Oriented Circles in Non-Euclidean Space, IV.

By

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1. The object of this paper is to apply to the geometry of circles the mothed of the discussion of line geometry in non-euclidean space.[1] As the Study coordinates of circles are analogous to those of the soma of the second sort[2], the present discussion is also analogous to that of the soma in non-euclidean space which was given by Prof. Nishiuchi[3].

In order to carry out this study, we shall confine ourselves to the discussion of such a system of circles that the planes of the circles pass through a fixed point P and that the power of the circles with respect to that point is constant.

Let d be the distance between the fixed point P and the center of a circle of our system whose radius is r, then the power will be

$$tan\frac{d+r}{2k}\,tan\frac{d-r}{2k}=K,$$

where K is constant.

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1 Study, 'Zur nicht euklidischen und linien Geometrie.' Jahresber. D. M. Ver., 11 (1902). Coolidge's dissertation, 'The Dual projective Geometry of Elliptic and Spherical Space,' Greifswald (1902). Beck's dissertation, 'Die Strahlenketten im hyperbolischen Raume,' Hannover (1905).

2 Study, Geometrie der Dynamen. Nishiuchi and Kashiwagi. 'Oriented Circles in Non-Euclidean Space,' Mem. Coll. Sci. Kyoto, 4 (1920).

3 'Geometry of Soma in Non-Euclidean Space,' Mem. Coll. Sci. Kyoto.

When K is positive, we have a fixed sphere orthogonal to circles of our system whose center is at P, and its radius R is given by

$$cos^2\frac{r}{k}cos^2\frac{R}{k}=cos^2\frac{d}{k}.$$

But in this case the radius of the circle which is in bi-involution with another circle will be imaginary, so for the present treatment we may abandon such a case, and take the case only where K is negative.

Here, we can put

$$\frac{cos\frac{d}{k}}{cos\frac{r}{k}}=\frac{1-K}{1+K}\left(=\frac{1}{cos\varphi}\right),$$

where φ is constant.

Suppose that the coordinates of the point P are

$$(k : 0 : 0 : 0),$$

then we have

$$a_0=k\ cos\frac{r}{k}\ sec\varphi, \quad (a\ a)=k^2,$$

$$b_0=0,\ (b\ b)=k^2,\ \ (ab)=0,$$

where (a) is the coordinates of the center of a circle and (b) those of the plane of the circle, and r is its radius.

In this case, if we take $(\mathfrak{X})$ as the Study coordinates of the circle, then we have

$$\mathfrak{X}_0=0, \quad \mathfrak{X}_j=\sqrt{-1}\frac{b_i}{k}\ cos\frac{r}{k},$$

$$\mathfrak{X}_{0i}=\frac{b_i}{k}\ cos\frac{r}{k}sec\varphi,$$

$$\mathfrak{X}_{ij}=\begin{vmatrix}\frac{a_i}{k} & \frac{a_j}{k}\\ \frac{b_i}{k} & \frac{b_j}{k}\end{vmatrix}, \quad (i,\ j=1,\ 2,\ 3).$$

And we easily see that

$$\rho\mathfrak{X}_{01}\equiv\frac{b_1}{k}\cos\frac{r}{k}\tan\varphi,\quad \rho\mathfrak{X}_{02}\equiv\frac{b_2}{k}\cos\frac{r}{k}\tan\varphi,\quad \rho\mathfrak{X}_{03}\equiv\frac{b_3}{k}\cos\frac{r}{k}\tan\varphi,$$

$$\rho\mathfrak{X}_{23}\equiv\frac{1}{k^2}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\quad \rho\mathfrak{X}_{31}\equiv\frac{1}{k^2}\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix},\quad \rho\mathfrak{X}_{12}\equiv\frac{1}{k^2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

shall be taken as the coordinates of a circle of our system.

It is evident that there exists the following relation

$$(\mathfrak{X}\,|\,\mathfrak{X})\equiv\sum{}'\mathfrak{X}_{ij}\mathfrak{X}_{kl}=0.$$

When the circle is null, we have

$$(\mathfrak{X}\mathfrak{X})\equiv\sum{}'\mathfrak{X}_{ij}\mathfrak{X}_{ij}=0.$$

Next, suppose that we have two circles of our system whose cooridnates are $(\mathfrak{X})$ and $(\mathfrak{X}')$ respectively, then their mutual power will be proportional to

$$(\mathfrak{X}\mathfrak{X}'),$$

and the cosine of the intersecting angle θ will be given by

$$\cos\theta=\frac{(\mathfrak{X}\mathfrak{X}')}{\sqrt{(\mathfrak{X}\mathfrak{X})}\sqrt{(\mathfrak{X}'\mathfrak{X}')}}.$$

The condition that they should be in involution is that

$$(\mathfrak{X}\mathfrak{X}')=0,$$

and the condition that they should be cospherical is that

$$(\mathfrak{X}\,|\,\mathfrak{X}')=0.$$

This is also the condition that the two circles intersect at two points.

Their moment and commoment will be

$$\sin\frac{d}{k}\sin\frac{k'}{k}=\frac{\tan\varphi}{k^3\sin\frac{r}{k}\sin\frac{r'}{k}}\begin{vmatrix} a_1\cos\frac{r'}{k}-a_1'\cos\frac{r}{k} & \dots & a_3\cos\frac{r'}{k}-a_3'\cos\frac{r}{k} \\ b_1 & \dots & b_3 \\ b_1' & \dots & b_3' \end{vmatrix}$$

$$=\frac{(\mathfrak{X}\,|\,\mathfrak{X}')}{\sqrt{(\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')}},$$

$$\cos\frac{d}{k}\cos\frac{d'}{k}=\frac{(\mathfrak{X}\mathfrak{X}')}{\sqrt{(\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')}}.$$

And their non-intersecting angle θ will be given by

$$(\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')\sin^4\theta+[(\mathfrak{X}\mathfrak{X}')^2-(\mathfrak{X}|\mathfrak{X}')^2-(\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')]\sin^2\theta+(\mathfrak{X}|\mathfrak{X}')^2=0.$$

And the condition that the circles should be paratactic is

$$\left\{\left((\mathfrak{X}|\mathfrak{X}')+(\mathfrak{X}\mathfrak{X}')\right)^2-(\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')\right\}\times \left\{\left((\mathfrak{X}|\mathfrak{X}')-(\mathfrak{X}\mathfrak{X}')\right)^2-(\mathfrak{X}\mathfrak{X})(\mathfrak{X}'\mathfrak{X}')\right\}=0.$$

The equations

$$\rho\mathfrak{Y}_{ij}=\lambda\mathfrak{X}_{ij}+\mu\mathfrak{X}'_{ij}$$
$$(i, j=0,\ 1,\ 2,\ 3),$$

will represent a circle when, and only when

$$(\mathfrak{X}|\mathfrak{X}')=0,$$

i. e. the two circles $(\mathfrak{X})$ and $(\mathfrak{X}')$ are cospherical or intersect at two points, and by varying $\lambda:\mu$ in this case we get all circles coaxal with $(\mathfrak{X})$ and $(\mathfrak{X}')$.

