A Theorem Relating to the Orders of a Transcendental Integral Function of Two Independent Variables.

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Definition. Let $f(z, z')$ be a transcendental integral function of z for any z' in a finite domain $D(z')$, and be a regular function of z' in $D(z')$ for any finite z.

If ρ be a finite number such that

i) for any value of z' in $D(z')$ and for any prescribed positive value ε, there corresponds a positive value $R \equiv R(\varepsilon, z')$ such that

$$\left| f(z,z') \right| \leqq e^{r^{\rho+\varepsilon}} \quad \text{for all } |z| = r \geqq R,$$

ii) for a certain value z' in $D(z')$ and for any prescribed positive value ε,

$$\left| f(z,z') \right| > e^{r^{\rho-\varepsilon}}$$

for infinitely many values of $|z| = r$ which diverges without limit.

ρ is called the *order* of $f(z,z')$ in z for z' in $D(z')$.

Similarly we may define the order of $f(z,z')$ in z'.

Definition. Let $E(z,z')$ be a field such that z' is any point in a finite domain $D(z')$ while z is any finite point in the whole Gauss plane, and $f(z,z')$ be a regular function of z and z' in $E(z,z')$. If $f(z,z')$ be a transcendental integral function of order ρ in z such that for any prescribed positive value ε and for all z' in $D(z')$, there corresponds a positive value R (independent of z') and we have

$$\left|f(z,z')\right| \leqq e^{r^{\rho+\varepsilon}} \qquad \text{for all } |z|=r\geqq R,$$

then $f(z,z')$ is called a *uniformly increasing function* of the ρ^{th} order in z for all z' in $D(z')$.

Note. If $f(z,z')$ be a uniformly increasing function of the ρ^{th} order in z for all values of z' in the vicinity of any point in $D(z')$, it will also be true, by an extension of the Heine–Borel theorem[1], for all z' in $D(z')$.

Theorem. Let $f(z,z')$ be a transcendental integral function of z and z' and be a uniformly increasing function of the ρ^{th} order in z for all z' in a finite domain $D(z')$. Let $S\equiv\{a_1, a_2, a_3, \ldots\ldots\}$ be a set of points in the z-plane. If $f(a_i, z')\equiv P_i(z')$, $(i=1, 2, 3,\ldots\ldots)$, where $P_i(z')$ is a polynomial of the n_i^{th} degree ($n_i< N$: a finite number), then there will be no limiting point of S at finiteness and the exponent of convergence of $|a_i|$, $(i=1, 2, 3, \ldots\ldots)$, will not be greater than ρ.

As $f(a_i, z')\equiv P_i(z')$, $(i=1, 2, 3,\ldots\ldots)$, $\dfrac{\partial^N f(a_i,z')}{\partial z'^N}\equiv 0$ for $i=1$, $2, 3,\ldots\ldots$ Accordingly S has no limiting point at finiteness, unless $f(z,z')$ should be a mere polynomial of z' for any value of z, which is contrary to the assumption. Thus the first part of the theorem is proved.

As $f(z,z')$ is a uniformly increasing function of the ρ^{th} order in z for all z' in $D(z')$, we have

$$\left|\frac{\partial f(z,z')}{\partial z'}\right|=\left|\frac{1}{2\pi\sqrt{-1}}\int_C\frac{f(z,\zeta)}{(\zeta-z')^2}d\zeta\right|\leqq\frac{e^{r^{\rho+\varepsilon}}}{r'}\leqq e^{r^{\rho+\varepsilon'}} \quad \text{for all } |z|=r\geqq R,$$

where C is a circle contained in $D(z')$, centre the point z' and radius equal to r'. For a fixed value of r', ε' may be taken as small as we please. Accordingly $\dfrac{\partial f(z,z')}{\partial z'}$ is a transcendental integral function at

[1] *cf.* E. W. Hobson: The theory of functions of a real varirble (1907), p. 88.

most of the ρ^{th} order in z for any fixed value z' in $D(z')$: and so also for $\dfrac{\partial^N f(z,z')}{\partial z'^N}$.

