# Projective Differential Geometry of Non-Developable Surfaces, III.

By

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(Received Nov. 7, 1923)

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## CHAPTER IV.

### CURVES OF DARBOUX AND SEGRE.

**1.** The curves of Darboux and Segre play a very important role in the theory of the projective differential geometry of surfaces. Many properties of them have been investigated by G. Darboux, C. Segre, G. Fubini, E. J. Wilczynski and G. M. Green from various points of view. In this chapter, I intend to add some new properties.

Let us now suppose parameter curves to be asymptotic curves. Then, in virtue of and (19) of **chap.** III, three families of Darboux curves are given by the equations

$$\sqrt[3]{C}\,du+w^i\sqrt[3]{B''}\,dv=0, \quad (i=1,\ 2,\ 3), \quad \cdots\cdots\cdots\cdots(1)$$

where $w$ is an imaginary cubic root of 1.

The three families of curves given by (1) are all distinct, if $C$ and $B''$ are not equal to zero. If one of $C$ and $B''$ is equal to zero, e.g. $B''=0$, (1) becomes $\sqrt[3]{C}\,du=0$.

Therfore we know that the *three families of Darboux curves are distinct for curved surfaces, while all of them coincide with the family of generating lines for ruled surfaces.* Also from (1) we can see at once that *no two of three families of the Darboux curves can be conjugate to each other* for curved surfaces.

Since three families of Segre curves are conjugate to three families of Darboux curves, we can say the same for the Segre curves.

In this chapter, we shall investigate the properties of the curves of Darboux and Segre for curved surfaces.

Any coojugate net may be defined by an equation of the form

$$\varphi^2 du^2 - dv^2 = 0,$$

where $\varphi$ is a function of $u$ and $v$.

Let us put

$$d\xi = \mu(\varphi du - dv),$$
$$d\zeta = \nu(\varphi du + dv),$$

where $\mu$ and $\nu$ are integrating factors of the equations

$$\varphi du - dv = 0$$

and

$$\varphi du + dv = 0$$

respectively.

Then we have

$$\left.\begin{aligned} du &= \frac{\nu d\xi + \mu d\zeta}{2\varphi\mu\nu}, \\ dv &= \frac{-\nu d\xi + \mu d\zeta}{2\varphi\mu\nu}. \end{aligned}\right\} \qquad \text{(2)}$$

Denote by $D_1$, $D_1'$, $D_1''$, .... the normal fundamental determinants in the case where the curves $\xi=$ const. and $\zeta=$ const. are parameter curves. Then we have from (17) of **chap.** I, (24) of **chap.** III and the above equation (2),

$$D_1 = -\frac{1}{D_1''} = -\frac{\nu}{\mu}. \qquad \text{(3)}$$

From (2) and (3), we have

$$\frac{\partial^2}{\partial\xi\partial\zeta} log \frac{D'}{D_1''} = 2\frac{\partial^2}{\partial\xi\partial\zeta} log\, D_1 = 2\frac{\partial^2}{\partial\xi\partial\zeta} log \frac{\nu}{\mu} = 2\frac{\partial^2}{\partial u \partial v} log \varphi. \qquad \text{(4)}$$

But the necessary and sufficient condition that the conjugate net given by (1) should be isothermal, is

$$\frac{\partial^2}{\partial\xi\partial\zeta} log \frac{D_1}{D_1''} = 0.$$

Therefore, from (1) and (4) we konw that if any one of three families of Darboux curves form an isothermal conjugate net with a family of curves of Segre conjugate to it, then

$$\frac{\partial^2 log}{\partial u \partial v}\left(\frac{C}{B''}\right)=0. \quad \dots\dots\dots\dots\dots\dots\dots (5)$$

But (5) is the necessary and sufficient condition that developable surfaces of the congruences of directrices and of canonical edges correspond to a conjugate net. Therefore, we konw that *three families of Darboux curves and three families of Segre curves constitute three isothermal conjugate net , if, and only if, developables of the congruences of directrices and of canonical edges correspond to a conjugate net.*

**2.** Put

$$\left.\begin{aligned} h&=\ \ \frac{1}{6}\frac{\partial}{\partial u}log\left(\frac{C}{B''}\right), \\ k&=-\frac{1}{6}\frac{\partial}{\partial v}log\left(\frac{C}{B''}\right). \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (6)$$

