Projective Differential Geometry of Non-Developable Surfaces. II.

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CHAPTER. III.

RECIPROCAL LINES.

1. Consider points

$$\left.\begin{aligned}
\rho&=\frac{\partial x}{\partial u}+\alpha x,\\
\sigma&=\frac{\partial x}{\partial v}+\beta x,\\
y&=D\sigma-D'\rho,\\
z&=D'\sigma-D''\rho,\\
\tau&=\frac{1}{2}\left\{\frac{\partial y}{\partial v}-\frac{\partial z}{\partial v}+\beta y-\alpha z\right\}+\gamma x,
\end{aligned}\right\}\quad(1)$$

where α, β, γ are analytic functions of u, v in the domain R.

The points ρ, σ, y and z are on the tangent plane of the given surface at the point x.

From (1) we have

$$|\, x\ \rho\ \sigma\ \tau \,| = -1.$$

Therefore, the points x, ρ, σ, τ are not coplanar. In other words, the point τ does not lie on the tangent plane at x.

By the same method as in **chap. II.** we can see that (x_i, y_i, z_i, τ_i) $(i=1, 2, 3, 4)$ are four independent systems of the solutions of the completely integrable system of the linear partial differential equations

$$
\left.
\begin{aligned}
&\frac{\partial x}{\partial u}=-D'y+Dz-\alpha x,\\
&\frac{\partial x}{\partial v}=-D''y+D'z-\beta x,\\
&\frac{\partial y}{\partial u}=\left\{D'(B-\frac{\partial D}{\partial v}+D'\alpha)-D(C''-\frac{\partial D''}{\partial u}+D'\beta)\right\}y\\
&\qquad-\left\{D'(C-\frac{\partial D}{\partial u}+D\alpha)-D(B-\frac{\partial D'}{\partial u}+D\beta)\right\}z-\mathfrak{A}_{\alpha,\beta}x,\\
&\frac{\partial y}{\partial v}=\tau+\left\{D''(B-\frac{\partial D}{\partial v}+D'\alpha)-D'(C''-\frac{\partial D''}{\partial u}+D'\beta)\right\}y\\
&\qquad-\left\{D''(C-\frac{\partial D}{\partial u}+D\alpha)-D'(B-\frac{\partial D'}{\partial u}+D\beta)\right\}z\\
&\qquad-(\mathfrak{A}'_{\alpha,\beta}+\gamma)x,\\
&\frac{\partial z}{\partial u}=-\tau+\left\{D'(C''-\frac{\partial D'}{\partial v}+D''\alpha)-D(B''-\frac{\partial D''}{\partial v}+D''\beta)\right\}y\\
&\qquad-\left\{D'(B-\frac{\partial D}{\partial v}+D'\alpha)-D(B-\frac{\partial D}{\partial u}+D'\beta)\right\}z\\
&\qquad-(\mathfrak{A}''_{\alpha,\beta}-\gamma)x,\\
&\frac{\partial z}{\partial v}=\left\{D''(C''-\frac{\partial D'}{\partial v}+D''\alpha)-D'(B''-\frac{\partial D''}{\partial v}+D''\beta)\right\}y\\
&\qquad-\left\{D''(B-\frac{\partial D}{\partial v}+D'\alpha)-D'(C''-\frac{\partial D''}{\partial u}+D'\beta)\right\}z-\mathfrak{A}''_{\alpha,\beta}x,\\
&\frac{\partial \tau}{\partial u}=\alpha\tau+(\mathfrak{A}_{\alpha,\beta}-\mathfrak{L}_{\alpha,\beta})(D''y-D'z)\\
&\qquad-(\mathfrak{A}'_{\alpha,\beta}-\mathfrak{L}'_{\alpha,\beta}+\theta_{x,\beta}+\gamma-\theta)(D'y-Dz)\\
&\qquad-\left\{\frac{\partial}{\partial v}\mathfrak{A}_{\alpha,\beta}-\frac{\partial}{\partial u}(\mathfrak{A}'_{\alpha,\beta}+\gamma)+\frac{1}{2}\left|\mathfrak{A}_{\alpha,\beta}\ D_u\ D\right|\right.\\
&\qquad\quad-\left|\mathfrak{A}_{\alpha,\beta}\ \mathfrak{C}\ D\right|-\left(\frac{\partial\alpha}{\partial v}-\frac{\partial\beta}{\partial u}\right)\left(\frac{1}{2}\frac{\partial D}{\partial v}-\frac{1}{2}\frac{\partial D'}{\partial u}\right.\\
&\qquad\quad\left.\left.+D\beta-D'\alpha\right)+2\gamma\alpha\right\}x.\\
&\frac{\partial \tau}{\partial v}=\beta\tau+(\mathfrak{A}'_{\alpha,\beta}-\mathfrak{L}'_{\alpha,\beta}+\theta-\theta_{\alpha,\beta}-\gamma)(D''y-D'z)\\
&\qquad-(\mathfrak{A}''_{\alpha,\beta}-\mathfrak{L}''_{\alpha,\beta})(D'y-Dz)\\
&\qquad-\left\{\frac{\partial}{\partial v}(\mathfrak{A}'_{\alpha,\beta}-\gamma)-\frac{\partial}{\partial u}\mathfrak{A}''_{\alpha,\beta}+\frac{1}{2}\left|\mathfrak{A}_{\alpha,\beta}D_v\ D\right|\right.\\
&\qquad\quad-\left|\mathfrak{A}_{\alpha,\beta}\ \mathfrak{B}\ D\right|-\left(\frac{\partial\alpha}{\partial v}-\frac{\partial\beta}{\partial u}\right)\left(\frac{1}{2}\frac{\partial D'}{\partial v}-\frac{1}{2}\frac{\partial D''}{\partial u}\right.\\
&\qquad\quad\left.\left.+D'\beta-D''\alpha\right)+2\gamma\beta\right\}x,
\end{aligned}
\right\}\quad(2)
$$

where

$$\mathfrak{A}_{\alpha,\beta}=A+B\alpha-C\beta-D\left(\frac{\partial\beta}{\partial u}-\alpha\beta\right)+D'\left(\frac{\partial\alpha}{\partial u}-\alpha^2\right),$$

$$\mathfrak{A}'_{\alpha,\beta}=A'+B'\alpha-C'\beta-\frac{1}{2}D\left(\frac{\partial\beta}{\partial v}-\beta^2\right)+\frac{1}{2}D''\left(\frac{\partial\alpha}{\partial u}-\alpha^2\right)$$
$$+\frac{1}{2}D'\left(\frac{\partial\alpha}{\partial v}-\frac{\partial\beta}{\partial u}\right),$$

$$\mathfrak{A}''_{\alpha,\beta}=A''+B''\alpha-C''\beta-D'\left(\frac{\partial\beta}{\partial v}-\beta^2\right)+D''\left(\frac{\partial\alpha}{\partial v}-\alpha\beta,\right),$$

$$\mathfrak{L}_{\alpha,\beta}=L+2\mathfrak{B}\alpha-2\mathfrak{C}\beta;$$

$$\mathfrak{L}'_{\alpha,\beta}=L'+2\mathfrak{B}'\alpha-2\mathfrak{C}'\beta,$$

$$\mathfrak{L}''_{\alpha,\beta}=L''+2\mathfrak{B}''\alpha-2\mathfrak{C}''\beta,$$

$$\theta_{\alpha,\beta}=\frac{1}{2}\frac{\partial^2 D}{\partial v^2}-\frac{\partial^2 D'}{\partial u\partial v}+\frac{1}{2}\frac{\partial^2 D''}{\partial u^2}-\frac{1}{4}\begin{vmatrix}D_u & D_v & D\end{vmatrix}+\frac{\partial}{\partial v}\left(D\beta-D'\alpha\right)$$
$$-\frac{\partial}{\partial u}\left(D'\beta-D''\alpha\right).$$

In virtue of (1), higher derivatives of x may be expressed by linear functions of x, y, z, τ.

