On the Projective Line Element upon a Hypersurface.

By

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ABSTRACT.

Let φ be a quadratic differential form such that all the curves upon a hypersurface which satisfy $\varphi=0$ are asymptotic, and g be a function of the referred curvilinear coordinates upon the hypersurface which makes $g\varphi$ an invariant form. In this paper, I investigate the geometrical meaning of the integral

$$\int\sqrt{g\varphi}$$

taken along a curve on the hypersurface and the property of the extremal curves of this integral, and I prove that a necessary and sufficient condition that a hypersurface may be represented upon a hypersurface of the second degree in such a manner that the asymptotic curves are in correspondence is that φ may be reduced to the form

$$\rho(u_1,\ldots\ldots, u_n)\ \{du_1^2+\ldots\ldots+du_n^2\}.$$

CHAPTER I

FUNDAMENTAL QUANTITIES.

1. Consider a regular hypersurface[1] (i.e. a hypersurface such that both of the manifoldness of points on it and the hyperplanes tangent to it are n) in the $n+1$ dimensional projective space defined by the equations

$$(1)\qquad x_i=x_i(u_1, u_2,\ldots\ldots, u_n),$$
$$(i=0, 1,\ldots\ldots, n+1),$$

where the functions x's are analytic functions of u's in a domain R.

1 J. Kanitani, These Memoires, **8**, 1 (1925).

Put

$$(2)\qquad h_{ij}=\left|\begin{matrix} x_o & \dfrac{\partial x_o}{\partial u_1}\cdots\cdots\dfrac{\partial x_o}{\partial u_n} & \dfrac{\partial^2 x_o}{\partial u_i \partial u_j} \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ x_{n+1} & \dfrac{\partial x_{n+1}}{\partial u_1}\cdots\cdots\dfrac{\partial x_{n+1}}{\partial u_n} & \dfrac{\partial^2 x_{n+1}}{\partial u_i \partial u_j} \end{matrix}\right| .$$

$$(3)\qquad h=\left| h_{ij} \right|$$

Hereafter, we shall denote the determinant as on the right side of (2) by

$$\left| x \ \frac{\partial x}{\partial u_1}\cdots\cdots\cdots\cdots\frac{\partial x}{\partial u_n} \ \frac{\partial^2 x}{\partial u_i \partial u_j} \right|$$

Put

$$(4)\qquad \varphi=\sum_{\sigma,\ \tau=1}^{n} h_{\sigma\tau}\, du_\sigma\, du_\tau,$$

$$(5)\qquad \psi=\frac{3}{2}d\varphi-\frac{3}{2(n+2)}\varphi d\log h-\left| x \ \frac{\partial x}{\partial u_1}\cdots\cdots\frac{\partial x}{\partial u_n} \ d^3 x \right|$$

$$=\sum_{\sigma,\tau,\rho=1}^{n} K_{\sigma\tau\rho}\, du_\sigma\, du_\tau\, du_\rho \quad (K_{ijl}=K_{ilj}=K_{jil}).$$

The curves which satisfy $\varphi=0$ are asymptotic curves and those which satisfy $\varphi=0$ are Darbou curves.[1]

Hereafter, we shall omit the symbol of the summation Σ and denote the indices which shall be summed up from 1 to n by greek letters α, β, γ, λ, μ, ν, σ, τ, ρ, etc.

2. Consider the transformation of the curvilinear coordinates

$$u_i=u_i(u_1',\ldots\ldots, u'_n),$$
$$(i=1,\ 2,\ldots\ldots,\ n),$$
$$w=\frac{\partial(u_1,\ u_2,\ldots\ldots,\ u_n)}{\partial(u_1',\ u_2',\ldots\ldots,\ u'_n)}\neq 0.$$

By this transformation, we have

$$h'_{ij}=wh_{\sigma\tau}\ \frac{\partial u_\sigma}{\partial u_i'}\ \frac{\partial u_\tau}{\partial u'_j},$$
$$h'=w^{n+2}h,$$
$$K'_{ijl}=K_{\sigma\tau\rho}\ \frac{\partial u_\sigma}{\partial u_i'}\ \frac{\partial u_\tau}{\partial u_j'}\ \frac{\partial u_\rho}{\partial u'_l}.$$

1 J. Kanitani, *loc. cit.*

If h^{ij} be the cofactor of h_{ij} in the determinant $|h_{ij}|$ divided by h,

(6) $h_{i\sigma}\, h^{j\sigma}=\varepsilon_{ij},\ (\varepsilon_{ii}=1,\ \varepsilon_{ij}=0,\ \text{if } i\neq j),$

(7) $h'^{ij}=\dfrac{1}{w}h^{\sigma\tau}\dfrac{\partial u_i'}{\partial u_\sigma}\dfrac{\partial u_j'}{\partial u_\tau}.$

Put

$$E_{ij}=K_{i\lambda\sigma}K_{j\mu\tau}h^{\lambda\mu}h^{\sigma\tau}$$

Then

$$E'_{ij}=E_{\sigma\tau}\frac{\partial u_\sigma}{\partial u_i'}\frac{\partial u_\tau}{\partial u_j'}.$$

Let ρ be any constant and put

$$\frac{1}{h}\begin{vmatrix} E_{11}+\rho h_{11} & E_{12}+\rho h_{12} & \cdots\cdots & E_{1n}+\rho h_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ E_{n1}+\rho h_{n1} & E_{n2}+\rho h_{n2} & \cdots\cdots & E_{nn}+\rho h_{nn} \end{vmatrix}$$

$$=E+\rho E_1+\rho^2E_2+\cdots\cdots+\rho^{n-1}E_{n-1}+\rho^n.$$

Then we have

$$E'=E,\ E_i'=\frac{1}{w^{n-i}}E.$$

3. Next, consider the transformation

$$x_i'=\lambda x_i$$
$$(i=0,\ 1,\ldots\ldots,\ n+1)$$

where λ is an analytic function of u's in the domain R.

By this transformation, we have

$$h'_{ij}=\lambda^{n+2}h_{ij},$$
$$h'=\lambda^{n(n+2)}h,$$
$$K'_{ijl}=\lambda^{n+2}K_{ijl},$$
$$h'^{ij}=\lambda^{-(n+2)}h^{ij}$$
$$E'_{ij}=E_{ij}.$$

Therefore, the differential forms

$$(E_i)^{\frac{1}{n-i}}\varphi,\ (E_i)^{\frac{1}{n-i}}\psi,\ E_{\sigma\tau}\,du_\sigma\,du_\tau$$

remain unchanged by any projective transformation of the hypersurface and by any transformation of the curvilinear coordinates and also by the multiplication of the point coordinates by any common factor.

CAHPTER II

PROJECTIVE LINE ELEMENTS.

4. Suppose that a congruence of $(n-1)$-flats (a system of ∞^n $(n-1)$-flats) is associated with the hypersurface in such a manner that to any point on the hypersurface corresponds one and only one $(n-1)$-flat which lies in the tangent hyperplane at this point.