When parameter curves are asymptotic curves and $\alpha=h$, $\beta=k$, $\gamma=0$, the system of equations (1) of **chap.** III becomes

$$\left.\begin{aligned} &\frac{\partial x}{\partial u}=\rho-hx, \\ &\frac{\partial x}{\partial v}=\sigma-kx, \\ &\frac{\partial \rho}{\partial u}=h\rho+C\sigma+\mathfrak{A}_{h,k}x, \\ &\frac{\partial \rho}{\partial v}=-\tau-k\rho+\mathfrak{A}'_{k,h}x, \\ &\frac{\partial \sigma}{\partial u}=-\tau-h\sigma-\mathfrak{A}'_{h,k}x, \\ &\frac{\partial \sigma}{\partial v}=B''\rho+k\sigma-\mathfrak{A}''_{h,k}x, \\ &\frac{\partial \sigma}{\partial u}=h\tau+(\mathfrak{A}'_{h,k}-\theta)\rho-(\mathfrak{A}_{h,k}-\mathfrak{L}_{h,k})\sigma \end{aligned}\right\} (7)$$

$$-\left(\frac{\partial}{\partial v}\mathfrak{A}_{h,k}-\frac{\partial}{\partial u}\mathfrak{A}'_{h,k}-C\mathfrak{A}''_{h,k}+6h\mathfrak{A}'_{h,k}\right)x,$$

$$\frac{\partial}{\partial v}=k\tau+(\mathfrak{A}''_{h,k}-\mathfrak{L}''_{h,k})\rho-(\mathfrak{A}'_{k,k}+\theta)\sigma$$

$$-\left(\frac{\partial}{\partial v}\mathfrak{A}'_{k,h}-\frac{\partial}{\partial u}\mathfrak{A}''_{k,k}+B''\mathfrak{A}_{h,k}-6k\mathfrak{A}'_{h,k}\right)x.$$

Let $\lambda$ be any one of the roots of the cubic equation

$$x^3=-\frac{C}{B''}.$$

Then the equation

$$\frac{dv}{du}=\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

difines a family of Darboux curves and the equation

$$\frac{dv}{du}=-\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

defines a family of Segre curves conjugate to the family of curves defined by (8).

Tangents of the family of curves defined by (8) constitute a congruence. On every tangent line of the congruence, there are two focal points. Let $t$ be a tangent line of the congruence which touches at a point $P(x)$ to a curve $C$ of the family defined by (8). One of the focal points on $t$ is $P$. The other focal point is called by Wilczynski the *ray* point of $P$ with respect to the curve $C$.

Any point on the line $t$, except the point $P(x)$, may be defined by an expression of the form

$$\phi=\rho+\lambda\sigma+\mu x.$$

For $\phi$ be the ray point on $t$ with respect to the curve $C$, it is necessary and sufficient that four points $x$, $\phi$, $\frac{\partial\phi}{\partial u}$, $\frac{\partial\phi}{\partial v}$ are co-planar. From this reason, by the same method as employed in **n° 5 chap.** III, making use of (7), we can easily see that the ray point may be defined by the expression

$$\rho+\lambda\sigma. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Similarly, we can see that the ray point of $P(x)$ on the tangent $t'$ at $P$ of the curve $C'$ of the family of curves defined by (9) with respect to $C'$ may be defined by the expression

$$\rho-\lambda\rho+B''\lambda^2x. \quad\dots\dots\dots\dots\dots\dots(11)$$

Therefore, the ray of $P(x)$ with respect to the conjugate curves $C$ and $C'$, (i.e. the line which connects the points (10) and (11) ), passes through the two points

$$\rho+\frac{B''\lambda^2}{2}x, \quad \sigma-\frac{B''\lambda}{2}x.$$

The reciprocal line of this ray passes through the point $x$ and through the point

$$\tau+\frac{B''\lambda}{2}(\rho-\lambda\sigma). \quad\dots\dots\dots\dots\dots\dots(11)$$

The osculating plane at $x$ of the curve $C$ is given by the equation

$$\left| \xi \ \ x \ \ \frac{dx}{du} \ \ \frac{d^2x}{du^2} \right| =0, \quad\dots\dots\dots\dots\dots\dots(12)$$

where $(\xi)$ are current coordinates, and $\frac{dx}{du}$ and $\frac{d^2x}{du^2}$ denote the differentiations along $C$.