Similarly, when we have three circles $(\mathfrak{X})$, $(\mathfrak{X}')$ and $(\mathfrak{X}'')$, then the equations

$$\rho\mathfrak{Y}_{ij}=\lambda\mathfrak{X}_{ij}+\mu\mathfrak{X}'_{ij}+\nu\mathfrak{X}''_{ij}$$
$$(i, j=0,\ 1,\ 2,\ 3),$$

will represent a circle when, and only when,

$$(\mathfrak{X}'|\mathfrak{X}'')=(\mathfrak{X}''|\mathfrak{X})=(\mathfrak{X}|\mathfrak{X}')=0,$$

i. e. two by two they are cospherical, and in this case either all pass through two points, or are on one sphere.

Next, we shall find a circle of our system which is in bi-involution with the circle $(\mathfrak{X})$.

Let $(\mathfrak{X}')$ be the coordinates of the required circle, then we have

$$a'_0=\lambda a_0+\mu k=k\cos\frac{r'}{k}\ \sec\varphi,\qquad a'_i=\lambda a_i,$$

$$b'_0=0,\qquad b'_i=\nu\frac{\partial}{\partial t}|t\ a\ b|$$

$$(i=1,\ 2,\ 3),$$

where

$$\lambda=\frac{-\cos\frac{r}{k}\cos\frac{r'}{k}\sin^2\varphi}{\cos^2\varphi-\cos^2\frac{r}{k}},\quad \mu=\frac{\sin^2\frac{r}{k}\cos\frac{r'}{k}\cos\varphi}{\cos^2\varphi-\cos^2\frac{r}{k}},$$

$$\nu=\frac{\cos\varphi}{k\sqrt{\cos^2\varphi-\cos^2\frac{r}{k}}},\quad \cos^2\frac{r'}{k}=\frac{\cos^2\varphi-\cos^2\frac{r}{k}}{\sin^2\frac{r}{k}-\cos^2\frac{r}{k}\sin^2\varphi}.$$

So we have

$$\rho'\mathfrak{X}'_{0i}=\frac{b'_i}{k}\cos\frac{r'}{k}\tan\varphi=\mathfrak{X}_{jk}\frac{\sin\varphi}{\sqrt{\sin^2\frac{r}{k}-\cos^2\frac{r}{k}\sin^2\varphi}},$$

$$\rho'\mathfrak{X}'_{jk}=\frac{1}{k^2}\begin{vmatrix} a'_j & a'_k \\ b'_j & b'_k \end{vmatrix}=\mathfrak{X}_{0i}\frac{\sin\varphi}{\sqrt{\sin^2\frac{r}{k}-\cos^2\frac{r}{k}\sin^2\varphi}}.$$

2. We shall define as a *complex* of circles of our systems one, the coordinates of each member of which are porportional to analytic functions of three independent variables, the ratios not being all functions of two variables.

According to this definition, a general complex may be written by the equation

$$f(\mathfrak{X})=0.$$

If $f(\mathfrak{X})$ be linear, the complex is said to be *linear* and the simplest linear complex may be given by

$$(\mathfrak{A}\mathfrak{X})\equiv\sum\mathfrak{A}_{ij}\mathfrak{X}_{ij}=0.$$

If

$$(\mathfrak{A}|\mathfrak{A})=0,$$

the complex consists in the totality of circles cospherical with the fixed circle whose coordinates $(\mathfrak{Y})$ are

$$\mathfrak{Y}_{ij}=\rho\mathfrak{A}_{ij}.$$

A system of circles whose coordinates are proportional to analytic functions of two independent variables, their ratios not being all functions of one variable, shall be called a *congruence*. It is evedent that our circles may form a focal congruence[1] and it may be given by the two equations

$$f(\mathfrak{X})=0, \quad \varphi(\mathfrak{X})=0.$$

Circles of our system which are in bi-involution with those of the given complex

$$(\mathfrak{A}\mathfrak{X})=0,$$

will generate a second complex

$$(\mathfrak{A}\,|\,\mathfrak{X})=0.$$

The focal congruence, whose equations are

$$(\mathfrak{A}\mathfrak{X})=(\mathfrak{A}\,|\,\mathfrak{X})=0,$$

will be composed of all circles of our complex and those which are in bi-involution with them, or common to all complexes

$$l(\mathfrak{A}\mathfrak{X})+m(\mathfrak{A}\,|\,\mathfrak{X})=0.$$

These complexes shall be said to form a *coaxal system.*

In the coaxal system, there are two special complexes for which $l:m$ is given by the relation

$$\sum(l\mathfrak{A}_{ij}+m\mathfrak{A}_{kl})(l\mathfrak{A}_{kl}+m\mathfrak{A}_{ij})=0,$$

i. e.

$$(l^2+m^2)(\mathfrak{A}\,|\,\mathfrak{A})+2lm(\mathfrak{A}\mathfrak{A})=0.$$

Let the two values of l/m be l_1/m_1, l_2/m_2, then

$$\frac{l_1}{m_1}=-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1}\,,\ \frac{l_2}{m_2}=-\mathfrak{J}-\sqrt{\mathfrak{J}^2-1}\,,\ \mathfrak{J}=\frac{(\mathfrak{A}\mathfrak{A})}{(\mathfrak{A}\,|\,\mathfrak{A})},$$

and the two special complexes are given by

$$l_1(\mathfrak{A}\mathfrak{X})+m_1(\mathfrak{A}\,|\,\mathfrak{X})=0,$$
$$l_2(\mathfrak{A}\mathfrak{X})+m_2(\mathfrak{A}\,|\,\mathfrak{X})=0,$$

respectively.

If

$$\mathfrak{J}\neq 1,$$

then the two are distinct and each of the complexes will be composed

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1 Kashiwagi, Mem. Col. Sci. Kyoto, **5**, 383, **6**, 97.

of all circles common to the two complexes

$$(\mathfrak{A}\mathfrak{X})=0,\ (\mathfrak{A}\,|\,\mathfrak{X})=0,$$

and consists in the totality of circles cospherical with the fixed circles

$$\rho\alpha_{ij}=\left(-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1}\right)\mathfrak{A}_{ij}+\mathfrak{A}_{kl},$$

$$\rho\alpha'_{ij}=\mathfrak{A}_{ij}+\left(-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1}\right)\mathfrak{A}_{kl},$$

respectively.

It is immediately evident that these two are in bi-involution.

Let $(\mathfrak{X})$ be any circle of our system, then the ratio of the moment and commoment of the circle with regard to those two circles will be

$$\frac{\sin\frac{d_1}{k}\sin\frac{d_2}{k}}{\sin\frac{d'_1}{k}\sin\frac{d'_2}{k}}=\frac{\left(-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1}\right)\left(\mathfrak{A}\,|\,\mathfrak{X}\right)+\left(\mathfrak{A}\mathfrak{X}\right)}{\left(\mathfrak{A}\,|\,\mathfrak{X}\right)+\left(-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1}\right)\left(\mathfrak{A}\mathfrak{X}\right)},$$

$$\frac{\cos\frac{d_1}{k}\cos\frac{d_2}{k}}{\cos\frac{d'_1}{k}\cos\frac{d'_2}{k}}=\frac{\left(-\mathfrak{J}+\sqrt{\mathfrak{J}_2-1}\right)\left(\mathfrak{A}\mathfrak{X}\right)+\left(\mathfrak{A}\,|\,\mathfrak{X}\right)}{\left(\mathfrak{A}\mathfrak{X}\right)+\left(-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1}\right)\left(\mathfrak{A}\,|\,\mathfrak{X}\right)},$$

respectively. For a circle of our complex

$$(\mathfrak{A}\mathfrak{X})=0,$$

these ratios will be reduced to

$$-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1},$$

and

$$-\mathfrak{J}-\sqrt{\mathfrak{J}_2-1},$$

respectively.