The $(n-1)$-flat corresponding to the point $B(x)$ on the hypersurface may be determined by the n points

$$B_i\left(\alpha_i(x)+\left(\frac{\partial x}{\partial u_i}\right)\right). \quad (i=1,\ 2,\ldots\ldots,\ n).$$

Let $B_{n+1}\ (y)$ be a point which satisfies

$$|B\ B_1\ldots\ldots\ldots B_n\ B_{n+1}\ |=1$$

Then we have the equations of the form

$$(1)\quad \begin{cases} dB=-\alpha_\sigma\ du_\sigma+du_\sigma\ B_\sigma \\ dB_i=d\theta_{io}B+d\theta_{i1}B+\ldots\ldots d\theta_{in+1}B_{n+1} \\ \quad (i=1,\ 2,\ldots\ldots,\ n) \end{cases}$$

where $d\theta_{ij}$ are Pfaffian expressions.[1]

From (1) we have

$$(2)\quad d\theta_{in+1}=|\ B\ B_1\ldots\ldots B_n\ dB_i\ |$$

$$=|\ x\ \frac{\partial x}{\partial u_1}\ldots\ldots\frac{\partial x}{\partial u_n}\quad d\left(\frac{\partial x}{\partial u_i}\right)|$$

$$=h_{i\sigma}\ du_\sigma,$$

$$0=(\delta,\ d)\ B$$

$$=(-\delta(\alpha_\sigma\ du_\sigma)+d(\alpha_\sigma\ \delta u_\sigma)+du_\sigma\ \delta\theta_{\sigma o}-\delta u_\sigma d\theta_{\sigma o})B+\ldots\ldots$$

where terms not written are linearly dependent on $B_1,\ B_2,\ldots\ldots,\ B_{n+1}$, and accordingly

$$(3)\qquad \frac{\partial\theta_{io}}{\partial u_j}-\frac{\partial\theta_{jo}}{\partial u_i}=\frac{\partial\alpha_i}{\partial u_j}-\frac{\partial\alpha_j}{\partial u_i}$$

If the point B moves along a curve on the hypersurface, the corresponding $(n-1)$-flat generates a hypersurface which we shall call a hypersurface of the congruence.

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1 In this and the subsequent chapters I will use the absolute differential calculus due to M. René Lagrange. (Ann. Toulouse, ser. 3, vol. 14, (1922) pp. 1-69) and denote Pfafian expressions by $d\omega$, $d\Omega$, $d\theta$, etc. instead of ω, Ω, θ, etc.

Through the $(n-1)$-flat $B_1\ B_2\ldots\ldots B_n$ pass ∞^n hypersurfaces of the congruence. Any one of these surfaces which has the property that the tangent hyperplanes to it at all the points on $B_1\ B_2\ldots\ldots B_n$ are the same, may be called the developable hypersurfaces of the congruence.

Any point on the $(n-1)$-flat $B_1\ B_2\ldots\ldots B_n$ may be put in the form

$$P=\mu_1 B_1+\ldots\ldots\ldots\ldots+\mu_n B_n$$

and from (1) we have

$$dP=\mu_\sigma\ d\theta_{\sigma o}B+\ldots\ldots\ldots\ldots+\mu_\sigma\ d\theta_{\sigma\ n+1}B_{n+1},$$

where terms not written are linearly dependent on $B_1,\ B_2,\ldots\ldots,\ B_n$.

The tangent hyperplane at P to a hypersurface of the congruence is determined by the $n+1$ points

$$B_1,\ldots\ldots,\ B_n,\ \mu_\sigma\ d\theta_{\sigma o}B+\mu_\sigma\ d\theta_{\sigma\ n+1}\ B_{n+1}.$$

Therefore, the curves corresponding to the developable hypersurfaces are determined by the equations

$$(4)\qquad \frac{d\theta_{1o}}{d\theta_{1\ n+1}}=\ldots\ldots\ldots\ldots=\frac{d\theta_{no}}{d\theta_{n,\ n+1}}=\rho.$$

From (4) we have

$$(5)\qquad \begin{cases} \left(\dfrac{\partial\theta_{1o}}{\partial u_1}-\rho h_{11}\right)du_1+\ldots\ldots+\left(\dfrac{\partial\theta_{1o}}{\partial u_n}-\rho h_{1n}\right)du_n=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \left(\dfrac{\partial\theta_{no}}{\partial u_1}-\rho h_{n1}\right)du_1+\ldots\ldots+\left(\dfrac{\partial\theta_{no}}{\partial u_n}-\rho h_{nn}\right)du_n=0. \end{cases}$$

From (5) we have

$$(6)\qquad \begin{vmatrix} \dfrac{\partial\theta_{1o}}{\partial u_1}-\rho h_{11}\ldots\ldots\ldots\dfrac{\partial\theta_{1o}}{\partial u_n}-\rho h_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial\theta_{no}}{\partial u_1}-\rho h_{n1}\ldots\ldots\ldots\dfrac{\partial\theta_{uo}}{\partial u_n}-\rho h_{nn} \end{vmatrix}=0.$$

A system of the equations of the form

$$\frac{du_1}{p_1}=\frac{du_2}{p_2}=\ldots\ldots\ldots=\frac{du_n}{p_n},$$

where p's are analytic functions of u's in the domain considered, defines a system of ∞^{n-1} curves such that through any point on the hypersurface passes a curve which belongs to it. We shall call such system the family of curves.

Since (6) is an equation of the degree n with respect to ρ, there are

n families of the curves which correspond to the developable hypersurfaces of the congruence.

Let ρ_1, ρ_2 be any two roots of (6) and

$$du_1 : du_2 : \dots\dots\dots\dots\dots\dots : du_n,$$
$$\delta u_1 : \delta u_2 : \dots\dots\dots\dots\dots\dots : \delta u_n$$

be the two directions corresponding to these roots.

Then we have.

$$d\theta_{io} = \rho_1 \, d\theta_{in+1},$$
$$\delta\theta_{io} = \rho_2 \, \delta\theta_{in+1},$$
$$du_\sigma \, \delta\vartheta_{\sigma o} - \delta u_\sigma \, d\theta_{\sigma o} = (\rho_1 - \rho_2) \, h_{\sigma\tau} \, du_\sigma \, \delta u_\tau.$$

Therefore, if

$$(7) \qquad \frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_j}{\partial u_i}, \quad (i, j = 1, 2, \dots\dots, n),$$

any two curves of the said families issuing from a point on the given hypersurface are conjugate to each other.

If a congruence of $(n-1)$-flats associated with the hypersurface in the said manner satisfies condition (7), we shall say this congruence is conjugate to the hypersurface.

5. Assume that condition (7) is satisfied and α to be a function which satisfies

$$d \log \alpha = \alpha_\sigma \, du_\sigma.$$

Then the coordinates of the point B_i are of the form

$$\frac{\partial \log \alpha}{\partial u_i}(x) + \left(\frac{\partial x}{\partial u_i}\right). \quad (i = 1, 2, \dots\dots, n).$$

If we make the transformation

$$x_i' = \lambda x_i,$$

we have

$$\lambda\left\{\frac{\partial \log \alpha}{du_i}(x) + \left(\frac{\partial x}{\partial u_i}\right)\right\} = \frac{\partial \log}{\partial u_i}\left(\frac{\alpha}{\lambda}\right)(x') + \left(\frac{\partial x'}{\partial u_i}\right)$$

and if we make the transformation

$$u_i' = u_i'(u_1, u_2, \dots\dots, u_n),$$
$$(i = 1, 2, \dots\dots, n),$$

we have

$$\frac{\partial \log \alpha}{\partial u_i}(x) + \left(\frac{\partial x}{\partial u_i}\right) = \frac{\partial u'_\sigma}{\partial u_i}\left\{\frac{\partial \log \alpha}{\partial u_\sigma'}(x) + \left(\frac{\partial x}{\partial u_\sigma'}\right)\right\}.$$

Accordingly, the $(n-1)$-flat of the congruence corresponding to the point $B(x)$ may be determined by the n points

$$B_i\left(\frac{\partial \log \alpha}{\partial u_i'}(x)+\left(\frac{\partial x}{\partial u_i'}\right)\right). \quad (i=1,\ 2,\ldots\ldots,\ n).$$

Therefore, if a congruence of $(n-1)$-flats conjugate to the hypersurface is given, we can determine a function α which becomes α/λ, if we multiply the point coordinates of the points on the hypersurfaces by a common factor λ, and which remain unaltered by any transformation of the curvilinear coordinates.