But we have from (7) and (8)

$$\left.\begin{aligned}
&\frac{dx}{du}=\rho+\lambda\sigma-(h+k\lambda)x,\\
&\frac{d^2x}{du^2}=-2\lambda\tau+(B''\lambda^2-2k\lambda)\rho+(C'-2k\lambda^2)\sigma\\
&\qquad\qquad+\{A-kC-\lambda^2(A''+hB'')+2k^2\lambda^2\}x.
\end{aligned}\right\}(13)$$

If we substitute the values of $\frac{dx}{du}$ and $\frac{d^2x}{du^2}$ given by (13) in (12), (12) becomes

$$\left| \xi \ \ x \ \ \rho+\sigma\lambda \ \ \tau-\frac{B''\lambda}{2}(\rho-\sigma\lambda) \right| =0. \quad\dots\dots\dots\dots(14)$$

Similarly, we can see that the osculating plane at $x$ of the curve $C'$ is given by the equation

$$| \ \xi \ \ x \ \ \rho-\lambda \ \ \tau \ | =0. \quad\dots\dots\dots\dots\dots\dots\dots\dots(15)$$

Therefore, the axis of $x$ with respect to the conjugate curves $C$ and $C'$, (i.e. the line of the intersection of the planes given by (14) and (15) ), passes through the point $x$ and through the point

$$\tau-\frac{B''\lambda}{2}\rho+\frac{B''\lambda^2}{2}\sigma.$$

The reciprocal line of this axis connects the points

$$\rho-\frac{B''\lambda^2}{2}x, \quad \sigma+\frac{B''\lambda}{2}x.$$

Summing up the above results, we have the following two theorems which are dual to each other. (Fig. 1).

Let $P$ be an ordinary point on a curved surface. Three Darboux curves and three Segre curves which meet at $P$ constitute three pairs of conjugate curves. Corresponding to these three pairs of conjugate curves, there are three rays and three axes of $P$.

*The Reciprocal line of any one of these axes passes through the corresponding ray point on the tangent at $P$ of the curve of Darboux passing through $P$.*

*Three ray points of $P$ with respect to the three curves of Darboux passing through $P$ lie on the line which is the harmonic conjugate of the tangent at $P$ of any one of these curves of Darboux with respect to the corresponding ray and the reciprocal line of the axis of $P$.*

*The Reciprocal line of any one of these rays lies on the corresponding osculating plane at $P$ of the curve of Segre passing through $P$.*

*Three osculating planes at $P$ of the three curves of Segre passing through $P$ intersect at the line which is the hormonic conjugate of the tangent at $P$ of any one of these curves of Segre with respect to the corresponding axis and the reciprocal line of the ray of $P$.*

**3.** Let $P(x)$ be a point on the surface. Two asymptotic tangents at $P$ and three rays of $P$ with respect to the three conjugate curves formed by curves of Darboux and Segre passing through $P$ determine a conic $K$ which touches these five lines. We shall now find the equation of $K$ referring to the moving coordinate frame of reference of which the vertices are at the points $x$, $\rho$, $\sigma$, and the unit point is at the point $x+\rho+\sigma$.

Line coordinates of the said rays referring to the moving coordinate of reference are

$$\left.\begin{array}{l}(1,\ -\tfrac{1}{2}B''\lambda^2,\ \tfrac{1}{2}B''\lambda),\\ (1,\ -\tfrac{1}{2}w^2B''\lambda^2,\ \tfrac{1}{2}wB''\lambda)\\ (1,\ -\tfrac{1}{2}wB''\lambda^2,\ \tfrac{1}{2}w^2B''\lambda)\end{array}\right\} \quad \cdots\cdots\cdots (16)$$

and the last respectively, where $w$ is an imaginary cubic root of 1.

Let $\nu_i$ $(i = 0,\ 1,\ 2)$ be line coordinates referring to the moving coordinate frame of reference. The equation in line coordinates of a conic which touches the line $x\rho$ and $x\sigma$ is of the form

$$\nu_0(a_0\nu_0+a_1\nu_1+a_2\nu_2)+a_3\nu_1\nu_2=0. \quad \cdots\cdots\cdots (17)$$