Let

$$\frac{\partial^{i+j}x}{\partial u^i\partial v^j}=p_{ij}\tau+q_{ij}y+r_{ij}z+s_{ij}x.$$

Then we have from (1)

$$p_{20}=-D,\quad p_{11}=-D',\quad p_{20}=-D'',$$
$$q_{20}=-(2C'-B-D\beta),$$
$$r_{20}=C-D\alpha,$$
$$q_{11}=-(C''-D'\beta),$$
$$r_{11}=(B-D'\alpha),$$
$$q_{02}=-(B''-D''\beta),$$
$$r_{02}=(2B'-C''-D''\alpha),$$
$$s_{20}=D'\mathfrak{A}_{\alpha,\beta}-D\left(\mathfrak{A}'_{\alpha,\beta}-\gamma\right)-\frac{\partial\alpha}{\partial u}+\alpha^2,$$
$$s_{11}=\frac{1}{2}\left(D''\mathfrak{A}_{\alpha,\beta}-D\mathfrak{A}''_{\alpha,\beta}-\frac{\partial\alpha}{\partial v}-\frac{\partial\beta}{\partial u}+2\alpha\beta+2D'\gamma\right),$$

$$
\left.
\begin{aligned}
s_{02}&=D''\left(\mathfrak{A}'_{\alpha,\beta}+\gamma\right)-D'\mathfrak{A}''_{\alpha,\beta}-\frac{\partial\beta}{\partial v}+\beta^2,\\
p_{30}&=-\left(C+\frac{\partial D}{\partial u}\right),\\
p_{21}&=-\left(B+\frac{\partial D'}{\partial v}\right),\\
r_{12}&=-\left(C''+\frac{\partial D'}{\partial v}\right),\\
r_{03}&=-\left(B''+\frac{\partial D''}{\partial v}\right),\\
q_{30}&=\frac{\partial}{\partial u}\left(p_{21}+\frac{\partial D}{\partial v}\right)-\beta p_{30}+2q_{20}(\mathfrak{B}D'-D\mathfrak{C}'')+2r_{20}(\mathfrak{B}D''-\mathfrak{C}''D')\\
&\quad-DD'(\theta-\theta_{\alpha,\beta}-2\gamma)+D(D''\mathfrak{L}_{\alpha,\beta}-D'\mathfrak{L}'_{\alpha,\beta})-(Ds_{11}+D's_{20})\\
&\quad-(Dq_{20}q_{11}+D'q_{20}r_{11}+D'q_{11}r_{20}+D''r_{20}r_{11}),\\
r_{30}&=-\frac{\partial}{\partial u}\left(p_{30}+\frac{\partial D}{\partial u}\right)+ap_{30}+2q_{20}(D\mathfrak{B}-D'\mathfrak{C})\\
&\quad+2r_{20}(D'\mathfrak{B}-\mathfrak{C}D'')+D^2(\theta-\theta_{\alpha,\beta}-2\gamma)-D(D'\mathfrak{L}_{\alpha,\beta}-D\mathfrak{L}'_{\alpha,\beta})\\
&\quad+2Ds_{20}+Dq_{20}^2+2D'q_{20}r_{20}+D''r_{20}^2,\\
q_{21}&=\frac{\partial}{\partial v}\left(p_{21}+\frac{\partial D}{\partial v}\right)-\beta p_{21}+2q_{20}(\mathfrak{C}''D'-D\mathfrak{B}'')+2r_{20}(\mathfrak{C}''D''\\
&\quad-D'\mathfrak{B}'')-DD''(\theta-\theta_{\alpha,\beta}-2\gamma)+D(D''\mathfrak{L}'_{\alpha,\beta}-D\mathfrak{L}''_{\alpha,\beta})\\
&\quad-(D''s_{20}+Ds_{02})-(Dq_{20}q_{02}+D'q_{20}r_{02}+D'r_{20}q_{02}+D''r_{20}r_{02})\\
&=\frac{\partial}{\partial u}\left(p_{12}+\frac{\partial D'}{\partial v}\right)-\beta p_{21}+2q_{11}(\mathfrak{B}D'-D\mathfrak{C}''),\\
&\quad+2r_{11}(\mathfrak{B}D''-D'\mathfrak{C}'')-D'^2(\theta-\theta_{\alpha,\beta}-2\gamma)+D'(D''\mathfrak{L}_{\alpha,\beta}-D'\mathfrak{L}'_{\alpha,\beta})\\
&\quad-2D's_{11}-(Dq_{11}^2+2D'q_{11}r_{11}+D''r_{11}^2),\\
r_{21}&=-\frac{\partial}{\partial v}\left(p_{30}+\frac{\partial D}{\partial u}\right)+ap_{12}+2q_{20}(D\mathfrak{C}''-D'\mathfrak{B})+2r_{20}(D'\mathfrak{C}''\\
&\quad-D''\mathfrak{B})+DD'(\theta-\theta_{\alpha,\beta}-2\gamma)-D(D'\mathfrak{L}'_{\alpha,\beta}-D\mathfrak{L}''_{\alpha,\beta})\\
&\quad+D's_{20}+Ds_{11}+Dq_{20}q_{11}+D'q_{20}r_{11}+D'q_{11}r_{20}+D''r_{11}r_{20},\\
q_{12}&=\frac{\partial}{\partial u}\left(r_{03}+\frac{\partial D''}{\partial v}\right)-\beta p_{12}+2q_{02}(D'\mathfrak{B}-D\mathfrak{C}'')+r_{02}(D''\mathfrak{B}-D'\mathfrak{C}'')\\
&\quad-D'D''(\theta-\theta_{\alpha,\beta}-2\gamma)+D''(D''\mathfrak{L}_{\alpha,\beta}-D'\mathfrak{L}'_{\alpha,\beta})\\
&\quad-(D's_{02}+D''s_{11})-(Dq_{11}q_{02}+D'r_{11}q_{02}+D'q_{11}r_{02}+D''r_{11}r_{02}),
\end{aligned}
\right\}\quad(3)
$$

$$r_{12}=-\frac{\partial}{\partial u}\left(p_{12}+\frac{\partial D''}{\partial u}\right)+\alpha p_{12}+2q_{02}(D\mathfrak{B}-D'\mathfrak{C})+2r_{02}(D'\mathfrak{B}$$
$$-D''\mathfrak{C}'')+DD''(\theta-\Theta_{\alpha,\beta}-2\gamma)-D''(D\mathfrak{L}_{\alpha,\beta}-D\mathfrak{L}'_{\alpha,\beta})$$
$$+(Ds_{02}+D''s_{20})+Dq_{20}q_{02}+D'q_{02}r_{20}+D'q_{20}r_{02}+D''q_{20}r_{20}$$
$$=-\frac{\partial}{\partial v}\left(p_{21}+\frac{\partial D'}{\partial u}\right)+\alpha p_{12}+2q_{11}(D\mathfrak{C}''-D'\mathfrak{B})+2r_{11}(D'\mathfrak{C}''$$
$$-D''\mathfrak{B})+D'^2(\theta-\Theta_{\alpha,\beta}-2\gamma)-D'(D'\mathfrak{L}'_{\alpha,\beta}-D\mathfrak{L}''_{\alpha,\beta})$$
$$+2D's_{11}+Dq_{11}^2+2D'q_{11}r_{11}+D''\gamma_{11}^2,$$