Reciprocally, if a function α which has such a property is given, we can determine a congruence of $(n-1)$ flats conjugate to the hypersurface, by making correspond the $(n-1)$ flat determined by the n points

$$B_i\left(\frac{\partial \log \alpha}{\partial u_i}(x)+\left(\frac{\partial x}{\partial u_i}\right)\right) \quad (i=1,\ 2,\ldots\ldots,\ n)$$

to the point $B(x)$ on the hypersurface.

If we put

$$g=\frac{\alpha^2}{\sqrt[n+2]{h}}$$

then $g\varphi$ is an invariant form.

Reciprocally, if θ is a function which makes $\theta\varphi$ an invariant form, we can determine by the function β such that

$$\beta=\sqrt[2(n+2)]{h}\,\sqrt{\theta}$$

a congruence of $(n-1)$-flats conjugate to the hypersurface by the way above mentioned.

We shall call the quantity ds defined by the equation

$$ds^2=\theta\varphi$$

a projective line element and the congruence of $(n-1)$-flats determined by the function β the congruence subjected to this projective line element.

6. The quadratic differential form $g\varphi$ may be reduced to the sum of the squares of the n independent Pfaffian expressions.

Let

$$(7) \quad dw_i=a_{i\sigma}\,du_\sigma$$

be these Pfaffian expressions. Then we have

$$gh_{ij}=a_{\sigma i}\,a_{\sigma j}$$

$$|a_{ij}|=\sqrt{g^n h}$$

Let a^{ij} be the cofactor of a_{ij} in the determinant $|a_{ij}|$ divided by the value of this determinant. Then we have

$$a_{i\sigma}\, a^{j\sigma} = a_{\sigma i} a^{\sigma j} = \varepsilon_{ij},$$
$$du_i = a^{\sigma i}\, dw_\sigma,$$
$$a_{ij} = g a^{i\sigma}\, h_{\sigma j},$$
$$g a^{ij} = a_{i\sigma}\, h^{\sigma j},$$
$$h^{ij} = g a^{\sigma i} a^{\sigma j}.$$

Put

$$k_{ij\rho} = g\ K_{\sigma\tau\rho}\ a^{i\sigma}\ a^{j\tau}\ a^{i\rho}$$
$$e_{ij} = E_{\sigma\tau}\ a^{i\sigma}\ a^{j\tau}$$

Then we have

$$g\psi = k_{\sigma\tau\rho}\ dw_\sigma\ dw_\tau\ dw_\rho,$$
$$E_{\sigma\tau}\ du_\sigma\ du_\tau = e_{\sigma\tau}\ dw_\sigma\ dw_\tau,$$
$$e_{ij} = k_{i\sigma\tau}\ k_{j\sigma\tau}.$$

7. We shall call any system of n Pfaffian expressions $d\Omega_1$, $d\Omega_2$ ......, $d\Omega_n$ which satisfy

$$g\varphi = d\Omega_1^2 + \ldots\ldots + d\Omega_n^2$$

fundamental.

The necessary and sufficient condition that a system of n Pfaffian expressions

(8) $\quad dw_i' = q^i_\sigma\ dw_\sigma,\ (i = 1,\ 2, \ldots\ldots, n),$

q^i_j being analytic function of u's in the domain R, be fundamental is that

(9) $\quad q^\sigma_i\ q^\sigma_j = \varepsilon_{ij}.$

From (9) we have

$$|\ q^i_j\ |^2 = 1.$$

Assume that the functions q^i_j are so chosen that

$$|\ q^i_j\ | = 1.$$

Then we have from (8)

$$dw_i = q^\sigma_i\ dw_i'.$$

8. Let $f(u_1, \ldots\ldots, u_n)$ be any function of u's.

Then

$$df = \frac{\partial f}{\partial u_\sigma} du_\sigma = \frac{\partial f}{\partial u_\sigma} a^{\tau\sigma}\ dw_\tau.$$

Denote by the symbol $\dfrac{\partial f}{\partial w_i}$ the sum

$$a^{i\sigma}\frac{\partial f}{\partial u_\sigma}.$$

Then we have

$$df=\frac{\partial f}{\partial w_\sigma}dw_\sigma.$$

Any Pfaffian expressions $d\Omega$ may be put in the form

$$d\Omega=b_\sigma\, dw_\sigma$$

We shall also denote b_i by $\frac{\partial\Omega}{\partial w_i}$ so that we have

$$d\Omega=\frac{\partial\Omega}{\partial w_\sigma}dw_\sigma.$$

From the equation

$$\frac{\partial f}{\partial w_i}=a^{i\sigma}\frac{\partial f}{\partial u_s}$$

we have

$$\frac{\partial f}{\partial u_i}=a_{\sigma i}\frac{\partial f}{\partial w_\sigma}.$$

9. Put

(10) $\sigma_{ijk}=a_{k\lambda}\left(\frac{\partial a^{j\lambda}}{\partial w_i}-\frac{\partial a^{i\lambda}}{\partial w_j}\right)$,

(11) $\tau_{ijk}=\frac{1}{2}(\sigma_{jki}+\sigma_{kij}-\sigma_{ijk})$.

Then we have

(12) $\sigma_{ijk}+\sigma_{jik}=0$,

(13) $\tau_{ijk}+\tau_{jik}=0$,

(14) $\tau_{jki}-\tau_{ikj}=\sigma_{ijk}$,

(15) $(\delta, d)\, w_i=\sigma_{\lambda\mu i}\, dw_\lambda\, \delta w_\mu$,

(16) $\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j}\right)f=\sigma_{ij\lambda}\frac{\partial f}{\partial w_\lambda}$,

(17) $(\delta, d)\ \Omega=\left(\frac{\partial b_\lambda}{\partial w_\mu}-\frac{\partial b_\mu}{\partial w_\lambda}+b_\nu\,\sigma_{\lambda\mu\nu}\right)dw_\lambda\,\delta w_\mu$.

If by a transformation of the fundamental system to which we refer, a *m-uple* system of the quantities $X_{r_1\ldots\ldots r_m}$ $(r_1,\ldots\ldots, r_m=1, 2,\ldots\ldots, n)$ defined by a sequence of operations is transformed to the system

$X'_{r_1\ldots\ldots r_n}$ $(r_1,\ldots\ldots, r_m=1, 2,\ldots\ldots, n)$ which is connected to the original system by the equations

$$X'_{r_1\ldots\ldots r_m}=q^{r_1}_{\sigma_1}\, q^{r_2}_{\sigma_2}\ldots\ldots q^{r_m}_{\sigma_m}\, X_{\sigma_1\,\sigma_2\,\ldots\ldots\sigma_m},$$

we shall call this system to be covariant.