If the conic given by (17) touch the three lines, the line coordinates of which are given by (16), the coefficients $a_0$, $a_1$, $a_2$, $a_3$ must satisfy the equations

$$\left.\begin{array}{l}B''a_1\lambda^2-B''a_2\lambda-2\left(a_0+\dfrac{\theta}{4}a_3\right)=0,\\ w^2B''a_1\lambda^2-wB''a_2\lambda-2\left(a_0+\dfrac{\theta}{4}a_3\right)=0,\\ wB''a_1\lambda^2-w^2B''a_2\lambda-2\left(a_0+\dfrac{\theta}{4}a_3\right)=0.\end{array}\right\} \quad \cdots\cdots\cdots (18)$$

From (18) we have

$$a_1=a_2=0,$$

$$\frac{a_0}{a_3}=-\frac{\theta}{4}.$$

Therefore, the equation in line coordinates of $K$ is

$$\theta\nu_0^2=4\nu_1\nu_2. \quad \cdots\cdots\cdots (19)$$

Line coordinates of three reciprocal lines of three axes of $P$ with respect to three pairs of the conjugate curves formed by curves of Darboux and Segre passing through $P$ are

$$(1,\ \tfrac{1}{2}B''\lambda^2,\ -\tfrac{1}{2}B''\lambda,).$$

$$(1,\ \tfrac{1}{2}w^2B''\lambda^2,\ -\tfrac{1}{2}B''w\lambda),$$

$$(1,\ \tfrac{1}{2}wB'\lambda,\ -\tfrac{1}{2}w^2B''\lambda),$$

respectively. Therefore, these three lines touch the conic $K$.

Let $w_i$ $(i = 0, 1, 2)$ be point coordinates referring to the moving coordinate frame of reference. Then, from (19) we see that the equation of $K$ in point coordinates is

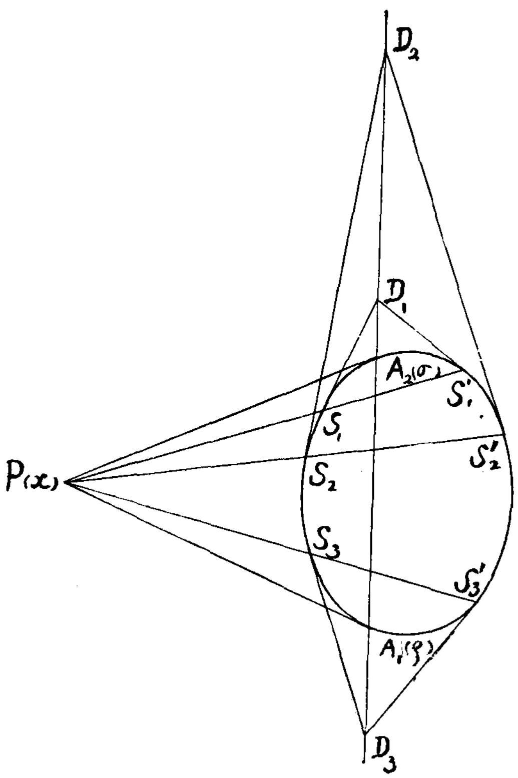
$$w_0^2 = \theta w_1 w_2. \quad \cdots\cdots\cdots\cdots\cdots\cdots (20)$$

From (19) or (20), we can see at once that the lines $x\rho$ and $x\sigma$ touch $K$ at the points $\rho$ and $\sigma$ respectively. Therefore, the line $\rho\sigma$ is the polar line of the point $x$ with respect to $K$. But three ray points $D_1$, $D_2$, $D_3$ of $P$ with respect to the three curves of Darboux passing through $P$ lie on the line $\rho\sigma$. Therefore, three polar lines of $D_1$, $D_2$ and $D_3$ pass through $P$ and they are tangents at $P$ of three Segre curves passing through $P$.

Therefore, we have the following theorem.

*Let $P$ be an ordinary point on a curved surface and $D_1$, $D_2$, $D_3$ be three ray points of $P$ with respect to the three curves of Darboux passing through $P$. Then the points $D_1$, $D_2$, $D_3$ lie on a line which intersects the asymptotic tangents at in the points* $A_1$ *and* $A_2$. *Let $S_1$, $S_2$, $S_3$ be three ray points of the three curves of Segre passing through $P$, $PS_1$, $PS_2$, $PS_3$ being tangents conjugate to the tangents $PD_1$, $PD_2$, $PD_3$ respectively. Let $D_1S'_1$, $D_2S'_2$, $D_3S'_3$ the reciprocal lines of axes of $P$ with respect to three pairs of the conjugate curves formed by the curve of Darboux and Scgre passing through $P$, $S_1'$, $S_2'$, $S_3'$ being the*

Fig. 1.



points on the tangents $PS_1$, $PS_2$, $PS_3$ respectively (Fig. 1).