Conversely, when one of these ratios is reduced to $-\mathfrak{J}+\sqrt{\mathfrak{J}^2-1}$ or $-\mathfrak{J}-\sqrt{\mathfrak{J}^2-1}$ the circle $(\mathfrak{X})$ will satisfy the relation

$$(\mathfrak{A}\mathfrak{X})=0,$$

that is, the circle belongs to our complex.

A pair of real circles which are in bi-involution with each other shall be said to form a *propre cross*, and the cross formed by these two circles (α) and (α') shall be *the axial cross* of the complex.

A linear complex where (α) and (α') are distinct, i. e.

$$\mathfrak{J}\neq 1$$

is said to *be general.*

CIRCLE CROSSES IN HYPERBOLIC SPACE.

3. We shall now continue the discussion of our systems of circles in the special direciton where the fundamental element is not, in general, a circle, but a pair of circles (circle cross) invariantly connected.

Let us start in the real domain of hyperbolic space, and for the sake of brevity, we put the measure of curvature of space $1/k^2$ equal to -1, and take as the coordinates of a point (x) four numbers $(\dot{x})$ such that

$$(\dot{x})=\dot{x}_0,\ \dot{x}_1,\ \dot{x}_2,\ \dot{x}_3,$$

$$-\dot{x}_0^2+\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2=-1.$$

The equation to the absolute may be written in the form

$$\dot{x}_0^2-\dot{x}_1^2-\dot{x}_2^2-\dot{x}_3^2=0$$

And for the coordinates our circle, we can take the following real numbers

$$\lambda\dot{\mathfrak{X}}_{0i}=\dot{b}_i cosh r tan\varphi,$$

$$\lambda\dot{\mathfrak{X}}_{kl}=-\begin{vmatrix}\dot{a}_k & \dot{a}_l\\ \dot{b}_k & \dot{b}_l\end{vmatrix}$$

$$(i,\ k,\ l=1,\ 2,\ 3.,\ i\neq k\neq l).$$

If $(\dot{\mathfrak{X}}')$ be the coordinates of our circle which is in bi-involution with the given circle $(\dot{\mathfrak{X}})$, then

$$\dot{\mathfrak{X}}'_{0i}=\mu\dot{\mathfrak{X}}_{kl},$$

$$\dot{\mathfrak{X}}'_{kl}=-\mu\dot{\mathfrak{X}}_{0i}.$$

Next, we shall consider a linear complex whose equation is

$$(\dot{\mathfrak{A}}\dot{\mathfrak{X}})\equiv\sum\dot{\mathfrak{A}}_{0i}\dot{\mathfrak{X}}_{0i}-\sum\dot{\mathfrak{A}}_{jk}\dot{\mathfrak{X}}_{jk}=0.$$

Then the coaxal system will be given by

$$l(\dot{\mathfrak{A}}\dot{\mathfrak{X}})+m(\dot{\mathfrak{A}}\,|\,\dot{\mathfrak{X}})=0$$

and the coordinates of the circle of the axial cross will have the form

$$\dot{\alpha}_{0i}=l\dot{\mathfrak{A}}_{0i}-m\dot{\mathfrak{A}}_{jk},\quad \dot{\alpha}_{jk}=l\dot{\mathfrak{A}}_{jk}+m\dot{\mathfrak{A}}_{0i}.$$

Let us now write

$$\begin{aligned}\dot{\mathfrak{A}}_{01}+\sqrt{-1}\,\dot{\mathfrak{A}}_{23}&=\rho X_1,\\ \dot{\mathfrak{A}}_{02}+\sqrt{-1}\,\dot{\mathfrak{A}}_{31}&=\rho X_2,\\ \dot{\mathfrak{A}}_{03}+\sqrt{-1}\,\dot{\mathfrak{A}}_{12}&=\rho X_3.\end{aligned}\qquad (1)$$

If we replace (X) by $l(X)+im(X)$, we get

$$\rho' X_i'=(l\dot{\mathfrak{A}}_{0i}-m\dot{\mathfrak{A}}_{jk})+\sqrt{-1}(l\dot{\mathfrak{A}}_{jk}+m\dot{\mathfrak{A}}_{0i}),$$

and the new linear complex

$$(\dot{\mathfrak{A}}'\dot{\mathfrak{X}})=0$$

where

$$\dot{\mathfrak{A}}'_{0i}=l\dot{\mathfrak{A}}_{0i}-m\dot{\mathfrak{A}}_{jk},\quad \mathfrak{A}'_{jk}=l\dot{\mathfrak{A}}_{jk}+m\dot{\mathfrak{A}}_{0i},$$

will be a coaxal system and have the same axial cross as the original one.

If either of the circles of a cross have the coordinates $(\dot{\mathfrak{A}})$ then the three numbers (X) given by equation (1) may be taken to represent the cross.

Suppose that we have a triad of coordinates (X), then the coordinates of the corresponding cross will be found from (1) by assuming for ρ such a value that the coordinates $(\dot{\mathfrak{A}})$ shall satisfy the identity

$$(\dot{\mathfrak{A}}\,|\,\dot{\mathfrak{A}})=0.$$

For that it is necessary and sufficient that the imaginary part of $\rho^2(XX)$ should vanish, i.e.

$$\sigma\dot{\mathfrak{A}}_{0i}=\left[\frac{X_i}{\sqrt{(XX)}}+\frac{\overline{X}_i}{\sqrt{(\overline{X}\overline{X})}}\right],$$
$$\sigma\dot{\mathfrak{A}}_{jk}=\sqrt{-1}\left[\frac{X_i}{\sqrt{(XX)}}-\frac{\overline{X}_i}{\sqrt{(\overline{X}\overline{X})}}\right]. \qquad (2)$$

with the condition

$$(XX)\neq 0, \quad (\overline{X}\overline{X})\neq 0.$$

Let the circles of a cross have the coordinates $(\dot{\alpha})$ and $(\dot{\alpha}')$, where

$$\dot{\alpha}_{0i}=-\lambda\dot{\alpha}'_{jk},$$
$$\dot{\alpha}_{jk}=\lambda\dot{\alpha}'_{0i}.$$

In the same way let a second cross be determined by the circles $(\dot{\beta})$ and $(\dot{\beta}')$. The condition that the circles $(\dot{\alpha})$ and $(\dot{\beta})$ should be cospherical, or that one should be cospherical with the mate of the other in a circle cross, is that

$$(\dot{\alpha}\,|\,\dot{\beta})=0,$$

i.e.

$$\frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}}=\pm\frac{(\overline{X}\overline{Y})}{\sqrt{(\overline{X}\overline{X})}\sqrt{(\overline{Y}\overline{Y})}}. \qquad (3)$$

They will be cospherical and in involution, if

$$(XY)=(\overline{X}\overline{Y})=0. \qquad (4)$$

We shall say that two crosses intersect orthogonally if their circles intersect and are in involution.