If $X_{r_1\ldots\ldots r_m}$ be an element of a *m-uple* system, we shall call the quantity defined by the equation

$$(18)\qquad \frac{\overline{\partial}\, X_{r_1\ldots\ldots r_m}}{\partial w_h}=\frac{\partial X_{r_1\ldots\ldots r_m}}{\partial w_h}-\tau_{c\lambda h}\, X_{r_1\ldots\ldots r_{\sigma-1}\lambda r_{\sigma+1}\ldots\ldots r_m}$$

the absolute partial derivative of $X_{r_1\ldots\ldots r_m}$ and the quantity defined by the equation

$$(19)\qquad \overline{d}\, X_{r_1\ldots\ldots r_m}=dX_{r_1\ldots\ldots r_m}-\tau_{r_\sigma\lambda\rho}\, X_{r_1\ldots\ldots r_{\sigma-1}\lambda r_{\sigma+1}\ldots\ldots r_m}dw_\rho$$

the absolute differential of $X_{r_1\ldots\ldots r_m}$.

By the transformation of the fundamental system, we have

$$(20)\qquad \sigma'_{ijk}=q^i_\lambda\, q^j_\mu\, q^k_\nu\, \sigma_{\lambda\mu\nu}+q^i_\lambda\, q^j_\mu\left(\frac{\partial q^k_\lambda}{\partial w_\mu}-\frac{\partial q^k_\mu}{\partial w_\lambda}\right),$$

$$(21)\qquad \tau'_{ijk}=q^i_\lambda\, q^j_\mu\, q^k_\nu\, \tau_{\lambda\mu\nu}+q^j_\lambda\, q^k_\mu\, \frac{\partial q^i_\lambda}{\partial w_\mu}.$$

In virtue of (21), we know that the assemblage of all the partial derivatives of all the elements of a covariant system forms also a covariant system.

The absolute differentiation obeys the following laws.

$$(22)\qquad \overline{d}(X_{r_1\ldots\ldots r_m}+Y_{r_1\ldots\ldots r_m})=\overline{d}\, X_{r_1\ldots\ldots r_m}+\overline{d}\, Y_{r_1\ldots\ldots r_m},$$

$$(23)\qquad \overline{d}(X_{r_1\ldots\ldots r_m}\, Y_{s_1\ldots\ldots s_l})$$

$$=Y_{s_1\ldots\ldots s_l}\,\overline{d}X_{r_1\ldots\ldots r_m}+X_{r_1\ldots\ldots r_m}\,\overline{d}\,Y_{s_1\ldots\ldots s_l}\,,$$

$$(24)\qquad \overline{d}(X_{r_1\ldots\ldots r_m\,\sigma_1\ldots\ldots\sigma_p}\, Y_{s_1\ldots\ldots s_l\,\sigma_1\ldots\ldots\sigma_p})$$

$$=Y_{s_1\ldots\ldots s_l\,\sigma_1\ldots\ldots\sigma_p}\,\overline{d}X_{r_1\ldots\ldots r_m\,\sigma_1\ldots\ldots\sigma_p}$$

$$+X_{r_1\ldots\ldots r_m\,\sigma_1\ldots\ldots\sigma_p}\,\overline{d}\,Y_{s_1\ldots\ldots s_l\,\sigma_1\ldots\ldots\sigma_p}\,.$$

10. From (15) and (16) we have

$$(25)\qquad \overline{\partial}(dw_i)-\overline{d}(\partial w_i)=0,$$

(26) $\overline{\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j}\right)} f=0$

Put

(27) $\tau_{ijkl}=\frac{\partial \tau_{ijk}}{\partial w_l}-\frac{\partial \tau_{ijl}}{\partial w_k}-\tau_{j\lambda l}\,\tau_{i\lambda k}+\tau_{j\lambda k}\,\tau_{i\lambda l}+\tau_{ij\lambda}\,\sigma_{kl\lambda}.$

Then we have

(28) $\overline{(\delta, d)}\, X_{r_1\ldots\ldots r_m}=(\delta, d)\, X_{r_1\ldots\ldots r_m}$

$$-\tau_{r_\sigma \rho\lambda\mu} X_{r_1\ldots\ldots r_{\sigma-1}\,\rho\, r_{\sigma+1}\ldots\ldots r_m}\, dw_\lambda\, \delta w_\mu,$$

(29) $\frac{\overline{\partial}}{\partial w_j}\left(\frac{\overline{\partial} X_{r_1\ldots\ldots r_m}}{\partial w_i}\right)-\frac{\overline{\partial}}{\partial w_i}\left(\frac{\overline{\partial} X_{s_1\ldots\ldots s_m}}{\partial w_j}\right)$

$$=\frac{(\delta, d)\, X_{r_1\ldots\ldots r_m}}{dw_i\, \partial w_j}-\tau_{r_\sigma \rho ij} X_{r_1\ldots\ldots r_{\sigma-1}\,\rho\, r_{\sigma+1}\ldots\ldots r_m},$$

(30) $\tau_{ijkl}+\tau_{jikl}=0,$

(31) $\tau_{ijkl}+\tau_{ijlk}=0,$

(32) $\tau_{ijkl}+\tau_{iklj}+\tau_{iljk}=0,$

(33) $\tau'_{ijkl}=q^i_\lambda\, q^j_\mu\, q^k_\nu\, q^l_\sigma\, \tau_{\lambda\mu\nu\sigma}.$

CHAPTER III

FUNDAMENTAL EQUATIONS.

11. Let us denote by A not only a point on the hypersurface but its coordinates multiplied by α. Then the $(n-1)$-flat corresponding to A of the congruence subjected to the projective line element $g\varphi$, is determined by the n points

$$A_i=\frac{\partial A}{\partial w_i}. \quad (i=1, 2, \ldots\ldots, n)$$

Consider the point

(1) $A_{n+1}=\frac{1}{n}\ \frac{\overline{\partial} A_\sigma}{\partial w_\sigma}.$

Then

$$\left| A\ A_1\ldots\ldots A_n\ A_{n+1} \right| = \frac{1}{n} \left| A\ \frac{\partial A}{\partial w_1}\ldots\ldots\frac{\partial A}{\partial w_n}\ \frac{\partial^2 A}{\partial w_\sigma^2} \right|$$

$$=\frac{a^{\sigma\lambda}\,a^{\sigma\mu}}{n\sqrt{hg^n}}\left|\,A\ \frac{\partial A}{\partial u_1}\cdots\cdots\frac{\partial A}{\partial u_n}\ \frac{\partial^2 A}{\partial u_\lambda\,\partial u_\mu}\right|$$

$$=\frac{1}{n}\ h^{\lambda\mu}\ h_{\lambda\mu}=1.$$

Hence, any point M in the space may be expressed in the form

$$M=\lambda_o A+\lambda_1 A_1+\cdots\cdots\cdots+\lambda_{n+1} A_{n+1}.$$

Suppose that the points dA, $\overline{d}A_i$, $(i=1, 2,\ldots\ldots, n)$, dA_{n+1} are expressed in the form

$$(2)\quad\begin{cases} dA=dw_{oo}\,A+dw_{o1}\,A_1+\cdots\cdots\cdots+dw_{on}\,A_{n+1},\\ \overline{d}A_1=dw_{1o}\,A+dw_{11}\,A_1+\cdots\cdots\cdots+dw_{1n}\,A_{n+1},\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ dA_{n+1}=dw_{n+1\,o}\,A+\cdots\cdots\cdots+dw_{n+1,\,n+1}\,A_{n+1}, \end{cases}$$

where dw_{ij} are Pfaffian expressions.

We shall call the equations (2) the fundamental equations.