*Then, there is a conic which touches eight lines* $PA_1$, $PA_2$; $D_1S_1$, $D_1S'_1$; $D_2S_2$, $D_2S_2'$; $D_3S_3$, $D_3S'_3$ *and* $A_1$, $A_2$; $S_1$ $S'_1$; $S_2$ $S'_2$; $S_3$ $S'_3$ *are the points of the contact.*

We shall call this coinc *the canonical ray conic* of $P$.

Similarly, we can prove the following theorem.

Let $p$ be a tangent plane at an ordinary point $P$ of a curved surface. Let $s_1$, $s_2$, $s_3$ be three osculating planes at $P$ of the three curves of Segre passing through $P$. Then $s_1$, $s_2$, $s_3$ intersect at a line which determines planes $a_1$, $a_2$ with asymptotic tangents at $P$. Let $d_1$, $d_2$, $d_3$ be three osculating planes at $P$ of the three curves of Darboux passing through $P$, $pd_1$. $pd_2$, $pd_3$ being tangents conjugate to the tangents $ps_1$, $ps_2$, $ps_3$ respectively. Let $d'_1s_1$, $d'_2s_2$, $d'_3s_3$ be the reciprocal lines of three rays of $P$ with respect to three conjugate curves formed by the curves of Darboux and Segre passing through $P$, $d'_1$ $d'_2$ $d'_3$ being planes containing the tangents $pd_1$, $pd_2$, $pd_3$ respectively,

*Then, there is a cone of the second degree which passes through eight lines* $pa_1$, $pa_2$; $d_1s_1$ $d'_1s_1$; $d_2s_2$, $d'_2s_2$; $d_3s_3$, $d'_3s_3$ *and the planes* $a_1$, $a_2$, $d_1$, $d'_1$; $d_2$, $d'_2$; $d_3$, $d'_3$ *are tangent planes of the cone.*

We shall call this cone *the canonical axis cone* of $P$.

In n°8 chap. III, we proved that, if

$$\alpha=l\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta},$$

$$\beta=l\frac{|\mathfrak{L}\ \mathfrak{B}\ D|}{\theta}+\frac{1}{2}\frac{\theta_v}{\theta},$$

$l$ being a constant, the lines $\rho\sigma$ and $x\tau$ are invariant lines. But when the parameter curves are asymptotic curves,

$$-\frac{1}{3}\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta}=\frac{1}{6}\frac{\partial}{\partial u}log\left(\frac{C}{B''}\right),$$

$$-\frac{1}{3}\frac{|\mathfrak{L}\ \mathfrak{B}\ D|}{\theta}+\frac{1}{2}\frac{\theta_v}{\theta}=-\frac{1}{6}\frac{\partial}{\partial v}log\left(\frac{C}{B''}\right).$$

Therefore we know that, referring to any system of parameter curves, if

$$\alpha=-\frac{1}{3}\frac{|\mathfrak{L}\ \mathfrak{C}\ D|}{\theta}+\frac{1}{2}\frac{\theta_u}{\theta},$$

$$\beta=-\frac{1}{3}\frac{|\mathfrak{L}\ \mathfrak{B}\ D|}{\theta}+\frac{1}{2}\frac{\theta_v}{\theta},$$

then $\rho\sigma$ is the line on which the three ray points of $x$ with respect to three Draboux curves passing through $x$ lie and $x\tau$ is the line at which intersect the three osculating planes at $x$ of the three curves of Segre passing through $x$.

**4.** To any point of the surface, we can associate a canonical ray conic. Therefore, these conics constitute a congruence. On each conic, there are six focal points[1]. We shall determine these focal points.

Let $(w_0, w_1, w_2)$ be projective coordinates of a point $Q$ on the conic $K$ which lie on the tangent plane at $P(x)$ on the surface, referring to the moving coordinate frame of reference.

Put

$$\frac{w_1}{w_0}=t.$$

Then we have from (20)

$$w_0 : w_1 : w_2=1 : t : \frac{1}{\theta t}.$$

Projective coordinates $(\xi)$ of $Q$ referring to the space fixed coordinate frame of reference may be expressed in the form

$$\xi=x+t\rho+\frac{1}{\theta t}\sigma, \quad \dots\dots\dots\dots\dots\dots (21)$$

if the proportional factor properly chosen, for the unit point of the moving coordinate frame of reference is $x+\rho+\sigma$.