$$q_{03}=\frac{\partial}{\partial v}\left(p_{03}+\frac{\partial D''}{\partial v}\right)-\beta p_{03}+q_{02}(D'\mathfrak{C}''-D\mathfrak{B}'')+2r_{02}(D''\mathfrak{C}''$$
$$-D'\mathfrak{B}'')-D''^2(\theta-\Theta_{\alpha,\beta}-2\gamma)+D''(D''\mathfrak{L}'_{\alpha,\beta}-D'\mathfrak{L}''_{\alpha,\beta})$$
$$-2D''s_{02}-(Dq_{02}^2+2D'q_{02}r_{02}+D''r_{02}^2),$$

$$r_{03}=-\frac{\partial}{\partial v}\left(p_{12}+\frac{\partial D''}{\partial u}\right)+\alpha p_{03}+2q_{02}(D\mathfrak{C}''-D'\mathfrak{B})+2r_{02}(D'\mathfrak{C}''$$
$$-D''\mathfrak{B})+D'D''(\theta-\Theta_{\alpha,\beta}-2\gamma)-D''(D'\mathfrak{L}'_{\alpha,\beta}-D\mathfrak{L}''_{\alpha,\beta})$$
$$+D's_{02}+D''s_{11}+Dq_{11}q_{02}+D'r_{11}q_{02}+D'r_{02}q_{11}+D''r_{11}r_{02},$$

$$p_{40}=\frac{\partial}{\partial u}p_{30}+\alpha p_{30}-r_{30},$$

$$p_{31}=\frac{\partial}{\partial v}p_{30}+\beta p_{30}+q_{30},$$

$$p_{22}=\frac{\partial}{\partial v}p_{21}+\beta p_{21}+q_{21},$$

$$p_{13}=\frac{\partial}{\partial u}p_{03}+\alpha p_{03}-r_{03},$$

$$p_{04}=\frac{\partial}{\partial v}p_{03}+\beta p_{03}+q_{03}.$$

In the vicinity of a point (u, v) in the domain R, we have

$$x\,(u+du,\,v+dv)$$
$$=x(u,v)+\left(du\frac{\partial}{\partial u}+dv\frac{\partial}{\partial u}\right)x(u,v)$$
$$+\frac{1}{2}\left(du\frac{\partial}{\partial u}+dv\frac{\partial}{\partial v}\right)^2 x(u,v)$$
$$+\frac{1}{6}\left(du\frac{\partial}{\partial u}+dv\frac{\partial}{\partial v}\right)^3 x(u,v)$$
$$+\frac{1}{24}\left(du\frac{\partial}{\partial u}+dv\frac{\partial}{\partial u}\right)^4 x(u,v)$$
$$+\cdots\cdots\cdots$$

$$
\left.\begin{aligned}
&= x\left[1-\kappa+\frac{1}{2}\begin{Bmatrix} s \\ 2 \end{Bmatrix}+\cdots\cdots\right] \\
&\quad +y\left[-D'du-D''dv+\frac{1}{2}\begin{Bmatrix} q \\ 2 \end{Bmatrix}+\frac{1}{6}\begin{Bmatrix} q \\ 3 \end{Bmatrix}+\cdots\cdots\right. \\
&\quad +z\left[Ddu+D'dv+\frac{1}{2}\begin{Bmatrix} r \\ 2 \end{Bmatrix}+\frac{1}{2}\begin{Bmatrix} r \\ 3 \end{Bmatrix}+\cdots\cdots\right] \\
&\quad +\tau\left[-\frac{1}{2}(Ddu^2+2D'du\,dv+D''dv^2+\frac{1}{6}\begin{Bmatrix} p \\ 3 \end{Bmatrix}\right. \\
&\qquad \left.+\frac{1}{24}\begin{Bmatrix} p \\ 4 \end{Bmatrix}+\cdots\cdots\right] \\
&= x\left[1-\kappa+\frac{1}{2}\begin{Bmatrix} s \\ 2 \end{Bmatrix}+\cdots\cdots\right] \\
&\quad +\rho\left[du-\frac{1}{2}\left(D'\begin{Bmatrix} q \\ 2 \end{Bmatrix}+D''\begin{Bmatrix} r \\ 2 \end{Bmatrix}\right)-\frac{1}{6}\left(D'\begin{Bmatrix} q \\ 3 \end{Bmatrix}+D''\begin{Bmatrix} r \\ 3 \end{Bmatrix}\right)\right. \\
&\qquad \left.+\cdots\cdots\right] \\
&\quad +\sigma\left[dv+\frac{1}{2}\left(D\begin{Bmatrix} q \\ 2 \end{Bmatrix}+D'\begin{Bmatrix} r \\ 2 \end{Bmatrix}\right)+\frac{1}{6}\left(D\begin{Bmatrix} q \\ 2 \end{Bmatrix}+D'\begin{Bmatrix} r \\ 3 \end{Bmatrix}\right)\right. \\
&\qquad \left.+\cdots\cdots\right] \\
&\quad +\tau\left[-\frac{1}{2}(Ddu^2+2D'du\,dv+D''dv^2)+\frac{1}{6}\begin{Bmatrix} p \\ 3 \end{Bmatrix}+\frac{1}{24}\begin{Bmatrix} p \\ 4 \end{Bmatrix}\right. \\
&\qquad \left.+\cdots\cdots\right],
\end{aligned}\right\} \quad (3)
$$

where

$$\kappa = \alpha\,du+\beta dv,$$

$$\begin{Bmatrix} p \\ n \end{Bmatrix} = \sum_{i=0}^{n} {}_nC_i p_{n-i,i} du^{n-i} dv^i,$$

etc.

Let w_i ($i=0, 1, 2 \ 3, 8$) be projective coordinates referred to the tetrahedron of reference with vertices at the points x, ρ, σ, τ. In virtue of (4), projective coordinates of any point of the given surface in the vicinity of the point x (u, v) referred to this tetrahedron of reference, are given by the equations

$$\left.\begin{aligned}
w_0&=1-\kappa+\frac{1}{2}\left\{\begin{matrix} s \\ 2 \end{matrix}\right\}+\cdots\cdots,\\
w_1&=du-\frac{1}{2}\left(D'\left\{\begin{matrix} q \\ 2 \end{matrix}\right\}+L''\left\{\begin{matrix} r \\ 2 \end{matrix}\right\}\right)-\frac{1}{6}\left(L'\left\{\begin{matrix} q \\ 3 \end{matrix}\right\}+L''\left\{\begin{matrix} r \\ 3 \end{matrix}\right\}\right)\\
&\quad+\cdots\cdots,\\
w_2&=dv+\frac{1}{2}\left(D\left\{\begin{matrix} q \\ 2 \end{matrix}\right\}+L'\left\{\begin{matrix} r \\ 2 \end{matrix}\right\}\right)+\frac{1}{6}\left(D\left\{\begin{matrix} q \\ 3 \end{matrix}\right\}+L'\left\{\begin{matrix} r \\ 3 \end{matrix}\right\}\right)\\
&\quad+\cdots\cdots,\\
w_3&=-\frac{1}{2}(Ddu^2+2L'du\,dv+D''dv^2)+\frac{1}{6}\left\{\begin{matrix} p \\ 3 \end{matrix}\right\}+\frac{1}{24}\left\{\begin{matrix} p \\ 4 \end{matrix}\right\}\\
&\quad+\cdots\cdots,
\end{aligned}\right\}\qquad(5)$$

if the unit point be properly chosen.