If the circle (α) be cospherical with the two circles (β) and (β') of a certain cross, then

$$(\dot{\alpha}\,|\,\dot{\beta})=(\dot{\alpha}\,\dot{\beta})=0,$$
$$(\dot{\alpha}\dot{\beta}')=(\dot{\alpha}\,|\,\dot{\beta}')=0.$$

And we have the following:

If a not null circle be cospherical with two cricles of a cross, it will cut each of them orthogonally and in involution with them.

Let θ_1, θ_2 and d_1, d_2 be the angles and the distances of the circles

$(\dot{\alpha})$ and $(\dot{\beta})$ respectively, then their commoment and moment will be

$$-coshd_1coshd_2=-cos\theta_1\ cos\theta_2=\frac{(\dot{\alpha}\dot{\beta})}{\sqrt{(\dot{\alpha}\dot{\alpha})}\sqrt{(\dot{\beta}\dot{\beta})}}$$

$$=\frac{1}{2}\left[\frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}}+\frac{(\overline{X}\overline{Y})}{\sqrt{(\overline{X}\overline{X})}\sqrt{(\overline{Y}\overline{Y})}}\right], \qquad (5)$$

$$sinhd_1sinhd_2=-sin\theta_1\ sin\theta_2=\frac{\sqrt{1-i}(\dot{\alpha}|\dot{\beta})}{\sqrt{(\dot{\alpha}\dot{\alpha})}\sqrt{(\dot{\beta}\dot{\beta})}}$$

$$=\frac{1}{2}\left[\frac{(XY)}{\sqrt{(XX)}\sqrt{(YX)}}-\frac{(\overline{X}\overline{Y})}{\sqrt{(\overline{X}\overline{X})}\sqrt{(\overline{Y}\overline{Y})}}\right].$$

Here, we can establish one to one correspondence between the totality of all circle crosses in hyperbolic space and the totality of all complex points (or lines) in an elliptic plane. Two intersecting crosses will correspond to points, and crosses intersecting orthogonally will correspond to orthogonal points of the elliptic plane.

The totality of crosses (U) which intersect a given cross (X) orthogonally will be given by means of the linear equation

$$(UX)=0.$$

A linear equation will be transformed linearly into another linear one, if the variables and coefficients be treated contragrediently. Now, for the sake of clearness, we shall assume our cross space is doubly overlaid. We shall say that a cross belongs to the *upper layer*, when it is represented by a complex line in the complex plane; when it is represented by a point in that plane, we shall speak of a cross of the *lower layer*.

The necessary and sufficient condition that two crosses of defferent layers should intersect orthogonally is that the corresponding point and line of the complex plane should be in united position. Thus two crosses of different layers which intersect orthogonally to each other may be represented by a complex line element (system of a point and a line through this point) after Lie.

4. The simplest one-dimensional family of crosses is the *circle-chain* composed of all crosses whose coordinates are linearly dependent, by means of real coefficients, on those of two given crosses, i. e.

$$\rho X_i = aY_i + bZ_i, \quad (i=1,\ 2,\ 3). \tag{6}$$

In the representing complex plane, we see that we have ∞^1 points of a line so related that the cross ratio of any four is real.

All crosses of the circle-chain will cut orthogonally by another cross (of the other layer) called the *axis* of the chain. Let (U) be the coordinates of the axis of the chain, then we have

$$(UY)=0,$$
$$(UZ)=0$$

and

$$U_i = \frac{\partial}{\partial t_i}\begin{vmatrix} t & Y & Z \end{vmatrix}.$$

Every circle-chain has at least one pair of real crosses, called the *principal crosses* of the chain, which intersect orthogonally. To find these, we must solve the equation

$$aa'(YY)+(ab'+a'b)(YZ)+bb'(ZZ)=0,$$
$$aa'(\bar{Y}\bar{Y})+(ab'+a'b)(\bar{Y}\bar{Z})+bb'(\bar{Z}\bar{Z})=0.$$

By the elimination of $a' : b'$ from these equations, we have a quadratic in $a : b$; i.e.

$$a^2\begin{vmatrix} (YY) & (YZ) \\ (\bar{Y}\bar{Y}) & (\bar{Y}\bar{Z}) \end{vmatrix} + ab\begin{vmatrix} (YY) & (ZZ) \\ (\bar{Y}\bar{Y}) & (\bar{Z}\bar{Z}) \end{vmatrix} + b^2\begin{vmatrix} (YZ) & (ZZ) \\ (\bar{Y}\bar{Z}) & (\bar{Z}\bar{Z}) \end{vmatrix} = 0.$$

It is easily seen that this equation has real and distinct roots. If we take the principal crosses as $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, we can express our chain in the simple form

$$X_1 = a(p+qi),\ X_2 = b(r+si),\ X_3 = 0,\ (\mathrm{i}=\sqrt{-1}). \tag{7}$$

Eliminating a/b, we get

$$(p\dot{\mathfrak{A}}_{31}+q\dot{\mathfrak{A}}_{02})(r\dot{\mathfrak{A}}_{10}-s\dot{\mathfrak{A}}_{23})=(p\dot{\mathfrak{A}}_{02}-q\dot{\mathfrak{A}}_{31})(r\dot{\mathfrak{A}}_{23}+s\dot{\mathfrak{A}}_{01}).$$

Let $(\dot{x})$ be the coordinates of a point on the circle of our system, then we have

$$-\dot{b}_0\dot{x}_0+\dot{b}_1\dot{x}_1+\dot{b}_2\dot{x}_2+\dot{b}_3\dot{x}_3=0,$$

$$(\dot{a}\dot{x})=-coshr.$$

Or

$$\lambda\dot{\mathfrak{X}}_{01}=\dot{x}_2\dot{x}_3\ sin\varphi\quad \lambda\dot{\mathfrak{X}}_{02}=-\dot{x}_1\dot{x}_3\ sin\varphi\quad \lambda\mathfrak{X}_{03}=0,$$

$$\lambda\dot{\mathfrak{X}}_{23}=\dot{x}_1(\dot{x}_0-cos\varphi),\quad \lambda\dot{\mathfrak{X}}_{31}=\dot{x}_2(\dot{x}_0-cos\varphi),\quad \lambda\mathfrak{X}_{12}=0.$$

Hence, the equation of the chain surface in point coordinates will be

$$(ps-qr)\Big[-(\dot{x}_0-cos\varphi)^2+\dot{x}_3^2 sin^2\varphi\Big]\dot{x}_1\dot{x}_2 \qquad (8)$$
$$+(pr+qs)(\dot{x}_1^2+\dot{x}_2^2)(\dot{x}_0-cos\varphi)\dot{x}_3\ sin\ \varphi=0,$$

where
$$(\dot{x}\dot{x})=-1.$$

If

$$(ps-qr)=0,\ \text{or}\ (pr+qs)=0,$$

we have two real and two imarginary circle systems.