As $a_i \neq a_j$ for $i \neq j$, we may assume that $|a_1| < |a_2| < |a_3| < \cdots$[1]. There are many transcendental integral functions of z, which have simple zero points at S and no others. Let $g(z)$ be one of the lowest order among them. Then the exponent of convergency of $|a_i|$, $(i=1, 2, 3, \cdots\cdots)$, is equal to the order of $g(z)$. As S is the set of the simple zero points of $g(z)$, $g'(a_i) \neq 0$, $(i=1, 2, 3, \cdots\cdots)$. Put

$$E_i(z,z') \equiv \left(\frac{1}{z-a_i} + \frac{1}{a_i} + \frac{z}{a_i^2} + \cdots\cdots + \frac{z^{p_i-1}}{a_i^{p_i}} \right) \frac{P_i(z')}{g'(a_i)},$$

and if $a_1 = 0$, put

$$E_1(z,z') \equiv \frac{P_1(z')}{g'(o)} \cdot \frac{1}{z}.$$

If $Max\ |P_i(z')|$ in $D(z')$ be A_i, we have

$$\left| E_i(z,z') \right| = \left| \left(\frac{1}{z-a_i} + \frac{1}{a_i} + \frac{z}{a_i^2} + \cdots\cdots + \frac{z^{p_i-1}}{a^{p_i}} \right) \frac{P_i(z')}{g'(a_i)} \right|$$

$$\leqq \frac{1}{1-\left|\dfrac{z}{a_i}\right|} \left| \frac{z^{p_i}}{a_i^{p_i+1}} \right| \cdot \left| \frac{A_i}{g'(a_i)} \right| \quad \text{for } |z| < |a_i|.$$

Hence, we determine p_i so as to satisfy

$$\left| E_i(z,z') \right| \leqq \varepsilon_i \quad \text{for} \quad |z| \leqq \frac{|a_i|}{2},$$

where $\varepsilon_i > 0$ and $\sum_{i=1}^{\infty} \varepsilon_i$ is convergent.

Then $\sum_{i=1}^{\infty} E_i(z,z')$ is absolutely and uniformly convergent in a field $E(z,z')$ (such that z' is any point in $D(z')$, while z is any finite point

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[1] If the exponent of convergence of $|a_i|$, $(i=1, 2, 3, \ldots\ldots)$, be transfinite, we take a subset $S_1 = \{b_1, b_2, b_3, \ldots\ldots\}$ of S, instead of S, for which $|b_i| < |b_j|$, $(i<j)$, and the exponent of convergence of $|b_i|$, $(i=1, 2, 3, \ldots\ldots)$, is finite and is greater than ρ.

in the whole Gauss plane) except the vicinity of $z=a_i$, $(i=1, 2, 3,\ldots\ldots\ldots)$.

Put $g(z)\sum_{i=1}^{\infty}E_i(z,z')\equiv F(z,z')$.

Then $F(z,z')$ is regular in $E(z,z')$ and takes the value $P_i(z')$ at $z=a_i$, $(i=1, 2, 3,\ldots\ldots)$. We now consider the function $f(z,z')-F(z,z')$. It is regular in $E(z,z')$ and vanishes at S. Hence we may put

$$f(z,z')-F(z,z')\equiv g(z)\cdot H(z,z')$$

where $H(z,z')$ is regular in $E(z,z')$, and is a transcendental integral function of z for any z' in $D(z')$.

$$\frac{\partial^N f(z,z')}{\partial z'^N}\equiv\frac{\partial^N F(z,z')}{\partial z'^N}+g(z)\cdot\frac{\partial^N H(z,z')}{\partial z'^N}.$$

$\frac{\partial^N f(z,z')}{\partial z'^N}$ is, as before, a transcendental integral function at most of the ρ^{th} order in z for any fixed value z' in $D(z')$. As $F(z,z')$ is a polynomial at most of the $(N-1)^{th}$ degree in z', $\frac{\partial^N F(z,z')}{\partial z'^N}\equiv 0$. As $H(z,z')$ is regular in $E(z,z')$, and is a transcendental integral function of z for any z' in $D(z')$, $\frac{\partial^N H(z,z')}{\partial z'^N}$ must be one also. For if $\frac{\partial^N H(z,z')}{\partial z'^N}\equiv 0$, $\frac{\partial^N f(z,z')}{\partial z'^N}$ would be also identically zero, which is absurd. Thus from the identity

$$\frac{\partial^N f(z,z')}{\partial z'^N}\equiv g(z)\cdot\frac{\partial^N H(z,z')}{\partial z'^N},$$

we infer that the order of $g(z)$ can not exceed that of $\frac{\partial^N f(z,z')}{\partial z'^N}$, *i. e.*, the exponent of convergency of $|a_i|$, $(i=1, 2, 3,\ldots\ldots\ldots)$, can not exceed ρ. *Q. E. D.*

The theorem may be easily extended to the case where ρ is transfinite but less than $\omega^{\omega}=\Omega$.