If $Q$ be a focal point on $K$, then, since all the surfaces of the congruence which passes through $K$ must have the tangent plane common at $Q$, four points $\xi, \frac{\partial\xi}{\partial t}, \frac{\partial\xi}{\partial u}, \frac{\partial\xi}{\partial v}$ should be coplanar.

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[1] Darboux, Leçous sur la théorie générale des surfaces. Vol. II. p. 5.

From this reason, by the usual method, we can see that for $Q$ to be a focal point, it is necessary and sufficient that

$$C(\theta t^2)^3 - B'' - 2t\{(\theta t^2)^2\mathfrak{A}_{h,k} - 2\mathfrak{A}'_{h,k}(\theta t^2) + \mathfrak{A}''_{h,k}\}$$

$$+\left(2h - \frac{\theta_u}{\theta}\right)(\theta t^2)^2 - \left(2k - \frac{\theta_v}{\theta}\right)(\theta t^2) = 0 \quad \text{..........(22)}$$

Six roots of this equation determine the six focal points on $K$.

Now let us suppose that six focal points of any canonical ray conic lie on three lines passing through the point at which the plane of the conic touches the surface so that on each one of the three lines lie two focal points.

Then, six roots of (22) muts be of the form

$$\pm t_1, \ \pm t_2, \ \pm t_3.$$

Accordingly,

$$(\theta t_i^2)^2\mathfrak{A}_{h,k} - 2\mathfrak{A}'_{h,k}(\theta t_i)^2 + \mathfrak{A}''_{h,k} = 0, \ (i=1,2,3) \quad \text{.....(23)}$$

From (23) we have

$$\mathfrak{A}_{h,k} = \mathfrak{A}''_{h,k} = 0, \quad \text{..........(24)}$$

$$\mathfrak{A}'_{h,k} = 0. \quad \text{..........(25)}$$

From (6) and (25) we have

$$\frac{\partial}{\partial u \partial v} \log \frac{C}{B''} = 0.$$

Therefore,

$$\frac{C}{B''} = \frac{U(u)}{V(v)}, \quad \text{..........(26)}$$

where $U(u)$ and $V(v)$ are functions of $u$ only and of $v$ only respectively.

Carry out the transformation

$$\bar{u} = \int \sqrt[3]{U}\, du, \qquad \bar{v} = \int \sqrt[3]{V}\, dv. \quad \text{..........(27)}$$

Then, in virtue of (20) in **chap.** I, we have

$$\frac{\overline{C}}{\overline{B''}} = 1.$$

Let us assume that this transformation has been made.

Then,

$$C = B'' \qquad h = k = 0, \quad \text{..........(28)}$$

and (24) becomes

$$A=A''=0. \qquad \dots\dots\dots\dots\dots\dots (29)$$

When the parameter curves are asymptotic curves, the system of the equations (4) in **chap.** II[1] becomes

$$\left.\begin{aligned}
&2\frac{\partial A}{\partial v}+\frac{\partial^2 C}{\partial v^2}=\frac{\partial \theta}{\partial u}+C\frac{\partial B''}{\partial u},\\
&-2\frac{\partial A''}{\partial u}+\frac{\partial^2 B''}{\partial u^2}=\frac{\partial \theta}{\partial v}+B''\frac{\partial C}{\partial v},\\
&\frac{\partial^2 A}{\partial v^2}+\frac{\partial^2 A''}{\partial u^2}-\frac{\partial (A''C)}{\partial v}-\frac{\partial (AB'')}{\partial u}-A''\frac{\partial C}{\partial v}-A\frac{\partial B''}{\partial v}=0.
\end{aligned}\right\}(30)$$

From (28), (29), and (30) we have

$$\left.\begin{aligned}
&\frac{3}{2}\frac{\partial \theta}{\partial u}=\frac{\partial^2 C}{\partial v^2},\\
&\frac{3}{2}\frac{\partial \theta}{\partial v}=\frac{\partial^2 C}{\partial u^2}.
\end{aligned}\right\} \dots\dots\dots\dots\dots\dots (31)$$

From (7), (28) and (29) we have the system of equations

$$\left.\begin{aligned}
&\frac{\partial \rho}{\partial u}=C\sigma,\\
&\frac{\partial \rho}{\partial v}=\frac{\partial \sigma}{\partial u}=-\tau,\\
&\frac{\partial \sigma}{\partial v}=C\rho,\\
&\frac{\partial \tau}{\partial u}=-\theta\rho-C_v\sigma,\\
&\frac{\partial \tau}{\partial v}=-C_u\rho-\theta\sigma,
\end{aligned}\right\} \dots\dots\dots\dots\dots\dots (32)$$

which is completely integrable in virtue of (31).