Next, suppose that we have a focal congruence of circles of such a nature that the corresponding cross coordinates are analytic functions of two real parameters u, v, then the cross cutting orthogonally the adjacent crosses (U) and $(U+dU)$ will be determined by the equation

$$X_i=\begin{vmatrix} U_j & U_k \\ \dfrac{\partial U_j}{\partial u} & \dfrac{\partial U_k}{\partial u}\end{vmatrix}du+\begin{vmatrix} U_j & U_k \\ \dfrac{\partial U_j}{\partial v} & \dfrac{\partial U_k}{\partial v}\end{vmatrix}dv. \qquad (9)$$

If

$$\left| U\ \frac{\partial U}{\partial u}\ \frac{\partial U}{\partial v}\right|\equiv 0, \qquad (10)$$

there is but one cross cutting orthogonally.

If we take u and v as the focal parameters, then we have

$$\frac{\partial\dot{b}_i}{\partial u}=\lambda\dot{a}_i sinhr\ \frac{\partial r}{\partial u}+\lambda\frac{\partial\dot{a}_i}{\partial u}\ coshr+\mu\dot{b}_i,$$

$$\frac{\partial \dot{b}_i}{\partial v}=\lambda' \dot{a}_i \sinh r \frac{\partial r}{\partial v}+\lambda' \frac{\partial \dot{a}_i}{\partial v} \cosh r+\mu' \dot{b}_i,^{(1)}$$

$$(i=0,\ 1,\ 2,\ 3)$$

and from the identity (10)

$$l\frac{\partial \dot{b}_i}{\partial u}+m\frac{\partial \dot{b}_i}{\partial v}=0,$$

$$p\dot{b}_i=l\frac{\partial \dot{a}_i}{\partial u}+m\frac{\partial \dot{a}_i}{\partial v}.$$

Hence, we get

$$l(\lambda-\lambda')\left(\dot{a}\frac{\partial \dot{b}}{\partial u}\right)\cosh r=0,$$

i.e.

$$\lambda=\lambda',$$

or

$$\left(\dot{a}\frac{\partial \dot{b}}{\partial u}\right)=\left(\dot{a}\frac{\partial \dot{b}}{\partial v}\right)=0.$$

But from the last condition, we have

$$\lambda=\lambda'=0, \text{ or } r=const.$$

Therefore, in this case, the focal congruence is pseudo-normal and the space axis of the circles of the congruence is a normal one, or the radius of the circles of the congruence is constant.

Let us exclude this case and pass over to the other, where

$$\left|U\frac{\partial U}{\partial u}\frac{\partial V}{\partial v}\right|\neq 0.$$

In this case we shall call the cross in *general position* in our congruence. Hence, we have the following:

The crosses cutting orthogonally a corss in general position in a focal congruence, and each adjacent one, will generate a chain.

The simplest two-dimensional system of circle crosses is the *circle chain congruence* whose coordinates (X) are linearly dependent on those of the crosses (Y), (Z), (T) with real coefficients a, b, c, i.e.

(1) Kashiwagi, Mem. Coll. Sci., **6**, 109 (1923).

$$X_i = aY_i + bZ_i + cT_i, \qquad (11)$$
$$(i=1,\ 2,\ 3).$$

(a)

$$|\ Y\ Z\ T\ | \neq 0.$$

In this case the chain congruence is represented by all points of the real domain of the representing plane.

The cross intersecting orthogonally pairs of crosses of the chain congruence will generate a second congruence of like sort in the other layer called the *conjugate* to the first.

The conjugate to the chain congruence (11) will have the equation

$$U_i = p\begin{vmatrix} Y_j & Y_k \\ Z & Y_k \end{vmatrix} + q\begin{vmatrix} Z_j & Z_k \\ T & T_k \end{vmatrix} + r\begin{vmatrix} T_j & T_k \\ Y_j & Y_k \end{vmatrix}.$$

The relation between the two is reciprocal. We can prove that there are three crosses in a chain congruence which belong at once to both systems. Let their coordinates be (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively, then the equations of our chain congruence may thus be reduced to the simple form

$$X_1 = a(p+qi),\ X_2 = b(r+si),\ X_3 = c(t+ri).$$

where a, b, c are real homogeneous variables and $i = \sqrt{-1}$.

(b)

$$|\ Y\ Z\ T\ | = 0 \qquad (12)$$

but (Y), (Z), (T) are not linearly dependent on real coefficients λ, μ, ν, i.e.

$$\lambda Y_i + \mu Z_i + \nu T_i \neq 0.$$

In this case, we have a cross (U) which cuts orthogonally all crosses of the system, where

$$U_i = \begin{vmatrix} Y_j & K_k \\ Z & Z_k \end{vmatrix}.$$

Conversely we can prove that every cross which cuts orthogonally the cross (U) may be expressed in this form.

CIRCLE CROSSES IN ELLIPTIC SPACE.

5. As we relate the geometry of the circle cross in hyperbolic space with that of a point in the complex plane, so we may relate a circle cross in elliptic space to a pair of real points in two planes.

First, let us choose such a unit of measure that $k=1$.

We have for absolute

$$x_0^2+x_1^2+x_2^2+x_3^2=0,$$

and the coordinates of a circle of our system will be

$$\rho\mathfrak{X}_{01}=b_1 \cos r \tan \varphi,\ \rho\mathfrak{X}_{02}=b_2 \cos r \tan \varphi,\ \rho\mathfrak{X}_{03}=b_3 \cos r \tan \varphi,$$

$$\rho\mathfrak{X}_{23}=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\quad \rho\mathfrak{X}_{31}=\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix},\quad \rho\mathfrak{X}_{12}=\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Next consider a linear complex whose equation is

$$\sum a_{ij}\mathfrak{X}_{ij}=0$$

and put

$$\mathfrak{A}_{01}+\mathfrak{A}_{23}=\rho_l X_1,\quad \mathfrak{A}_{01}-\mathfrak{A}_{23}=\sigma_r X_1,$$
$$\mathfrak{A}_{02}+\mathfrak{A}_{31}=\rho_l X_2,\quad \mathfrak{A}_{02}-\mathfrak{A}_{31}=\sigma_r X_2,$$
$$\mathfrak{A}_{03}+\mathfrak{A}_{12}=\rho_l X_3,\quad \mathfrak{A}_{03}-\mathfrak{A}_{12}=\sigma_r X_3.$$

If we replace our complex by another coaxal therewith we shall merely multiply $({}_lX)$, $({}_rX)$ by two different constants. When we wish to move back from independent sets of coordinates $({}_lX)$, $({}_rX)$ to the degenerate complexes of the system, i.e. to the circles of the cross defined thereby, we have to take for ρ and σ such values that

$$(\mathfrak{A}\,|\,\mathfrak{A})=0.$$

For this, it is necessary and sufficient that

$$\rho({}_lX{}_lX)-\sigma({}_rX{}_rX)=0.$$

And we have

$$\tau\mathfrak{A}_{oi}={}_lX_i\sqrt{({}_rX{}_rX)}+{}_rX_i\sqrt{({}_lX{}_lX)},$$

$$\tau\mathfrak{A}_{ji} = {}_lX_i\sqrt{({}_rX_rX)} - {}_rX_i\sqrt{({}_lX_lX)}.$$

Thus we see that $({}_lX)$, $({}_rX)$ may be taken as two separate triads of coordinates to determine the cross.