Evidently,

(3) $dw_{oo}=dw_{on}=0$,

(4) $dw_{oi}=dw_i$. $(i=1, 2,\ldots\ldots, n)$.

From (2) we have

(5) $dw_{in}=|\,A\,A_1\ldots\ldots A_n\ \overline{d}A_i\,|$

$$=\frac{a^{i\sigma}}{\sqrt{hg^n}}\left|\,A\ \frac{\partial A}{\partial u_1}\cdots\cdots\frac{\partial A}{\partial u_n}\quad d\left(\frac{\partial A}{\partial u_\sigma}\right)\right|$$

$$=g a^{i\sigma}\ h_{\sigma\lambda}a_{i\lambda}\ du_\lambda=dw_i$$

12. The system of the equations (2) is equivalent to system

$$(2')\quad\begin{cases} dA=dw_\lambda\ A_\lambda,\\ dA_i=dw_{io}A+(dw_{i\lambda}+\tau_{i\lambda\rho}\ dw_\rho)\ A_\lambda+dw_i\ A_{n+1},\\ dA_{n+1}=dw_{n+1,\,o}A+dw_{n+1\lambda}\ A_\lambda+dw_{n+1,\,n+1}\ A_{n+1}. \end{cases}$$

The necessary and sufficient condition that the system of the total differential equations (2′) and accordingly (2) may be completelyi ntegrable, is that the equations

(6) $(\delta, d)\ A=0$,

(7) $(\delta, d)\ A_i=0$, $(i=1, 2,\ldots\ldots, n)$,

(8) $(\delta, d)\ A_{n+1}=0$.

follow as the consequence of (2).

The equation (6) is equivalent to

$$(9)\quad dw_\sigma\,\delta w_{\sigma k}-\delta w_\sigma\,dw_{\sigma k}=0.$$

From (9) we have

$$(10)\quad \frac{\partial w_{ik}}{\partial w_j}=\frac{\partial w_{jk}}{\partial w_i}.\quad \begin{pmatrix} i,\ j=1,\ 2,\ldots\ldots,\ n \\ k=0,\ 1,\ldots\ldots,\ n \end{pmatrix}$$

From (2) and (5) we have

$$(11)\quad \frac{\overline{\partial} A_i}{\partial w_j}=\frac{\partial w_{io}}{\partial w_j}A+\ldots\ldots\ldots+\frac{\partial w_{in}}{\partial w_j}A_n+\varepsilon_{ij}\,A_{n+1}.$$

From (10) we have

$$\frac{\partial w_{\sigma k}}{\partial w_\sigma}=0.\quad (k=0,\ 1,\ldots\ldots,\ n)$$

From (4), (5) and (11) we have

$$\frac{\overline{\partial}}{\partial w_i}\left(\frac{\overline{\partial} A_i}{\partial w_j}\right)=\ldots\ldots\ldots+\left(\frac{\partial w_{ii}}{\partial w_j}+\varepsilon_{ij}\,\frac{\partial w_{n+1,\ n+1}}{\partial w_i}\right)A_{n+1},$$

where terms not written are linearly dependent on $A, A_1,\ldots\ldots, A_n$

But

$$(12)\quad \frac{\overline{\partial}}{\partial w_i}\left(\frac{\overline{\partial} A_i}{\partial w_j}\right)-\frac{\overline{\partial}}{\partial w_j}\left(\frac{\overline{\partial} A_i}{\partial w_i}\right)=-\tau_{i\lambda ji}\,A_\lambda.$$

Therefore, we have

$$n\,dA_{n+1}=\ldots\ldots\ldots+(dw_{\sigma\sigma}+dw_{n+1,\ n+1})\,A_{n+1}$$

and accordingly

$$(n-1)\,dw_{n+1}=dw_{\sigma\sigma}+dw_{n+1,\ n+1}.$$

On the other hand, we have

$$\begin{aligned} 0&=d\,|\,A\ A_1\ldots\ldots A_n\ A_{n+1}\,| \\ &=dw_{\sigma\sigma}+dw_{n+1\ \ n+1}. \end{aligned}$$

Hence, we have

$$dw_{n+1,\ n+1}=0.$$

13. In virtue of (28) in article 10, the system of the equations (7) is equivalent to

$$(13)\quad \overline{\delta}(dw_{io})-\overline{d}(\delta w_{io})+\sum_{\sigma=1}^{n+1}(dw_{i\sigma}\,\delta w_{\sigma o}-\delta w_{i\sigma}\,dw_{\sigma o})=0,$$

$$(14)\quad \overline{\delta}(dw_{ij})-\overline{d}(\delta w_{ij})+\sum_{\sigma=1}^{n+1}(dw_{i\sigma}\,\delta w_{\sigma j}-\delta w_{i\sigma}\,dw_{\sigma j})$$

$$+\tau_{ij\lambda\mu}\,dw_\lambda\,\delta w_\mu=0,$$

$$(15)\quad dw_{i\sigma}\,\delta w_\sigma-\delta w_{i\sigma}\,dw_\sigma=0.$$

From (15) we have

(16) $$\frac{\partial w_{hi}}{\partial w_j}=\frac{\partial w_{hj}}{\partial w_i}. \quad (h=1, 2, \ldots\ldots, n)$$

From (10) and (16) we have

(17) $$\frac{\partial w_{ij}}{\partial w_h}=\frac{\partial w_{ji}}{\partial w_h}.$$

From (2) we have

(18) $$d^2A=dw_\sigma dw_{\sigma o}A+\left\{dw_\sigma dw_{\sigma\tau}+\overline{d}(dw_\tau)\right\}A_\tau+dw_\sigma\ dw_\sigma\ A_{n+1},$$

(19) $$d^3A=\ldots\ldots\ldots\ldots+\left\{dw_\sigma\ dw_\tau\ dw_{\sigma\tau}+\tfrac{3}{2}\,d(dw_\sigma\ dw_\sigma)\right\}A_{n+1},$$

where terms not written are linearly dependent on $A, A_2, \ldots\ldots, A_n$

From (19) we have

$$-dw_\sigma\ dw_\sigma\ dw_{\sigma\tau}=g\psi=k_{\sigma\tau\rho}\ dw_\sigma\ dw_\tau\ dw_\rho$$

and accordingly,

(20) $$\frac{\partial w_{ij}}{\partial w_l}=-k_{ijl}.$$

From (12) and (20) we have

(21) $$k_{\sigma\sigma i}=0.$$

14. From (14) we have

(22) $$\frac{\overline{\partial}k_{ijl}}{\partial w_h}-\frac{\overline{\partial}k_{ijh}}{\partial w_l}+k_{i\sigma h}\ k_{\sigma jl}-k_{i\sigma l}\ k_{\sigma jh}$$
$$+\varepsilon_{jl}\frac{\partial w_{io}}{\partial w_h}-\varepsilon_{jh}\frac{\partial w_{io}}{\partial w_l}+\varepsilon_{ih}\frac{\partial w_{n+1\,j}}{\partial w_l}-\varepsilon_{il}\frac{\partial w_{n+1\,j}}{\partial w_h}+\tau_{ijhl}=0.$$

