality, that all a's are zero and may thus prove the second part of the theorem analogously as in §§3–7 of my paper, *loc. cit.* For the first part of the theorem, the analogous proof to §§1–2, §§8–21 and §§23–24 of the same paper may hold; but since $\frac{(z_{i1}+a_i)+(z_{i2}+a_i)}{2}=\frac{z_{i1}+z_{i2}}{2}+a_i$, as before, it is unnecessary that all a's $e^{\alpha_{i-s+1}(z_{r_i+1}+a_{r_i+1})\ldots\ldots\ldots\ldots(z_{r_{i+1}}+a_{r_{i+1}})}$, $(i=s,\ s+1,\ \ldots\ldots\ldots,\ t-1)$, and in $\wp((z_{r_i+1}+a_{r_i+1})\ldots\ldots\ldots(z_{r_{i+1}}+a_{r_{i+1}}),\ \omega_1^{(i-t+1)}, \omega_2^{(i-t+1)})$, $(i=t, \ldots\ldots)$, are zero and that all a's in $(z_{r_i+1}+a_{r_i+1})^{p_{r_i+1}}\ldots\ldots\ldots\ldots\ldots\ldots(z_{r_{i+1}}+a_{r_{i+1}})^{p_{r_{i+1}}}$, $(i=0,\ 1,\ \ldots\ldots\ldots\ldots,\ s-1;\ r_0=0)$, are or are not zero simultaneously. Accordingly, *an analytic function of many independent variables which has an algebraic addition-theorem has also a characteristic equation, but the converse does not hold.*