Let the circles of the cross have the coordinates (α) and (α'); then we have

$$\alpha_{ij} = \rho\alpha'_{kl}.$$

And let (β), (β') be the coordinates of circles of the other cross and let θ, θ'; d, d' be the angles and distances of the cirles (α) and (β); then,

$$\cos\theta\cos\theta' = \cos d\cos d' = \frac{(\alpha\beta)}{\sqrt{(\alpha\alpha)}\sqrt{(\beta\beta)}},$$

$$\sin\theta\sin\theta' = \sin d\sin d' = \frac{(\alpha|\beta)}{\sqrt{(\alpha\alpha)}\sqrt{(\beta\beta)}},$$

Hence, we easily find

$$\cos(d+d') = \frac{({}_rX_rY)}{\sqrt{({}_rX_rX)}\sqrt{({}_rY_rY)}},\quad \cos(d-d') = \frac{({}_lX_lY)}{\sqrt{({}_lX_lX)}\sqrt{({}_lY_lY)}},$$

or else

$$\cos(d+d') = \frac{({}_lX_lY)}{\sqrt{({}_lX_lX)}\sqrt{({}_lY_lY)}},\quad \cos(d-d') = \frac{(X_rY_r)}{\sqrt{({}_rX_rX)}\sqrt{({}_rY_rY)}},$$

The quantities ${}_lD$ and ${}_rD$ which are given by the equation

$$\cos{}_lD = \frac{({}_lX_lY)}{\sqrt{({}_lX_lX)}\sqrt{({}_lY_lY)}},$$

$$\cos{}_rD = \frac{(X_rY_r)}{\sqrt{({}_rX_rX)}\sqrt{({}_rY_rY)}},$$

shall be called *left and right distances* of the two crosses respectively. These are the algebraic sum of their moment and commoment and are analogous to the Clifford angles of two line crosses in elliptic space.

The condition that the circles (α) and (β) should be cospherical, or that one should be cospherical with the mate of the other in a circle cross, is that

$$\cos_l D = \pm \cos_r D$$

i.e.

$$\frac{({}_lX,{}_lY)}{\sqrt{({}_lX,{}_lX)}\sqrt{({}_lY,{}_lY)}} = \pm \frac{({}_rX,{}_rY)}{\sqrt{({}_rX,{}_rX)}\sqrt{({}_rY,{}_rY)}},$$

They will be cospherical and in involution, i.e. the two crosses cut orthogonally, if

$$\cos_l D = \cos_r D = 0$$

i.e.

$$({}_lX,{}_lY) = (X_r Y_r) = 0.$$

The condition for parataxy will be

$$\sin_l D \sin_r D = 0,$$

i.e.

$$\left[({}_lX,{}_lX)({}_lY,{}_lY) - ({}_lX,{}_lY)^2\right] \cdot \left[({}_rX,{}_rX)({}_rY,{}_rY) - ({}_rX,{}_rY)^2\right] = 0.$$

This may be written

$$\left\| \begin{matrix} {}_lX_1 & {}_lX_2 & {}_lX_3 \\ {}_lY_1 & {}_lY_2 & {}_lY_3 \end{matrix} \right\|^2 \cdot \left\| \begin{matrix} {}_rX_1 & {}_rX_2 & {}_rX_3 \\ {}_rY_1 & {}_rY_2 & {}_rY_3 \end{matrix} \right\|^2 = 0.$$

The only real solutions will be

$${}_lY_i = \rho_l X_i, \quad {}_rY_i = \sigma_r X_i.$$

We can establish one to one correspondence between the totality of all real crosses in elliptic space and the totality of all pairs of points (or lines), one in each of two real planes.

As in the hyperbolic case, so here, we shall look upon cross space as doubly overlaid, and assign a cross to the upper layer if it be determined by two points in the representing planes, while it shall be assigned to the lower layer if it be determined by two lines. Hence, we have the following:

In order that two crosses of different layers should intersect orthogonally, it is necessary and sufficient that they should be represented by line elements in the two planes.

6. The one-dimensional circle chain has the equations

$$ {}_lX_i = a_l Y_i + b_l Z_i \quad {}_rX_i = a_r Y_i + b_r Z_i $$
$$ (a,\ b : real,\quad i=1,\ 2,\ 3). $$

Let us consider the case in which

$$ {}_lY_i \neq \rho_l Z_i, \quad {}_rY_i \neq \sigma_r Z_i. $$

A fixed cross whose equations are

$$ \lambda_l U_i \equiv \begin{vmatrix} {}_lY_j & {}_lZ_j \\ {}_lY_k & {}_lZ_k \end{vmatrix}, \quad \mu_r U_i \equiv \begin{vmatrix} {}_rY_j & {}_rZ_j \\ {}_rY_k & {}_rZ_k \end{vmatrix}, $$

cuts orthogonally all the crosses of the system considered.

Suppose that we have a focal congruence of circles of our system. We may exprese them parametrically

$$ {}_lX_i \equiv {}_lX_i(u,\ v), $$
$$ {}_rX_i \equiv {}_rX_i(u,\ v). $$

Let us assume that

$$ \left| {}_lX \frac{\partial_l X}{\partial u} \frac{\partial_l X}{\partial v} \right| \times \left| {}_rX \frac{\partial_r X}{\partial u} \frac{\partial_r X}{\partial v} \right| \neq 0. $$

The cross cutting orthogonally the adjacent crosses $({}_lX), ({}_rX)$ and $({}_lX + d_l X), ({}_rX + d_r X)$ will be determined by the equations

$$ \rho_l W_i \equiv \begin{vmatrix} {}_lX_j & {}_lX_k \\ \frac{\partial_l X_j}{\partial u} & \frac{\partial_l X_k}{\partial u} \end{vmatrix} du + \begin{vmatrix} {}_lX_j & {}_lX_k \\ \frac{\partial_l X_j}{\partial v} & \frac{\partial_l X_k}{\partial v} \end{vmatrix} dv, $$

$$ \sigma_r W_i = \begin{vmatrix} {}_rX_j & {}_rX_k \\ \frac{\partial_r X_j}{\partial u} & \frac{\partial_r X_k}{\partial u} \end{vmatrix} du + \begin{vmatrix} {}_rX_j & {}_rX_k \\ \frac{\partial_r X_j}{\partial v} & \frac{\partial_r X_k}{\partial v} \end{vmatrix} dv. $$

We shall mean by the *general position* of a circle in our congruence, one where

$$ \left| {}_lX \frac{\partial_l X}{\partial u} \frac{\partial_l X}{\partial v} \right| \times \left| {}_rX \frac{\partial_r X}{\partial u} \frac{\partial_r X}{\partial v} \right| \neq 0. $$

In the general position, if the crosses $({}_lX)$, $({}_rX)$ and $({}_lX+d_lX)$, $({}_rX+d_rX)$ are fixed, then the totality of the orthogonal crosses to these crosses forms a circle chain.