Let us now introduce non-homogeneous coordinates by putting

$$\xi=\frac{w_1}{w_0},\quad \eta=\frac{w_2}{w_0},\quad \zeta=\frac{w_3}{w_0}.$$

Then we have from (5)

$$\left.\begin{aligned}
\xi&=du-\frac{1}{2}\left(L'\left\{\begin{matrix} q \\ 2 \end{matrix}\right\}+L''\left\{\begin{matrix} r \\ 2 \end{matrix}\right\}\right)+\kappa du-\frac{1}{6}\left(D'\left\{\begin{matrix} q \\ 3 \end{matrix}\right\}+D''\left\{\begin{matrix} r \\ 3 \end{matrix}\right\}\right)\\
&\quad-\frac{\kappa}{2}\left(L'\left\{\begin{matrix} q \\ 2 \end{matrix}\right\}+L''\left\{\begin{matrix} r \\ 2 \end{matrix}\right\}\right)-du\left(\frac{1}{2}\left\{\begin{matrix} s \\ 2 \end{matrix}\right\}-\kappa^2\right)+\cdots\cdots,\\
\eta&=dv+\frac{1}{2}\left(D\left\{\begin{matrix} q \\ 2 \end{matrix}\right\}+L'\left\{\begin{matrix} r \\ 2 \end{matrix}\right\}\right)+\kappa dv+\frac{1}{6}\left(D\left\{\begin{matrix} q \\ 3 \end{matrix}\right\}+D'\left(\begin{matrix} r \\ 3 \end{matrix}\right\}\right)\\
&\quad+\frac{\kappa}{2}\left(D\left\{\begin{matrix} q \\ 2 \end{matrix}\right\}+L'\left\{\begin{matrix} r \\ 2 \end{matrix}\right\}\right)-dv\left(\frac{1}{2}\left\{\begin{matrix} s \\ 2 \end{matrix}\right\}-\kappa^2\right)+\cdots\cdots,\\
\zeta&=-\frac{1}{2}(Ddu^2+2D'du\,dv+D''dv^2)+\frac{1}{6}\left\{\begin{matrix} p \\ 3 \end{matrix}\right\}-\frac{\kappa}{2}(Ddu^2\\
&\quad+2D'dudv+D''dv^2)+\frac{1}{24}\left\{\begin{matrix} p \\ 4 \end{matrix}\right\}+\frac{\kappa}{6}\left\{\begin{matrix} p \\ 3 \end{matrix}\right\}+\frac{1}{2}\left(\frac{1}{2}\left\{\begin{matrix} s \\ 2 \end{matrix}\right\}\right.\\
&\quad\left.-\kappa^2\right)(Ddu^2+2D'dudv+D''dv^2)+\cdots\cdots,
\end{aligned}\right\}\qquad(6)$$

Therefore, we can develope ζ in a paver series of ξ, η of the form

$$\left.\begin{aligned}\zeta=&a_{20}\xi^2+a_{11}\xi\eta+a_{02}\eta^2\\&+a_{30}\xi^3+a_{21}\xi^2\eta+a_{12}\xi\eta^2+a_{03}\eta^3\\&+a_{40}\xi^4+a_{31}\xi^3\eta+a_{22}\xi^2\eta^2+a_{13}\xi\eta^3+a_{04}\eta^4+\cdots\cdots\end{aligned}\right\}(7)$$

Substituting the values of ξ, η, ζ given by (6) in (7), and equating the coefficients of the terms of the same degree with respect to u and v in both sides of the resulting equation, we can determine the coefficients in (7).

First, comparing the terms of the second degree with respect to u and v in both sides of (7), we have

$$a_{20}=-\frac{D}{2},\quad a_{11}=-D',\quad a_{02}=-\frac{D''}{2}.$$

Next, comparing the terms of the third degree with respect to u and v in both sides of (7), we have

$$a_{30}du^3+a_{21}du^2dv+a_{12}dudv^2+a_{03}dv^3=\frac{1}{6}\left(3\left\{\begin{matrix}r\\2\end{matrix}\right\}du-3\left\{\begin{matrix}q\\2\end{matrix}\right\}dv\right.$$

$$\left.+\left\{\begin{matrix}p\\3\end{matrix}\right\}\right)+\frac{\kappa}{2}(Ddu^2+2D'dudv+D''dv^2).$$

Therefore, in virtue of (3), we get

$$a_{30}=\frac{1}{3}\mathfrak{C},\quad a_{21}=\mathfrak{B},\quad a_{12}=\mathfrak{C}'',\quad a_{03}=\frac{1}{3}\mathfrak{B}''.$$

Finally, comparing the terms of the third degree with respect to u and v in both sides of (7), we have

$$a_{40}du^4+a_{31}du^3dv+a_{22}du^2dv^2+a_{13}dudv^3+a_{04}dv^4$$

$$=\frac{1}{24}\left(4\left\{\begin{matrix}r\\3\end{matrix}\right\}du-4\left\{\begin{matrix}q\\3\end{matrix}\right\}dv+\left\{\begin{matrix}p\\4\end{matrix}\right\}\right)$$

$$-\frac{1}{4}\left\{\begin{matrix}s\\2\end{matrix}\right\}(Ddu^2+2D'dudv+D''dv^2)$$

$$-\frac{1}{8}\left(D\left\{\begin{matrix}q\\2\end{matrix}\right\}^2+2D'\left\{\begin{matrix}q\\2\end{matrix}\right\}\left\{\begin{matrix}r\\2\end{matrix}\right\}+D''\left\{\begin{matrix}r\\2\end{matrix}\right\}^2\right.$$

$$-\frac{\kappa}{6}\left[\left\{\begin{matrix}p\\3\end{matrix}\right\}+2(\mathfrak{C}du^3+3\mathfrak{B}du^2dv+3\mathfrak{C}''dudv^2+\mathfrak{B}''dv^3)\right]$$

$$-\frac{1}{2}\left\{\begin{matrix} q \\ 2 \end{matrix}\right\}\begin{vmatrix} D & \mathfrak{C} & dv^2 \\ D' & \mathfrak{C}' & -dudv \\ D'' & \mathfrak{C}'' & du \end{vmatrix} -\frac{1}{2}\left\{\begin{matrix} r \\ 2 \end{matrix}\right\}\begin{vmatrix} D & \mathfrak{B} & dv^2 \\ D' & \mathfrak{B}' & -dudv \\ D'' & \mathfrak{B}'' & du^2 \end{vmatrix}.$$

In virtue of (3), the right hand side of this equation is equal to

$$\frac{1}{12}\left(du\frac{\partial}{\partial u}+dv\frac{\partial}{\partial v}-4\kappa\right)\left(\mathfrak{C}du^3+3\mathfrak{B}du^2dv+3\mathfrak{C}''du^2dv+\mathfrak{B}''dv^3\right)$$

$$+\frac{1}{8}\left\{Ddu^2+2D'du\,dv+D''dv^2\right\}$$

$$\times\left\{\frac{\partial}{\partial v}\begin{vmatrix} D & \mathfrak{C} \\ D' & \mathfrak{C}' \end{vmatrix}du^2+\frac{\partial}{\partial v}\begin{vmatrix} D & \mathfrak{C} \\ D'' & \mathfrak{C}'' \end{vmatrix}dudv+\frac{\partial}{\partial v}\begin{vmatrix} D' & \mathfrak{C}' \\ D'' & \mathfrak{C}'' \end{vmatrix}dv^2\right.$$

$$\left.-\frac{\partial}{\partial u}\begin{vmatrix} D & \mathfrak{B} \\ D' & \mathfrak{B}' \end{vmatrix}du^2-\frac{\partial}{\partial u}\begin{vmatrix} D & \mathfrak{B} \\ D'' & \mathfrak{B}'' \end{vmatrix}dudv-\frac{\partial}{\partial u}\begin{vmatrix} D' & \mathfrak{B}' \\ D'' & \mathfrak{B}'' \end{vmatrix}dv^2\right\}$$

$$-\frac{1}{8}(\theta_{\alpha,\beta}+2\gamma)(Ddu^2+2Ddu\,dv+D''dv^2)^2$$

$$-\frac{1}{8}\left\{\frac{\partial D}{\partial v}du^2+2\frac{\partial D'}{\partial v}du\,dv+\frac{\partial D''}{\partial v}dv^2\right\}\begin{vmatrix} D & \mathfrak{C} & dv^2 \\ D' & \mathfrak{C}' & -du\,dv \\ D'' & \mathfrak{C}'' & du^2 \end{vmatrix}$$

$$+\frac{1}{8}\left\{\frac{\partial D}{\partial u}du^2+2\frac{\partial D'}{\partial u}du\,dv+\frac{\partial D''}{\partial u}dv^2\right\}\begin{vmatrix} D & \mathfrak{B} & dv^2 \\ D' & \mathfrak{B}' & -du\,dv \\ D'' & \mathfrak{B}'' & du^2 \end{vmatrix}$$

$$+\frac{1}{4}\left\{D\,du^2+2\,D'\,du\,dv+D''\,dv^2\right\}\begin{vmatrix} D & \mathfrak{L}_{\alpha,\beta} & dv^2 \\ D' & \mathfrak{L}_{\alpha,\beta} & -du\,dv \\ D'' & \mathfrak{L}_{\alpha,\beta} & dv^2 \end{vmatrix}.$$