From (22) we have

(23) $$2\frac{\overline{\partial}k_{ijl}}{\partial w_h}-2\frac{\overline{\partial}k_{ijh}}{\partial w_l}+\varepsilon_{jl}\left(\frac{\partial w_{io}}{\partial w_h}-\frac{\partial w_{n+1\,i}}{\partial w_h}\right)$$
$$-\varepsilon_{jh}\left(\frac{\partial w_{io}}{\partial w_l}-\frac{\partial w_{n+1\,i}}{\partial w_l}\right)-\varepsilon_{ih}\left(\frac{\partial w_{jo}}{\partial w_l}-\frac{\partial w_{n+1\,j}}{\partial w_l}\right)$$
$$+\varepsilon_{il}\left(\frac{\partial w_{jo}}{\partial w_h}-\frac{\partial w_{n+1\,j}}{\partial w_h}\right)=0,$$

(24) $$2k_{ih\sigma}\ k_{jl\sigma}-2k_{il\sigma}\ k_{jh\sigma}+2\tau_{ijhl}$$
$$+\varepsilon_{jl}\left(\frac{\partial w_{io}}{\partial w_h}+\frac{\partial w_{n+1\,i}}{\partial w_h}\right)-\varepsilon_{jh}\left(\frac{\partial w_{io}}{\partial w_l}+\frac{\partial w_{n+1\,i}}{\partial w_l}\right)$$
$$+\varepsilon_{ih}\left(\frac{\partial w_{jo}}{\partial w_l}+\frac{\partial w_{n+1\,j}}{\partial w_l}\right)-\varepsilon_{il}\left(\frac{\partial w_{jo}}{\partial w_h}+\frac{\partial w_{n+1\,j}}{\partial w_h}\right)=0.$$

From (21) and (23) we have

$$(25)\qquad n\left(\frac{\partial w_{jo}}{\partial w_h}-\frac{\partial w_{n+1\,j}}{\partial w_h}\right)=2\frac{\overline{\partial} k_{\sigma jh}}{\partial w_\sigma}-\varepsilon_{ji}\frac{\partial w_{n+1\sigma}}{\partial w_\sigma}.$$

From (21) and (24) we have

$$(26)\qquad (n-2)\left(\frac{\partial w_{jo}}{\partial w_h}+\frac{\partial w_{n+1\,j}}{\partial w_h}\right)=2e_{hj}+2\,\tau_{\sigma j\,\sigma}-\varepsilon_{jh}\frac{\partial w_{n+1\sigma}}{\partial w_\sigma}.$$

From (26) we have

$$(27)\qquad \frac{\partial w_{n+1\,\sigma}}{\partial w_\sigma}=e_{\sigma\sigma}+\tau_{\sigma\lambda\lambda\sigma}.$$

From (13) we have

$$(28)\qquad \frac{\overline{\partial}}{\partial w_l}\left(\frac{\partial w_{io}}{\partial w_h}\right)-\frac{\overline{\partial}}{\partial w_h}\left(\frac{\partial w_{io}}{\partial w_l}\right)-k_{ihj}\frac{\partial w_{\sigma o}}{\partial w_l}+k_{il\sigma}\frac{\partial w_{\sigma o}}{\partial w_h}$$

$$\varepsilon_{ih}\frac{\partial w_{n+1,\,o}}{\partial w_l}-\varepsilon_{il}\frac{\partial w_{n+1,\,o}}{\partial w_h}=0$$

From (28) we have

$$(29)\qquad (n-1)\frac{\partial w_{n+1,\,o}}{\partial w_h}=\frac{\overline{\partial}}{\partial w_\sigma}\left(\frac{\partial w_{\sigma o}}{\partial w_h}\right)-k_{\cdot\sigma\tau}\frac{\partial w_{\sigma o}}{\partial w_\tau}$$

CHAPTER IV

PROJECTIVE NORMALS GEODESIC CURVES.

15. Any point M on the hypersurface in the vicinity of A is given by the equation

$$M=A+dA+\tfrac{1}{2}d^2A+\tfrac{1}{6}d^3A+\ldots\ldots$$
$$=A[1+\tfrac{1}{2}dw_\sigma\, dw_{\sigma o}+\ldots\ldots$$
$$+A_\lambda[dw_\lambda+\tfrac{1}{2}\{dw_\sigma\, dw_{\sigma\lambda}+\overline{d}(dw_\lambda)\}+\ldots\ldots]$$
$$+A_{n+1}[\tfrac{1}{2}dw_\sigma\, dw_\sigma+\tfrac{1}{6}\{dw_\sigma\, dw_\tau\, dw_{\sigma\tau}+\tfrac{3}{2}d(dw_\sigma\, dw_\sigma)\}+\ldots\ldots]$$

Let $(z_1, z_2, \ldots\ldots, z_n)$ be nonsymmetrical coordinates referred to the system $[A;\, A_1, \ldots\ldots, A_{n+1}]$.[1] Then

$$(1)\qquad \begin{cases} z_i=dw_i+\tfrac{1}{2}\{dw_\sigma\, dw_{\sigma i}+\overline{d}(dw_i)\}+\ldots\ldots \\ \quad (i=1,\ 2,\ldots\ldots,\ n) \\ z_{n+1}=\tfrac{1}{2}dw_\sigma\, dw_\sigma+\tfrac{1}{6}\{dw_\sigma\, dw_\tau\, dw_{\sigma\tau}+\tfrac{3}{2}d(dw_\sigma\, dw_\sigma)\}+\ldots\ldots \end{cases}$$

1 Jôyô Kanitani, *loc. cit.*

The equations (1) give for the development in the power series of $z_1, \ldots\ldots, z_n$ to the third degree,

$$z_{n+1}=\tfrac{1}{2}(z_1^2+\ldots\ldots+z_n^2)+\tfrac{1}{3}k_{\sigma\tau\rho}\,z_\sigma\, z_\tau\, z_\rho+\ldots\ldots$$

Let $(\xi_o, \xi_1, \ldots\ldots, \xi_{n+1})$ be projective coordinates referred to the coordinate frame of reference whose vertices and unit point are respectively

$$A,\ A_1, \ldots\ldots,\ A_{n+1},\ A+A_1+\ldots\ldots+A_{n+1}.$$

Then the osculating quadric at A is

$$\xi_o\ \xi_{n+1}=\tfrac{1}{2}(\xi_1^2+\ldots\ldots+\xi_n^2+a\xi_{n+1}^2).$$

Therefore, the line AA_{n+1} is reciprocal to the $(n-1)$-flat $A_1\ A_2\ldots\ldots A_n$. We shall call this line the projective normal subjected to the projective line element given by the equation

$$(2)\quad ds^2=dw_1^2+\ldots\ldots\ldots+dw_n^2.$$

16. The tangent hyperplane to a developable hypersurface of the congruence subjected to the projective line element given by (2) which passes through $A_1\ A_2\ldots\ldots A_n$ is determined by the points

$$A_1,\ A_2, \ldots\ldots,\ A_n,\ \rho_i A+A_{n+1},$$

where ρ_i is a root of the equation

$$\begin{vmatrix} \dfrac{\partial w_{1o}}{\partial w_1}-\rho, & \dfrac{\partial w_{1o}}{\partial w_2} & \ldots\ldots\ldots & \dfrac{\partial w_{1o}}{\partial w_n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \dfrac{\partial w_{no}}{\partial w_1} & \ldots\ldots\ldots & \ldots\ldots\ldots & \dfrac{\partial w_{no}}{\partial w_n}-\rho \end{vmatrix}=0$$

Since

$$\rho_\sigma=\frac{\partial w_{\sigma o}}{\partial w_\sigma}=0,$$

the point A_{n+1} is the harmonic conjugate of A with respect to the n points at which the n tangent hyperplanes at any point on $A_1\ A_2\ldots\ldots A_n$ to the n developable hypersurfaces passing through $A_1\ A_2\ldots\ldots A_n$.