Specially, in the case

$$\left| {}_lX \frac{\partial_l X}{\partial u} \frac{\partial_l X}{\partial v} \right| = 0. \quad \left| {}_rX \frac{\partial_r X}{\partial u} \frac{\partial_r X}{\partial v} \right| = 0,$$

the cross whose coordinates are

$$\rho_l W_i \equiv \begin{vmatrix} \frac{\partial_l X_j}{\partial u} & \frac{\partial_l X_k}{\partial u} \\ \frac{\partial_l X_j}{\partial v} & \frac{\partial_l X_k}{\partial v} \end{vmatrix}, \quad \rho_r W_i \equiv \begin{vmatrix} \frac{\partial_l X_j}{\partial u} & \frac{\partial_r X_k}{\partial u} \\ \frac{\partial_r X_j}{\partial v} & \frac{\partial_l X_k}{\partial v} \end{vmatrix}$$

is the common orthogonal cross to a cross $({}_lX)$, $({}_rX)$ and every one of the crossen adjacent to it.

As in hyperbolic space, there is but one common perpendicular to a cross and its adjacent crosses. And the congruence is pseudo-normal or the radius of the circle of the congruence is constant.

Also, in the elliptic case, every circle chain has at least one pair of real crosses, called *principal crosses*, which are orthogonal. To find these, we must solve the equation

$$aa'({}_lY_lY)+(ab'+b'a)({}_lY_lZ)+bb'({}_lZ_lZ)=0,$$
$$aa'({}_rY_rY)+(ab'+b'a)({}_rY_rZ)+bb'({}_rZ_rZ)=0.$$

Eliminating $a' : b'$, we get

$$a^2\begin{vmatrix} ({}_lY_lY) & ({}_lY_lZ) \\ ({}_rY_rY) & ({}_rY_rZ) \end{vmatrix} + ab\begin{vmatrix} ({}_lY_lY) & ({}_lZ_lZ) \\ ({}_rY_rY) & ({}_rZ_rZ) \end{vmatrix} + b^2\begin{vmatrix} ({}_lY_lZ) & ({}_lZ_lZ) \\ ({}_rY_rZ) & ({}_rZ_rZ) \end{vmatrix} = 0.$$

The discriminant of the equation is

$$\begin{vmatrix} ({}_lY_lY) & ({}_lZ_lZ) \\ ({}_rY_rY) & ({}_rZ_rZ) \end{vmatrix}^2 + 4({}_lY_lY)({}_lZ_lZ)[({}_rY_rZ)-({}_lY_lZ)]^2 \not< 0.$$

Therefore the roots cannot be be complex. The discriminant may be considered as the simultaneous invariant of the form

$$(a_lY+b_lZ,\ a_lY+b_lZ)=0,$$
$$(a_rY+b_rZ,\ a_rY+b_rZ)=0,$$

for $a:b$. When our crosses have real coordinates, the roots of these two quadratic equations are conjugate complex pairs. The two quadratics can have no common roots unless they are identical. Hence the simultaneous invariant is not zero. Thus the crosses are distinct ones.

Let us take the crosses corresponding to them as (0, 1, 0) (0, 1, 0), (0, 0, 1) (0, 0, 1). Then we may express our chain in the simple form

$$
\begin{aligned}
&{}_lX_1={}_rX_1=0,\\
&{}_lX_2=p{}_rX_2,\\
&{}_lX_3=q{}_rX_3. \qquad (p,q : \text{constant.})
\end{aligned}
$$

To find the equation of surface generated, let $(\mathfrak{X})$ be the coordinates of a circles of our system, then

$$\mathfrak{X}_{01}=\mathfrak{X}_{23}=0,$$

$$q(\mathfrak{X}_{02}-\mathfrak{X}_{31})(\mathfrak{X}_{03}+\mathfrak{X}_{12})=p(\mathfrak{X}_{02}+{}_{31})(\mathfrak{X}_{03}-\mathfrak{X}_{12}).$$

A point (x) will lie on this circle if

$$(bx)=0,\ (ax)=\cos r\sqrt{(xx)},$$

$$(aa)=1,\ (xx)=1.$$

Hence, we have

$$
\begin{aligned}
&\mathfrak{X}_{02}=\rho x_3 \tan\varphi,\\
&\mathfrak{X}_{03}=-\rho x_2 \tan\varphi,\\
&\mathfrak{X}_{31}=\rho\frac{x_2}{x_1}\left(\sqrt{(xx)}-\frac{x_0}{\cos\varphi}\right),\\
&\mathfrak{X}_{12}=\rho\frac{x_3}{x_1}\left(\sqrt{(xx)}-\frac{x_0}{\cos\varphi}\right).
\end{aligned}
$$

So the required surface is given by

$$
\begin{aligned}
&q\left[x_1x_3\tan\varphi-x_2\left(\sqrt{(xx)}-\frac{x_0}{\cos\varphi}\right)\right]\\
&\qquad\left[-x_1x_2\tan\varphi+x_3\left(\sqrt{(xx)}-\frac{x_0}{\cos\varphi}\right)\right]\\
&=-p\left[x_1x_3\tan\varphi+x_2\left(\sqrt{(xx)}-\frac{x_0}{\cos\varphi}\right)\right]\\
&\qquad\left[x_1x_2\tan\varphi+x_3\left(\sqrt{(xx)}-\frac{x_0}{\cos\varphi}\right)\right]
\end{aligned}
$$

Or

$$(p+q)(x_2^2+x_3^2)x_1\left(\sqrt{xx}-\frac{x_0}{\cos\varphi}\right)\tan\varphi$$

$$+(p-q)\left[\left(\sqrt{xx}-\frac{x_0}{\cos\varphi}\right)^2+x_1^2\tan^2\varphi\right]x_2x_3=0.$$

Suppose, next, that we have

$$\rho_l X_i = a_l Y_i + b_l Z_i, \quad \sigma_r X_i = a_r Y_i + b_r Z_i,$$

where

$$_lY_i = p_l Z_i, \quad |_rY\ _rZ\ _rT| \neq 0,$$

or

$$_rY_i = q_r Z_i, \quad |_lY\ _lZ\ _lT| \neq 0.$$

This is a new one-parameter family of circle crosscs called a *circle-strip*, or, more exactly, a *left strip* (or a *right strip*), and all crosses of the system are paratactic.

Let $({}_lX')$ $({}_rX')$ be the common perpendicular a to pair of crosses of the left circle-strip, then

$$({}_lX'\,{}_lY)=0,$$
$$({}_rX'\,{}_rZ)=({}_rX'\,{}_rY)=0.$$

Hence, we may write

$$_lX'_i=\lambda_l Y'_i+\mu_l Z'_i, \quad _rX'_i={}_rY'_i={}_rZ'_i.$$

We thus have a right circle-strip, and each circle-strip cuts each of the others orthogonally.

A left strip of the upper layer will be represented by a point in the left plane, and a linear range in the right plane. The right strip in the lower layer will be represented by the pencil through the point in the left plane, and the line of the range in the right.