Therefore, denoting φ_n the terms of n-th the degree with respect to ξ and η in the development of ζ, we have

$$
\left.\begin{aligned}
\varphi_2 &= -\frac{1}{2}(D\xi^2+2D'\xi\eta+D''\eta^2),\\
\varphi_3 &= \frac{1}{3}(\mathfrak{C}\xi^3+3\mathfrak{B}\xi^2\eta+3\mathfrak{C}''\xi\eta^2+\mathfrak{B}''\eta^3),\\
\varphi_4 &= \frac{1}{12}\left\{\xi\left(\frac{\partial\mathfrak{C}}{\partial u}\xi^2+3\frac{\partial\mathfrak{B}}{\partial u}\xi^2\eta+3\frac{\partial\mathfrak{C}''}{\partial u}\xi\eta^2+\frac{\partial\mathfrak{B}''}{\partial u}\eta^3\right)\right.\\
&\qquad\left.\eta\left(\frac{\partial\mathfrak{C}}{\partial v}\xi^3+3\frac{\partial\mathfrak{B}}{\partial v}\xi^2\eta+3\frac{\partial\mathfrak{C}''}{\partial v}\xi\eta^2+\frac{\partial\mathfrak{B}''}{\partial v}\eta^3\right)\right\}\\
&\quad -\frac{1}{3}\left(\alpha\xi+\beta\eta\right)\varphi_3-\frac{1}{2}\left(\theta_{\alpha,\beta}+2\gamma\right)\varphi_2^2\\
&\quad -\frac{1}{4}\varphi_2\left\{\frac{\partial}{\partial v}\begin{vmatrix} D & \mathfrak{C}\\ D' & \mathfrak{C}'\end{vmatrix}\xi^2+\frac{\partial}{\partial v}\begin{vmatrix} D & \mathfrak{C}\\ D'' & \mathfrak{C}''\end{vmatrix}\xi\eta+\frac{\partial}{\partial v}\begin{vmatrix} D' & \mathfrak{C}'\\ D'' & \mathfrak{C}''\end{vmatrix}\eta^2\right.\\
&\qquad\left.-\frac{\partial}{\partial u}\begin{vmatrix} D & \mathfrak{B}\\ D' & \mathfrak{B}'\end{vmatrix}\xi^2-\frac{\partial}{\partial u}\begin{vmatrix} D & \mathfrak{B}\\ D'' & \mathfrak{B}''\end{vmatrix}\xi\eta-\frac{\partial}{\partial u}\begin{vmatrix} D' & \mathfrak{B}'\\ D'' & \mathfrak{B}''\end{vmatrix}\eta^2\right\}\\
&\quad -\frac{1}{8}\left(\frac{\partial D}{\partial v}\xi^2+2\frac{\partial D'}{\partial v}\xi\eta+\frac{\partial D''}{\partial v}\eta^2\right)\begin{vmatrix} D & \mathfrak{C} & \eta^2\\ D' & \mathfrak{C}' & -\xi\eta\\ D'' & \mathfrak{C}'' & \xi^2\end{vmatrix}\\
&\quad +\frac{1}{8}\left(\frac{\partial D}{\partial u}\xi^2+2\frac{\partial D'}{\partial u}\xi\eta+\frac{\partial D''}{\partial u}\eta^2\right)\begin{vmatrix} D & \mathfrak{B} & \eta^2\\ D' & \mathfrak{B}' & -\xi\eta\\ D'' & \mathfrak{B}'' & \xi^2\end{vmatrix}\\
&\quad -\frac{1}{2}\varphi^2\begin{vmatrix} D & \mathfrak{L}_{\alpha,\beta} & \eta^2\\ D' & \mathfrak{L}'_{\alpha,\beta} & -\xi\eta\\ D'' & \mathfrak{L}''_{\alpha,\beta} & \xi^2\end{vmatrix}
\end{aligned}\right\}\quad(9)
$$

2. From (9) we can see that any quadric which has contact of the second order with the give surface at the point x is of the form

$$\zeta=\varphi_2+\zeta(\lambda\xi+\mu\eta+\nu\zeta),$$

referring to the non-homogeneous coordinates introduced in **n° 1,** which give, for the development of ζ to the terms of the fourth degree inclusive,

$$\zeta=\varphi_2+\varphi_2(\lambda\xi+\mu\eta)+\varphi_2(\lambda\xi+\mu\eta)2+\nu\varphi_2^2.$$

The projection from the point τ of the curve of the intersection of the ξ-η plane, *i. e.*, the tangent plane at x, has for its equation

$$o=\varphi_3-\varphi_2(\lambda\xi+\mu\eta)+\varphi_4-\varphi_2(\lambda\xi+\mu\eta)^2-\nu\varphi_2^2+\cdots\cdots\cdots$$

Hence this curve has a triple point at the origin and the tangents at this point are given by the equation

$$\varphi_3-\varphi_2(\lambda\xi+\mu\eta)=o.$$

For these three tangents to coincide, λ and μ must satisfy the equation

$$\left.\begin{aligned}&\frac{1}{3}(\mathfrak{C}k^3+3\mathfrak{B}k^2+3\mathfrak{C}''k+\mathfrak{B}'')+\frac{1}{2}(Dk^2+2D'k+D'')(\lambda k+\mu)=0,\\&\mathfrak{C}k^2+2\mathfrak{B}k+\mathfrak{C}''+(Dk+D')(\lambda k+\mu)+\frac{1}{2}\lambda(Dk^2+2D'k+D'')=0,\\&\mathfrak{C}k+\mathfrak{B}+\frac{1}{2}D(\lambda k+\mu)+\lambda(Dk+D')=0,\end{aligned}\right\}(10)$$

where $k=\dfrac{\xi}{y}$.