Let $(r_i, s_i, t_i)$ $(i=1, 2, 3)$ be a system of solutions such that

$$|\, r\ s\ t\,| \neq 0.$$

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[1] J. Kanitani, Mem. Coll. Sci. Kyoto, **6**, p. **214**. These equations are misprinted in that page. The left-hand sides of the first and the second of them should be respectively equal to $f^{(1)}$ ank $f^{(2)}$ given in p. **210** and p. **211** of the Mem. cited.

Then,

$$\rho_i=\sum_{j=1}^{3}C_{ij}r_j, \qquad (i=1,\ 2,\ 3,\ 4),$$

$$\sigma_i=\sum_{j=1}^{3}C_{ij}s_j,$$

$$\tau_i=\sum_{j=1}^{3}C_{ij}t_j.$$

Therefore, the points $\rho$ and $\sigma$ lie on the plane

$$\begin{vmatrix} \eta_1 & C_{11} & C_{12} & C_{13} \\ \eta_2 & C_{21} & C_{22} & C_{23} \\ \eta_3 & C_{31} & C_{32} & C_{33} \\ \eta_4 & C_{41} & C_{42} & C_{43} \end{vmatrix}=0,$$

i.e. the line $\rho\sigma$ on which three ray point of the point $x$ with respect to the curves of Darboux passing through $x$, describes a plane, when $x$ moves on the surface.

Next, let us assume that the line $\rho\sigma$ describes a plane when $x$ moves on the surface. Then the points $\rho$, $\sigma, \frac{\partial\rho}{\partial u}, \frac{\partial\sigma}{\partial v}, \frac{\partial\sigma}{\partial u}, \frac{\partial\rho}{\partial v}$ should lie in a plane. Accordingly, in virtue of (7),

$$\mathfrak{A}_{h,k}=\mathfrak{A}'_{h,k}=\mathfrak{A}''_{h,k}=0.$$

Therefore, we know that *six focal points on a canonical ray conic K lie on three lines passing through the point P at which the plane of K touches the surface, so that on any one of the three lines lie two focal points, if, and only if, the line on which the three ray points of P with respect to the curves of Darboux passing through P lie describes a plane, when P moves on the surface.*

**5.** To any point of the surface, we can associate a canonical axis cone. Therefore, to the totality of the points of the surface, correspond an assemblage of $\infty^2$ axis cones which is the space dual of the congruence of conics. We shall call also this assemblage the *congruence* of the axis cones. Tangent planes of the manifoldness of $\infty^2$ of $\infty^1$ canonical axis cones corresponding to $\infty^1$ points of a curve $L$ on the surface determine a surface which is the envelope of these $\infty^2$ tangent planes and which we shall call the surface of the congruence corresponding to the curve $L$.

Among $\infty^1$ tangent planes of a canonical axis cone $C$, there are six focal planes any one of which touches all the surfaces of the congruence circumscribed by $C$ at the same point. We shall now find these six focal planes of $C$.

In the same manner as in n°4, we can see that, referring to the moving coordinate frame of reference of which the vertices are at the points $x$, $\rho$, $\sigma$ $\tau$ and the unit point is at the point $x+\rho+\sigma+\tau$, the equation in point coordinates of $C$ is

$$\theta w_3^2=4w_1w_2. \quad \cdots\cdots\cdots\cdots (33)$$

Put

$$\frac{w_1}{w_3}=t.$$

Then, for the points on $C$, in virtue of (33)

$$w_1 : w_2 : w_3 = t : \frac{\theta}{4t} : 1.$$

Projective coordinates ($\eta$) of a point $Q$ on a canonical axis cone corresponding to a point $P$ on the surface, referring to the space fixed coordinates frame of reference, may be expressed in the form

$$\eta=\tau+t\,\rho+\frac{\theta}{4t}\sigma+\nu x,$$

if the proportional factor be properly chosen.