17 The assemblage of the projective normals at all the points on the hypersurface forms a congruence. When the point moves on a curve on the hypersurface, the projective normal at A generates a surface which we shall call a surface of the congruence. Through AA_{n+1} pass ∞^n surfaces of the congruence. Any one of them such that the tangent planes to it at all the points on AA_{n+1} are the same may be called the developable surface of the congruence.

The curves corresponding to the developable surfaces are defined by the equations

$$\frac{dw_{n+1,\,1}}{dw_1} = \ldots\ldots = \frac{dw_{n+1,\,n}}{dw_n}.$$

By the same method as in article 4, we can prove that through AA_{n+1} pass n developable surfaces of the congruence and that two curves corresponding to any two of them are conjugate to each other.

18 The equation of a quadric of the $n-1$ dimensions on the tangent hyperplane at A which touches the cone of the asymptotic tangents at A over the variety at which it intersects with $A_1\ A_2 \ldots\ldots A_n$ is of the form

$$k^2\xi_o^2 + \xi_1^2 + \ldots\ldots + \xi_n^2 = 0.$$

If this quadric is regarded as the absolute, the distance d between A and a point on the tangent hyperplane at A in the vicinity of A, is

$$d = \frac{k}{2i} \log \frac{k + i\sqrt{z_1^2 + \ldots\ldots + z_n^2}}{k - i\sqrt{z_1^2 + \ldots\ldots + z_n^2}}$$

$$= \sqrt{z_1^2 + \ldots\ldots + z_n^2} + \ldots\ldots\ldots\ldots\ldots$$

where $z_1, \ldots\ldots, z_n$ are the nonsymmetrical coordinates of the point referred to the system $[A;\ A_1, \ldots\ldots, A_n]$.

Now, let P be a point on the hypersurface in the vicinity of A, and Q be the projection of P from a point on AA_{n+1} upon the tangent hyperplane at A, then we have

$$\sqrt{z_1^2 + \ldots\ldots + z_n^2} = \sqrt{dw_1^2 + \ldots\ldots + dw_n^2} + \ldots\ldots$$

Therefore, we have the following theorem.

Let

$$u_1 = f_1(t), \ldots\ldots\ldots\ldots\ldots\ldots, u_n = f_n(t),$$

be the equations of a curve on the hypersurface and $P_o, P_1, \ldots\ldots, P_n$ *be the points on this curve which correspond to the values* $t_o,\ t_1, \ldots\ldots, t_n$ *of* t *such that*

$$t_o < t_1 < \ldots\ldots\ldots\ldots < t_n.$$

Let Q_i $(1 \leqq i \leqq n)$ *be the projection of* P_i *from a point on the projective normal at* P_{i-1} *subjected to the projective line element given by* (2) *upon the tangent hyperplane at* P_{i-1} *and* d_i *be the distance between* P_{i-1} *and* Q_i *in the case where a quadric of the* $n-1$ *dimensions on the tangent hyperplane at* P_{i-1} *which touches the cone of the asymptotic tangents at* P_{i-1} *over the variety at which it intersects with the* $(n-1)$*-flat corresponding to* P_{i-1} *of the congruence subjected to the projective line element given by* (2) *is the absolute. Then*

$$s=\int_{t_0}^{t_1}\sqrt{dw_1^2+\ldots\ldots+dw_n^2}=\lim_{t_i-t_{i-1}\to o}(d_1+\ldots\ldots+d_n)$$

19. Let us call the extremal curves of the integral

$$s=\int\sqrt{dw_1^2+\ldots\ldots+dw_n^2}$$

the geodesic curves.

The equations of the geodesic curves are

$$(3)\qquad \frac{d^2u_i}{ds^2}+\left\{\begin{matrix}\sigma\tau\\ i\end{matrix}\right\}\frac{du_\sigma}{ds}\frac{du_\tau}{ds}=0,$$

$$(i=1,\ 2,\ldots\ldots,\ n),$$

where $\{{}^{ij}_{k}\}$ are Cristoffel symbols formed with respect to $g\varphi$.

In virtue of (10) in article 9, we have

$$(4)\quad [{}^{ij}_{k}]=a_{\lambda i}\,a_{\mu j}\,a_{\nu k}\,\tau_{\mu\nu\lambda}+a_{\sigma k}\frac{\partial a_{\sigma j}}{\partial u_i},$$

$$(5)\quad \{{}^{ij}_{k}\}=a_{\lambda i}\,a_{\mu j}\,a^{\nu k}\,\tau_{\mu\nu\lambda}+a^{\sigma k}\frac{\partial a_{\sigma j}}{\partial u_i}.$$

From (3) and (5) we have

(6) $d(dw_i)=dw_\lambda\,dw_\mu\,\tau_{i\lambda\mu}$.

In virtue of (6), we have

$$\overline{d}(dw_i)=d(dw_i)-\tau_{i\lambda\mu}\,dw_\lambda\,dw_\mu=0.$$

Therefore, the equations of the geodesic curves may be written

(7) $\overline{d}(dw_i)=0.\quad (i=1,\ 2,\ldots\ldots,\ n).$

20. Consider a family of curves defined by the equations

$$\frac{du_1}{p_1}=\frac{du_2}{p_2}=\ldots\ldots\ldots\ldots=\frac{du_n}{p_n}$$

which are not asymptotic. Let C be a curve of this family which passes through A.

Choose the fundamental system to which we refer so that the equation of the above family of the curves becomes[1]

$$dw_2=dw_3=\ldots\ldots dw_n=0.$$

The osculating plane of the curve C at A is determined by the points

1 Jôyô Kanitani, *loc. cit.*

$$A,\ A_1,\ \sum_{\sigma=2}^{n} (-k_{11\sigma}+\tau_{1\sigma 1})A_\sigma + A_{n+1}.$$

The $(n-2)$-flat reciprocal to this osculating plane is determined by the $n-1$ points

$$(-k_{112}+\tau_{121})A + A_2, \ldots\ldots, (k_{11n}+\tau_{1n1})A + A_n.$$

The characteristic of the curve C at A is the intersection of the three hyperplanes

$$| M\ A\ A_1 \ldots\ldots\ldots\ldots A_n | = 0,$$
$$d\, | M\ A\ A_1 \ldots\ldots\ldots\ldots A_n | = 0,$$
$$d^2\, | M\ A\ A_1 \ldots\ldots\ldots\ldots A_n | = 0,$$

where M is a current point in the space and d denotes the differentiation along the curve C.

This intersection may be determined by the three hyperplanes

$$| M\ A\ A_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A_n | = 0,$$
$$| M\ A\ A_{n+1}\ A_2\ \ldots\ldots\ldots\ldots\ldots\ldots A_n | = 0,$$
$$| M\ A_1\ A_{n+1}\ A_2\ \ldots\ldots\ldots\ldots\ldots\ldots A_n |$$
$$+ \sum_{\sigma=2}^{n} (-k_{11\sigma}+\tau_{\sigma 11})\ | M\, A\, A_{n+1}\, A_2 \ldots\ldots A_{\sigma-1} A_1 A_{\sigma+1} \ldots\ldots A_n | = 0.$$

and, accordingly, by the $n-1$ points

$$(k_{112}+\tau_{121})A + A_2, \ldots\ldots, (k_{11n}+\tau_{1n1})A + A_n$$

A line l which passes through A and through the point

$$\sum_{\sigma=2}^{n} \mu_\sigma A_\sigma$$

intersects the characteristic and the reciprocal of the osculating plane at the points

$$\sum_{\sigma=2}^{n} \mu_\sigma \{A_\sigma + (k_{112}+\tau_{121})A\}$$

and

$$\sum_{\sigma=2}^{n} \mu_\sigma \{A_\sigma + (-k_{112}+\tau_{121})A\}$$

respectively.