The system of the equations (10) is equivalent to the system

$$\left.\begin{aligned}&\mathfrak{C}k+\mathfrak{B}+\frac{1}{2}D(\lambda k+\mu)+\lambda(Dk+D')=0,\\&\mathfrak{B}k+\mathfrak{C}''+\frac{1}{2}D'(\lambda k+\mu)+\frac{1}{2}\mu(Dk+D')+\frac{1}{2}\lambda(D'k+D'')=0,\\&\mathfrak{C}''k+\mathfrak{B}''+\frac{1}{2}D''(\lambda k+\mu)+\mu(D'k+D'')=0.\end{aligned}\right\}(11)$$

Multiply the first equation (11) by D'', the second of it by $-2D'$, the third of it by D, and add them, then we have, in virtue of the equations (11) and (22) **chap. 1,**

$$(\lambda k+\mu)=o.$$

Therefore the system (11) is equivalent to the system

$$\left.\begin{aligned}&\mathfrak{C}k^3+\mathfrak{B}k^2-\mu k(Dk+D')=0,\\&2\mathfrak{B}k^2+2\mathfrak{C}''k+\mu k(Dk+D')-\mu(D'k+D'')=0.\\&\mathfrak{C}'k+\mathfrak{B}''+\mu(D'k+D'')=0.\end{aligned}\right\}(12)$$

From (12) we have

$$\mathfrak{C}k^3+3\mathfrak{B}k^2+3\mathfrak{C}''+\mathfrak{B}''=0.$$

Therefore, we know that if the tangents at the triple point coincide, they must coincide with one of the tangents defined by the equation

$$\mathfrak{C}\xi^3+3\mathfrak{B}\xi^2\eta+3\mathfrak{C}''\xi\eta^2+\mathfrak{B}''\eta^3=0,$$

which are called *the tangents of the quadric osculatin* by Darboux[1]. Hence the equation

$$\mathfrak{C}du^3+3\mathfrak{B}du^2dv+3\mathfrak{C}''dudv^3+\mathfrak{B}''dv^3=0$$

gives the Darboux's *curve of the quadric osculation.*

3. If the tangents at the said triple point themselves are those of the quadric osculation, λ and μ must be equal to zero. Therefore, the quadric Q given by the equation

$$\zeta=\varphi_2+\nu\zeta^2 \quad \ldots\ldots\ldots(13)$$

has the following properties :—

1° It has contact of the second order with the given surface S at the point x, and accordingly, intersects with S at a curve of which the point x is a triple point.

2° The tangents of the curve of intersection of S and Q at the point x are those of the quadric osculation.

We shall call this quadric the *semi-canonical qaudric.* From (13) we know that there are ∞^1 semi-canonical quadrics at a point of S.

Referring to the homogeneous co-ordinates introduced in **n°1**, the equation of the semi-cononical quadric Q is

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[1] Bull. sci. math. Fr. ser. 2 vol. 4 p. 385 (1880).

$$w_3 w_0 = -\frac{1}{2}(Dw_1^2 + 2D'w_1w_2 + D''w_2^2) + \nu w_3^2.$$

The polar line of the point $(o,\ a_1,\ a_2,\ o)$ with respect to Q is

$$(a_1D + a_2D')w' + (a_1D' + a_2D'')w_2 = 0$$

and that of the point $(b_0,\ 0,\ 0,\ b_3)$ is

$$b_3w_0 + b_0w_3 - 2\nu b_3 w_3 = 0.$$

Therefore either of the lines $\rho\ \sigma$ ane $x\ \tau$ in the reciprocal polar of the other with respect to Q. We shall speak of the lines $\rho\ \sigma$ and $x\ \tau$ as the reciprocal line of the other.

4. When the point x moves on the given sufface the lines $\rho\ \sigma$ and $x\ \tau$ form congruences which are called respectively the congruence Γ and the congruence Γ' by Green[1].

The points y and z lie on the line $\rho\ \sigma$. Hence if the point x moves so that line $\rho\ \sigma$ describes a developable surface, the points y, z, dy and dz necessarily lie in a plane ; accordingly, in virtue of (2),

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ dv & 0 & 0 & -\mathfrak{A}_{\alpha,\beta}du - (\mathfrak{A}'_{\alpha,\beta} + \gamma)dv \\ -du & 0 & 0 & -(\mathfrak{A}'_{\alpha,\beta} - \gamma)du - \mathfrak{A}''_{\alpha,\beta}dv \end{vmatrix}$$

$$= -(\mathfrak{A}_{\alpha,\beta}du^2 + 2\mathfrak{A}'_{\alpha,\beta}dudv + \mathfrak{A}''_{\alpha,\beta}dv^2) = 0 \qquad (14)$$

Therefore the net of curves given by (14) corresponds to the developable sulfaces of the congruence Γ.

The necessary and sufficient condition that this net may from a conjugate net is

$$D\mathfrak{A}''_{\alpha,\beta} - 2D'\mathfrak{A}'_{\alpha,\beta} + D''\mathfrak{A}_{\alpha,\beta} = -\left(\frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial u}\right) = 0,$$

or

$$\frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial u}.$$

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[1] Trans. Amer Math. Soc **20**, p. 79.

5. Let the point

$$\phi = y + \lambda z$$

be one of the focal points on the line $\rho\ \sigma$.

Then all the surfaces of the congruence Γ which contain the line $\rho\ \sigma$ have the tangent plane at ϕ common. In other words the points $\left(y, z, \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right)$, and accordingly, in virtue of (2), the points

$$\begin{gathered} y, \quad z, \\ \lambda\tau - \left\{\mathfrak{A}_{\alpha,\beta} + \lambda(\mathfrak{A}'_{\alpha,\beta} + \gamma)\right\}x, \\ -\tau - \left\{(\mathfrak{A}_{\alpha,\beta} - \gamma) + \lambda\mathfrak{A}''_{\alpha,\beta}\right\}x \end{gathered}$$

must be coplanar. Therefore, λ must satisfy the equation

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & \mathfrak{A}_{\alpha,\beta} - \lambda(\mathfrak{A}_{\alpha,\beta} + \gamma) \\ -1 & 0 & 0 & -(\mathfrak{A}'_{\alpha,\beta} - \lambda\gamma) - \lambda\mathfrak{A}''_{\alpha,\beta} \end{vmatrix} = 0,$$

or

$$\lambda^2\mathfrak{A}''_{\alpha,\beta} + 2\lambda\mathfrak{A}_{\alpha,\beta} + \mathfrak{A}_{\alpha,\beta} = 0,$$

6. If the point x moves so that line $x\ \tau$ describes a developable surface of the congruence Γ', the points x, τ, dx and $d\tau$ and acc ordingly, invirtue of (2) the points

$$\left.\begin{aligned} & x, \qquad \tau, \qquad \rho\, du + \sigma\, dv, \\ & \left\{(\mathfrak{A}'_{\alpha,\beta} - \mathfrak{L}'_{\alpha,\beta})du + (\mathfrak{A}''_{\alpha,\beta} - \mathfrak{L}''_{\alpha,\beta})dv\right\}\rho \\ & \qquad - \left\{(\mathfrak{A}_{\alpha,\beta} - \mathfrak{L}_{\alpha,\beta})du - (\mathfrak{A}'_{\alpha,\beta} - \mathfrak{L}'_{\alpha,\beta})du\right\}\sigma \end{aligned}\right\} \quad (15)$$

must be coplanar. But the points x, ρ, σ, τ are not coplanar. Therefore, the last two points of (15) must coincide with the point of the intersection of the line $\rho\ \sigma$ and the plane on which points (15) lie. Accordingly, we must have

$$(\mathfrak{A}_{\alpha,\beta} - \mathfrak{L}_{\alpha,\beta})du^2 + 2(\mathfrak{A}'_{\alpha,\beta} - \mathfrak{L}'_{\alpha,\beta})du\,dv + (\mathfrak{A}''_{\alpha,\beta} - \mathfrak{L}''_{\alpha,\beta})dv^2 = 0 \ldots\ldots (16)$$

Therefore, the net of the curves defined by (16) corresponds to the developable surfaces of the congruence Γ'.