If $Q$ be a point on the curve along which the cone $C$ circumacribes the surface of the congruence corresponding to the curve on the surface which is defined by the equation

$$\varphi(u,v)=0,$$

then four points $x$, $\eta$, $\frac{\partial\eta}{\partial t}$, $\frac{\partial\eta}{\partial u}du+\frac{\partial\eta}{\partial v}dv$ sohuld be coplanar.

From this reason, by the usual method, we can see that

$$\left[Ct^3-\left(\mathfrak{A}_{h,k}-\mathfrak{L}_{h,k}\right)t^2+\left(\mathfrak{A}'_{h,k}-\theta\right)\frac{\theta}{4}+\left(\frac{\theta_u}{\theta}-2h\right)\frac{\theta t}{4}+\left(\nu+\frac{\theta}{2}\right)\frac{\theta}{4}\right]du$$

$$+\left[\frac{B''\theta^2}{16}-\left(\mathfrak{A}'_{h,k}+\theta\right)t^3+\left(\mathfrak{A}''_{h,k}-\mathfrak{L}''_{h,k}\right)\frac{\theta t}{4}+\left(\frac{\theta_v}{\theta}-2k\right)\frac{t^2}{4}+\left(\nu+\frac{\theta}{2}\right)t^3\right]dv=0 \quad ..(31)$$

If $Q$ be the point of the contact of one of the focal planes, the value of $\nu$ given by (31) should de independent of $du:dv$. Therefore, in this case, $t$ must be a root of the equation.

$$Ct^6-B''\left(\frac{\theta}{4}\right)^3-t\left\{\left(\mathfrak{A}_{h,k}-\mathfrak{L}_{h,k}\right)t^4-2\mathfrak{A}'_{h,k}\frac{\theta t^2}{4}+\left(\mathfrak{A}''_{h,k}-\mathfrak{L}''_{h,k}\right)\left(\frac{\theta}{4}\right)^2\right\}$$

$$+\frac{\theta t^2}{4}\left\{\left(\frac{\theta_u}{\theta}-2h\right)t^2-\left(\frac{\theta_v}{\theta}-2k\right)\frac{\theta}{4}\right\}=0 \quad \ldots\ldots(32)$$

Six roots of this equation determine the six focal plane of $C$.

Now let us suppose that six focal planes of any canonical axis cone intersect at three lines with the tangent plane of the surface at the vertex of the cone so that at each one of these lines, intersect two focal planes. Then, in the same manner as in **n°5**, we can see that

$$\mathfrak{A}_{h,k}-\mathfrak{L}_{h,k}=\mathfrak{A}''_{h,k}-\mathfrak{L}''_{h,k}=0, \quad \ldots\ldots\ldots(33)$$

$$\mathfrak{A}'_{h,k}=0.$$

Therefore, also in this case, $\frac{C}{B''}$ must be of the form of (26).

Suppose that the transformation of the form of (27) has been made.

Then,

$$C=B'', \quad h=k=0. \quad \ldots\ldots\ldots\ldots(34)$$

From (33) and (34) we have

$$A+\frac{\partial C}{\partial v}=0, \quad A''-\frac{\partial C}{\partial u}=0. \quad \ldots\ldots(35)$$

From (30), (34) and (35) we have

$$\left.\begin{aligned}\frac{3}{2}\frac{\partial\theta}{\partial u}&=-\frac{\partial^2 C}{\partial v^2},\\ \frac{3}{2}\frac{\partial\theta}{\partial v}&=-\frac{\partial^2 C}{\partial u^2}.\end{aligned}\right\}\qquad\qquad(36)$$

From (7), (34), (35) and (36) we have

$$\frac{\partial}{\partial n}(\tau+\theta x)=0,\quad \frac{\partial}{\partial v}(\tau+\theta x)=0.$$

Therefore, the lines $x\tau$ constitute a bundle of lines. Reciprocally, if the lines $x\tau$ constitute a bundle of lines, making use of (7), we can easily see that

$$\mathfrak{A}_{h,k}-\mathfrak{L}_{h,k}=\mathfrak{A}'_{h,k}=\mathfrak{A}''_{h,k}-\mathfrak{L}''_{h,k}=0.$$

Therefore, we know that *six focal planes of any canonical axis cone intersect at three lines with the tangent plane of the surface at the vertex of the cone so that at each one of these lines intersect two focal planes, if and only if, the lines at which intersect three osculating planes of three Segre curves which meet at a point on the surface, constitute a bundle of lines.*