The simplest two dimensional system of circle-crosses will be, as before, the chain congruence

$$_lX_i=a_l Y_i+b_l Z_i+c_l T_i, \qquad \text{(i)}$$
$$_rX_i=a_r Y_i+b_r Y_i+c_r T_i$$
$$|_lY_l Z_l T| \times |_rY_r Z_r T| \neq 0. \qquad \text{(ii)}$$

We may solve the first three equations for a, b, c and substitute in the last

$$\rho_r X_i = \sum_j a_{ij}\,{}_l X_j, \quad |a_{ij}| \neq 0. \qquad \text{(iii)}$$

The crosses intersecting orthogonally pairs of crosses of this congruence will generate a second congruence of like sort called the conjugate to the first. And its equations are

$$\sigma_l U_i = \sum_j a_{ji}\,{}_r U_j. \qquad \text{(iv)}$$

The relation between the two is *reciprocal.* If we seek a cross that belongs at once to both systems, we replace $({}_l U)$ and $({}_r U)$ in the equation (iv) by $(\rho_l X)$ and $(\rho_r X)$ respectively and substitute the value $({}_r X)$ expressed by $({}_l X)$ in the right hand side of the resulting equation, then we have

$$\tau_l X_i = \sum_j b_{ij}\,{}_l X_j, \qquad \text{(v)}$$

where

$$b_{ij} = b_{ji} = \sum_k a_{ik} a_{jk}.$$

Therefore, we have a symmetric determinant equation by eliminating $({}_l X)$ from the equation (v):

$$\begin{vmatrix} b_{11}-\tau & b_{12} & b_{13} \\ b_{21} & b_{22}-\tau & b_{23} \\ b_{31} & b_{32} & b_{33}-\tau \end{vmatrix} = 0. \qquad \text{(vi)}$$

But as a symmetric determinant equation has always real roots, we have three real roots for τ. The crosses corresponding to the three roots of the equation (vi) are orthogonal to one another.

We, now, give the name *general chain congruence* only to a congruence of our present type where the cubic above has distinct roots. If these three crosses be looked upon as (100) (100), (010) (010), (001) (001), then the equations to our chain congruence may be reduced to the form

$${}_l X_i = a_i\,{}_r X_i, \quad a_1 a_2 a_3 \neq 0, \quad i = 1, 2, 3.$$

and the equations of the reciprocal congruence will be

$$ {}_lU_i = \frac{1}{a_i}\,{}_rU_i, \quad i=1, 2, 3. $$

Hence we have the following:

The circle crosses cutting orthogonally pairs of crosses of a general chain congruence generate a second such congruence. The relation between the two is reciprocal, and they have in common three orthogonal intersecting crosses.

The condition that the two circles of our two systems should be cospherical, or that one should cospherical with the circle in bi-involution with the other, is

$$ ({}_lU_lX)\left[\sqrt{\sum\left(\frac{1}{a_i^2}\,{}_lX_i^2\right)}\,\sqrt{\sum(a_i^2\,{}_lU_i^2)} \pm \sqrt{({}_lX_lX)}\,\sqrt{({}_lU_lU)}\right]=0. $$

If the first factor vanish,

$$ ({}_lU_lX)=({}_rU_rX)=0, $$

the two cut orthogonally. If the second factor vanish, every circle of a cross of the first congruence for which

$$ \beta\sum\left(\frac{1}{a_i^2}\,{}_lX_i\right)=\alpha({}_lX_lX) $$

is cospherical with a circle of every cross of the second for which

$$ \alpha\sum(a_{ii}U_i)=\beta({}_lU_lU). $$

A different sort of congruence will arise in the case where

$$ |{}_lY_lZ_lT|=0, \quad |{}_rY_rZ_rT|\neq 0, \quad (|{}_rY_rZ_rT|=0, \quad |{}_lY_lZ_lT|\neq 0). \qquad \text{(vii)} $$

In this case the coordinates $({}_lY)$, $({}_lZ)$, $({}_lT)$, $({}_rY)$, $({}_rZ)$, $({}_rT)$ of the given three crosses are linearly dependent; thus the three points correspondent to $({}_lY)$, $({}_lZ)$, $({}_lT)$ [$({}_rY)$, $({}_rZ)$, $({}_rT)$] in the left (right) representing plane are colinear, i.e.

$$ {}_lT_i=l_lY_i+m_lZ_i, \quad ({}_rT_i=\lambda_rY_i+\mu_rZ_i). \qquad \text{(viii)} $$

Therefore

$$\rho_l X_i = (a+lc)_l Y_i + (b+mc)_l Z_i,$$
$$[\sigma_r X_i = (a+\lambda c)_r Y_i + (b+\mu c)_r Z_i].$$

Let us now put

$$a+lc=p, \quad b+mc=q,$$
$$a+\lambda c=p', \quad b+\mu c=q',$$

then we have

$$\rho_l X_i = p_l Y_i + q_l Z_i, \quad \rho_r X_i = p_r Y_i + q_r Z_i - c(l_r Y_i + m_r Z_i - {}_r T_i),$$
$$[\sigma_r X_i = p'_r Y_i + q'_r Z_i, \quad \sigma_l X_i = p'_l Y_i + q'_l Z_i - c(\lambda_l Y_i + \mu_l Z_i - {}_l T_i)].$$

Hence the congruence will contain ∞^1 circle-strip, whose reciprocals generate the reciprocal congruence. The canonical form will be

$${}_l X_1 = a_1\, {}_r X_1,$$
$${}_l X_2 = a_2\, {}_r X_2,$$
$${}_l X_3 = 0.$$

When in addition to (vii) all the first minors of the determinant

$$|{}_l Y\ {}_l Z\ {}_l T| \quad (|{}_r Y\ {}_r Z\ {}_r T|)$$

vanishes, we have a system of paratactic circle crosses.

If, on the other hand, we have

$$|{}_l Y\ {}_l Z\ {}_l T| = 0, \quad |{}_r Y\ {}_r Z\ {}_r T| = 0,$$

without the vanishing of the first minors of either determinant we have ∞^2 crosses which cut a given cross orthogonally. These crosses may be represented by the equations (in the lower layer)

$$\rho\, {}_l U_i = \lambda \begin{vmatrix} {}_l Y_j & {}_l Y_k \\ {}_l Z_j & {}_l Z_k \end{vmatrix} + \mu \begin{vmatrix} {}_l Z_j & {}_l Z_k \\ {}_l T_j & {}_l T_k \end{vmatrix} + \nu \begin{vmatrix} {}_l T_j & {}_l T_k \\ {}_l Y_j & {}_l Y_k \end{vmatrix},$$
$$\rho\, {}_r U_i = \lambda \begin{vmatrix} {}_r Y_j & {}_r Y_k \\ {}_r Z_j & {}_r Z_k \end{vmatrix} + \mu \begin{vmatrix} {}_r Z_j & {}_r Z_k \\ {}_r T_j & {}_r T_k \end{vmatrix} + \nu \begin{vmatrix} {}_r T_j & {}_r T_k \\ {}_r Y_j & {}_r Y_k \end{vmatrix}.$$

And the equations of this congruence may be reduced to the canonical form

$$\rho\, {}_l X_1 = a, \quad \sigma\, {}_r X_1 = b,$$
$$\rho\, {}_l X_2 = b, \quad \sigma\, {}_r X_2 = c,$$
$$\rho\, {}_l X_3 = o, \quad \sigma\, {}_r X_3 = o,$$

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