The necessary and sufficient condition that this net form a conjugate net is

$$D(\mathfrak{A}''_{\alpha,\beta}-\mathfrak{L}''_{\alpha,\beta})-2D'(\mathfrak{A}'_{\alpha,\beta}-\mathfrak{L}'_{\alpha,\beta})+D''(\mathfrak{A}''_{\alpha,\beta}-\mathfrak{L}''_{\alpha,\beta})$$

$$=-\left(\frac{\partial\alpha}{\partial v}-\frac{\partial\beta}{\partial u}\right)=0,$$

or

$$\frac{\partial\alpha}{\partial v}=\frac{\partial\beta}{\partial u}.$$

In the same manner as in n°**5**, we can see that in order that the point $\tau+\mu x$ may be one of the focal points on the line $x\tau$, the points

$$x, \qquad \tau,$$

$$(\mathfrak{A}'_{\alpha,\beta}-\mathfrak{L}'_{\alpha,\beta}+\mu-\theta+\theta_{\alpha,\beta}+\gamma)\rho-(\mathfrak{A}_{\alpha,\beta}-\mathfrak{L}_{\alpha,\beta})\sigma,$$

$$(\mathfrak{A}''_{\alpha,\beta}-\mathfrak{L}''_{\alpha,\beta})\rho+\left\{\mu-\theta+\theta_{\alpha,\beta}+\gamma-(\mathfrak{A}'_{\alpha,\beta}-\mathfrak{L}'_{\alpha,\beta})\right\}$$

must be coplaner. Accordingly,

$$\mu=\theta-\theta_{\alpha,\beta}-\gamma\pm\sqrt{(\mathfrak{A}'_{\alpha,\beta}-\mathfrak{L}'_{\alpha,\beta})^2-(\mathfrak{A}_{\alpha,\beta}-\mathfrak{L}_{\alpha,\beta})(\mathfrak{A}''_{\alpha,\beta}-\mathfrak{L}''_{\alpha,\beta})}.$$

The harmonic conjugate of the point x with respect to the focal points is the point

$$\tau+(\theta-\theta_{\alpha,\beta}-\gamma)x.$$

If the two focal points coincide, this point itself is the focal point.

7. Now let us suppose the parameter curves to be asymptotic curves. In this case, have

$$D=D''=0 \qquad \cdots\cdots(17)$$

Accordingly, in virtue of (21), (26) **chap. 1**,

$$\left.\begin{aligned}&\mathfrak{C}'=\mathfrak{B}=\mathfrak{B}'=\mathfrak{C}''=\mathfrak{A}'=0,\\&D'^2=1.\end{aligned}\right\}\cdots\cdots(18)$$

From the last equation of (18) we have

$$D'=\pm1.$$

If $D'=-1$, then carry out the transformation

$$\overline{\mathfrak{X}_i}=e^{\frac{i\pi}{4}}\mathfrak{X}_i.$$

By this transformation, we have

$$\overline{D'}=-D'.$$

Therefore, we may suppose, without loss of generality, D' to be equal to 1.

If we make this supposition, we have

$$\left.\begin{array}{lll} B=\mathfrak{B}=0, & B'=\mathfrak{B}'=0, & \\ C'=\mathfrak{C}'=0, & C''=\mathfrak{C}''=0, & \\ B''=\mathfrak{B}'' & C=\mathfrak{C}, & \\ \theta=B''C, & & \\ L=-\dfrac{\partial C}{\partial v}, & L'=0, & L''=\dfrac{\partial B''}{\partial u}, \\ \mathfrak{L}=-\dfrac{\partial C}{\partial v}-\dfrac{\theta_v}{\theta}C, & \mathfrak{L}'=0, & \mathfrak{L}''=\dfrac{\partial B''}{\partial u}+\dfrac{\theta_u}{\theta}B'', \end{array}\right\}\quad(19)$$

from (17), (18) and their definitions.

Let us denote by R_u the ruled surface formed by tangents to the curve C_u ($v=$const.), and by R_v the similar parametric ruled surface.

The tangent plane at the point σ to R_v is determined by the points x, σ, $\dfrac{\partial\sigma}{\partial u}$. But we have from (2), (17), (18),

$$\frac{\partial\sigma}{\partial u}=\frac{\partial z}{\partial u}=-\tau-(\mathfrak{A}'_{\alpha,\beta}-\gamma)x.$$

Therefore, this plane contains the point τ. Similarly the tangent plane at the point ρ to R_u contains the point τ. Therefore, *the two tangents to R_u and R_v at the points ρ and σ respectively intersect at the line $x\ \tau$.* Green has defined reciprocal lines by this property.

In virtue of (17) and the definitions of the fundamental determinants, the normal co-ordinates x_i ($i=1, 2, 3, 4$) are solutions of the system of equations

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial u^2} &= C\frac{\partial x}{\partial v}+Ax,\\ \frac{\partial^2 x}{\partial v^2} &= B''\frac{\partial x}{\partial u}-A''x.\end{aligned}\right\}\cdots\cdots\;\cdots\cdots(20)$$

Accordingly, if the given surface is a ruled surface,

$$B'' \text{ or } C=0,$$

in other words,

$$\theta=0.$$

But θ is a relative invariant. Therefore, we know that *the necessary and sufficient condition for the ruled surface is* $\theta=0$.

From (17) and (20) we know that when

$$\alpha=-\frac{1}{2}\frac{B''_u}{B}, \quad \beta=-\frac{1}{2}\frac{C_v}{C},$$

the lines $\rho\ \sigma$ and $x\ \tau$ are the directrices[1] of the first and second kind respectively, and that when

$$\alpha=\frac{1}{4}\frac{C''_u}{C}, \quad \beta=\frac{1}{4}\frac{B''_v}{B},$$

the lines $\rho\ \sigma$ and $x\ \tau$ are the canonical edges[2] of the first and second kind respectively.

From the equations (9), (16), (17), (23), (33), and (36) in the **chap. 1,** and (1) in this chapter, we can easily see that when

$$\alpha = \lambda\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta},$$

$$\beta = \lambda\frac{|\mathfrak{L}\ \mathfrak{B}\ D|}{\theta}+\frac{1}{2}\frac{\theta_v}{\theta},$$

$$\gamma = \mu\theta+\nu\theta,$$

where λ, μ, ν are constants, the line $\rho\ \sigma$ is an invariant line and the

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1 Wilcynshi, Trans. Amer. Math Soc. (**16**) 311 (1916).

2 Green, loc. cit.

point τ is an invariant point. But in virtue of (17) and (19) when the parameter-curves are asymptotic lines,

$$-\frac{1}{2}\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta}=-\frac{1}{2}\frac{B''_u}{B},$$

$$-\frac{1}{2}\frac{|\mathfrak{L}\ \mathfrak{B}\ D|}{\theta}+\frac{1}{2}\frac{\theta_v}{\theta}=-\frac{1}{2}\frac{C_v}{C},$$

$$-\frac{1}{4}\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta}=\frac{1}{4}\frac{C''_u}{C},$$

$$-\frac{1}{4}\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_v}{\theta}=\frac{1}{4}\frac{B''_v}{B}.$$

Therefore we know that, *referring to any parameter-curves, the lines $\rho\ \sigma$ and $x\ \tau$ are directrices of the first and second respectively, if*

$$\alpha=-\frac{1}{2}\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta},$$

$$\beta=-\frac{1}{2}\frac{|\mathfrak{L}\ \mathfrak{B}\ D|}{\partial}+\frac{1}{2}\frac{\theta_v}{\theta}$$

and they are canonical edges of the first and second kind respectively, if

$$\alpha=-\frac{1}{4}\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta},$$

$$\beta=-\frac{1}{4}\frac{|\mathfrak{L}\ \mathfrak{B}\ D|}{\theta}+\frac{1}{2}\frac{\theta_v}{\theta}.$$

8. Next, let us suppose that the parameter-curves form a conjugacte net.

In this case, we have

$$D'=0, \qquad\qquad (23)$$

accordingly,

$$\left.\begin{aligned}&D\ D''=-1,\\ &\frac{B}{D}=-\frac{B''}{D''},\ \frac{C}{D}=-\frac{C''}{D''},\ \frac{A}{D}=-\frac{A''}{D''}.\end{aligned}\right\}(24)$$

Let us put

$$\left.\begin{aligned} l=\frac{B}{D}=-\frac{B''}{D''},\quad m=-\frac{C}{D}=\frac{C''}{D''},\\ n=-\frac{A}{D}=\frac{A''}{D''}. \end{aligned}\right\}(25)$$

Then

$$\left.\begin{aligned} \frac{\mathfrak{B}}{D}=-\frac{\mathfrak{B}''}{D''}=\left(l-\frac{1}{2}\frac{\partial\, log\, D}{\partial v}\right),\\ \frac{\mathfrak{C}}{D}=-\frac{\mathfrak{C}''}{D''}=-\left(m+\frac{1}{2}\frac{\partial\, log\, D}{\partial u}\right). \end{aligned}\right\}(26)$$

In virtue of the definition of the normal fundamental determinants, the normal co-ordinates x_i (i=1, 2, 3, 4) are the solutions of the equation

$$\frac{\partial^2 x}{\partial u\partial v}=l\frac{\partial x}{\partial u}+m\frac{\partial x}{\partial v}+nx.,$$

Therefore, if

$$\alpha=-m,\quad \beta=-l,$$

the line $\rho\ \sigma$ is the ray[1] of the point x.

Now let us put

$$H=-\frac{\mathfrak{A}_{-m,-l}}{D},\qquad H'=\mathfrak{A}'_{-m,-l},\qquad H''=\frac{\mathfrak{A}''_{-m,-l}}{D''}.$$

Then

$$\left.\begin{aligned} &H=n+l-\frac{\partial l}{\partial u},\\ &H'=A'-\mathfrak{C}''m+\mathfrak{B}l+\frac{1}{2}D\left(\frac{\partial l}{\partial v}+l^2\right)-\frac{1}{2}D''\left(\frac{\partial m}{\partial u}+m^2\right),\\ &H''=n+lm-\frac{\partial m}{\partial v}. \end{aligned}\right\}(27)$$

Therefore, H and H'' are Laplace-Darboux invariants.

From (14) and (27) we know that the ray-curve[2] is given by

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[1], [2] Wilcynshi, loci. cit.

the equation

$$DH du^2 - 2H' du dv - D'' H'' = 0.$$

The condition for the conjugacy of the ray curves is

$$\frac{\partial m}{\partial v} = \frac{\partial l}{\partial u},$$

which is equivalent to

$$H - H'' = 0.^{(1)}$$

If

$$\alpha = m + \frac{\partial \log D}{\partial u}, \quad \beta = l - \frac{\partial \log D}{\partial v},$$

then we have from (13), (23), (24), (25) and (26)

$$q_{20} = 0, \ r_{20} = 2\mathfrak{C}, \ q_{02} = -2\mathfrak{B}'', \ r_{02} = 0, \ldots\ldots\ldots\ldots\ldots (28)$$

$$\left.\begin{aligned}
-\frac{\mathfrak{A}_{\alpha,\beta}}{D} &= H + 2\frac{\partial}{\partial u}(D\mathfrak{B}'') + 4\mathfrak{B}''\mathfrak{C}, \\
\mathfrak{A}'_{\alpha,\beta} &= H' - \frac{\partial \mathfrak{B}}{\partial v} + \frac{\partial \mathfrak{C}''}{\partial u} + 2D''\mathfrak{B}^2 - 2D\mathfrak{C}''^2, \\
\frac{\mathfrak{A}''_{\alpha,\beta}}{D''} &= H'' + 2\frac{\partial}{\partial v}(D''\mathfrak{C}) + 4\mathfrak{B}''\mathfrak{C}, \\
-\frac{\mathfrak{L}_{\alpha,\beta}}{D} = \frac{\mathfrak{L}''_{\alpha,\beta}}{D''} &= -\frac{\partial}{\partial u}(D\mathfrak{B}'') - \frac{\partial}{\partial v}(D''\mathfrak{C}) + 4\mathfrak{B}''\mathfrak{C}, \\
\mathfrak{L}'_{\alpha,\beta} &= \frac{\partial \mathfrak{C}''}{\partial u} - \frac{\partial \mathfrak{B}}{\partial v} + 2D''\mathfrak{B}^2 - 2D\mathfrak{C}''^2.
\end{aligned}\right\} (29)$$

From (28) we know that the points τ, $\frac{\partial^2 x}{\partial u^2}$, z, x, as well as the points τ, $\frac{\partial^2 x}{\partial v^2}$, y, x, are coplanar. But, since D' is equal to zero, the plane determined by the points $\frac{\partial^2 x}{\partial u^2}$, y, x, and that determined by the points $\frac{\partial^2 x}{\partial v^2}$, y, x are osculating planes at x of the curves C_u, C_v respectively which meet at x. Therefore, the line $x\,\tau$ is the axis[2] of x.

1, 2 Wilcynshi, loc. cit.

From (16) and (29) we know that the axis-curves are given by the equation

$$D\left\{H+3\frac{\partial}{\partial u}(D\mathfrak{B}'')+\frac{\partial}{\partial v}(D''\mathfrak{C})\right\}du^2-2H'dudv$$

$$-D''\left\{H''+\frac{\partial}{\partial u}(D\mathfrak{B}'')+3\frac{\partial}{\partial v}(D\mathfrak{C}'')\right\}dv^2=0.$$

The condition for the conjugacy of the axis-curve is

$$\frac{\partial}{\partial v}\left(m+\frac{\partial \log D}{\partial u}\right)=\frac{\partial}{\partial u}\left(l-\frac{\partial \log D}{\partial v}\right),$$

which is equivalent to

$$H-H''+2\frac{\partial^2 \log D}{\partial u \partial v}=0.$$

We shall now find another property of the conjugate net with conjugate axis-curve.

Consider a cone of the second order passing through the tangents drawn to the curves C_u, C_v at a point x at which they meet.

The equation of this cone is of the form

$$a_1\zeta^2+a_2\xi\eta+a_3\xi\zeta+a_4\eta\zeta=0,$$

referring to the non-homogeneous co-ordinates introduced in **n°1.**

Let

$$\alpha=m+\frac{\partial \log D}{\partial u}, \quad \beta=l-\frac{\partial \log D}{\partial v}.$$

Then, in virtue of (6), (23), (24) and (28) non-homogeneous coordinates of points on the curves C_u and C_v passing through x, in the vicinity of x, are

$$\left.\begin{aligned}\xi&=du+\cdots\cdots,\\ \eta&=\frac{1}{6}Dq_{30}du^3+\cdots\cdots,\\ \zeta&=-\frac{1}{2}Ddu^2+\cdots\cdots,\end{aligned}\right\}\cdots\cdots(30)$$

and

$$\left.\begin{aligned}\xi &= -\frac{1}{6}D''r_{03}dv^3, \dots\dots\dots\dots\dots\dots, \\ \eta &= dv + \dots\dots\dots\dots\dots\dots\dots\dots, \\ \zeta &= -\frac{1}{2}D''dv^2 + \dots\dots \dots\dots\dots,\end{aligned}\right\}\dots\dots\dots(31)$$

respectively.

From (30) and (31) we can see that when the said cone has the contact of the third order with both of the curves C_u, C_v at the point x, we must have

$$Dr_{03}+D''q_{30}=0.$$

In virtue of (3), (23). (24) and (28), this equation is equivalent to the equation

$$H-H''+2\frac{\partial^2 \log D}{\partial u\,\partial v}=0.$$

Therefore we have the following theorem :—

With any point P on a surface S as vertex, we can draw a cone of the second degree so that it has the contact of the second order at P with both of the curves of a conjugate net N on S which meet at P, if, and only if, the axis curves of N also form a conjugate net.

To be continued.