The harmonic conjugate of A with respect to these two points is

$$\sum_{\sigma=2}^{n} \mu_\sigma (A_\sigma + \tau_{121}A).$$

If the line l moves about A in the $(n-1)$-flat conjugate to C at A, this point generates a $(n-2)$-flat which is determined by the $n-1$ points

$$\tau_{121}A+A_2,\ldots\ldots,\ \tau_{1n1}+A_n$$

and which we shall call harmonic $(n-1)$-flat of C at A.

If C is a geodesic curve, we have

$$\tau_{211}=\tau_{311}=\ldots\ldots=\tau_{n11}=0.$$

Therefore, we know that *the harmonic $(n-2)$-flat of a geodesic curve at any point A on it lies in the $(n-1)$-flat corresponding to A of the congruence subjected to the referred projective line element.* The relation dual to it is also true: *the plane reciprocal to the harmonic $(n-2)$-flat of a geodesic curve at any point A on it passes through the projective normal at A.*

21. Suppose that all the $(n-1)$-flats of the congruence subjected to the projective line element (2) are upon a hyperplane. In this case, the $2n$ points $A_1,\ldots\ldots,\ A_n,\ dA_1,\ldots\ldots,\ dA_n$ must be on a hyperplane whatever the values of $dw_1,\ldots\ldots,\ dw_n$ may be.

Suppose that this hyperplane is determined by the points

$$A_1,\ A_2,\ldots\ldots,\ A_n,\ \lambda A+\mu A_{n+1}.$$

Then we have

$$(8)\qquad \begin{cases} \mu\dfrac{\partial w_{io}}{\partial w_i}=\lambda, \\[2ex] \dfrac{\partial w_{io}}{\partial w_j}=0, \text{ if } i\neq j. \end{cases}$$

From (8) we have

$$\lambda=\frac{\mu}{n}\ \frac{\partial w_{\sigma o}}{\partial w_\sigma}=0$$

and, accordingly,

$$\frac{\partial w_{io}}{\partial w_i}=0.\quad (i=1,\ 2,\ldots\ldots,\ n).$$

Therefore, in virtue of (29) in the article 13, we have

(9) $dw_{ko}=0.\quad (k=1,\ 2,\ldots\ldots,\ n+1).$

Reciprocally, if the condition (9) is satisfied, the points $A_1,\ A_2,\ldots\ldots,\ A_{n+1}$ lie always upon a fixed hyperplane.

In fact, if this condition is satisfied, we have the following completely integrable system of the total differential equations.

$$(10)\quad \begin{cases} dA_i=(dw_{i\lambda}+v_{i\lambda\rho}\, dw_\rho)A_\lambda+dw_i\, A_{n+1}, (i=1,\ 2,\ldots\ldots n), \\ dA_{n+1}=dw_{n+1\,\lambda}\, A_\lambda. \end{cases}$$

Let

$$(y_i^{(1)},\ y_i^{(2)},\ldots\ldots,\ y_i^{(n+1)})\quad (i=1,\ 2,\ldots\ldots,\ n+1)$$

be $n+1$ independent system of the solutions of (10). Then any system of the solutions of (10) $(Y^{(1)},\ \ldots\ldots Y^{(n+1)})$ may be put in the form

$$Y^{(i)}=\sum_{\sigma=1}^{n+1} C_\sigma\, y_\sigma^{(1)}$$

Now, if the coordinates of the points $A,\ A_1,\ldots\ldots,\ A_{n+1}$ are

$$(t_o,\ t_1,\ldots\ldots,\ t_{n+1}),$$
$$(t_o^{(i)},\ t_1^{(i)}\ldots\ldots,\ t_{n+1}^{(i)}),\ (i=1,\ 2,\ldots\ldots,\ n+1)$$

then

$$(t_j^{(1)},\ t_j^{(2)},\ldots\ldots,\ t_j^{(n+1)})\ (j=0,\ 1,\ 2,\ldots\ldots,\ n+1)$$

are $(n+2)$ systems of the solutions of (10).

Accordingly, we have

$$t_j^{(i)}=C_{j\sigma}\, y_\sigma^{(i)}.$$

Therefore, the points $A_1,\ldots\ldots,\ A_{n+1}$ are on the hyperplane

$$\begin{vmatrix} x_o & C_{o1}\ldots\ldots C_{o\ n+1} \\ x_1 & C_{11}\ldots\ldots C_{1\ n+1} \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n+1} & C_{n+1,\ 1}\ldots\ldots C_{n+1,\ 1\ n+1} \end{vmatrix}=0$$

22. If

$$dw_{ko}=0,\ (k=1,\ 2,\ldots\ldots,\ n+1)$$

then

$$(1,\ 0,\ 0,\ldots\ldots,\ 0)$$

is a system of the solutions of (10).

But

$$(t_j,\ t_j^{(1)}\ldots\ldots,\ t_j^{(n+1)})\ (j=0,\ 1,\ 2,\ldots\ldots,\ n+1)$$

are $n+2$ independent systems of the solutions.

Hence we have

$$\begin{aligned} 1&=C_o\, t_o+C_1\, t_1+\ldots\ldots+C_{n+1}\, t_{n+1}) \\ &=\alpha(C_o\, x_o+C_1\, x_1+\ldots\ldots+C_n\, x_{n+1}). \end{aligned}$$

Therefore, we know that the congruence of the $(n-1)$-flat conjugate to the hypersurface such that all the $(n-1)$-flats of it are upon a fixed hyperplane, is determined by the function

$$\alpha = \frac{1}{C_o x_o + \dots\dots + C_{n+1} x_{n+1}}$$

where the C's are constants.

CHAPTER V

HYPERSURFACES WHICH CAN BE REPRESENTED UPON A HYPERSURFACE OF THE SECOND DEGREE SO THAT THE ASYMPOTIC CURVES ARE IN CORRESPONDENCE.

23. The equation referred to the nonsymmetrical coordinates of an osculating quadric at the point A is of the form

$$z_{n+1} = \tfrac{1}{2}(z_1^2 + \dots\dots + z_n^2 + az_{n+1}^2) + z_{n+1}(a_1 z_1 + \dots\dots + a_n z_n)$$

which gives for the development of z_{n+1} in the power series of $z_1, \dots\dots, z_n$ to the third order

$$z_{n+1} = \tfrac{1}{2}(z_1^2 + \dots\dots + z_n^2) + \tfrac{1}{2}(z_1^2 + \dots\dots + z_n^2)(a_1 z_1 + \dots\dots + a_n z_n) + \dots\dots$$

If the given hypersurface is a quadric, there must be a quadric which has the contact of the third order with the given hypersurface, and accordingly we have

(1) $a_i = k_{11i} = k_{22i} \dots\dots = k_{nni},$

$k_{ijk} = 0. \quad (i \neq j \neq k).$

From (1) we have

$$a_i = \frac{1}{n} k_{\sigma\sigma i} = 0.$$

Therefore, we know that if the given hypersurface is a quadric, we must have

(2) $k_{ijk} = 0. \quad (i,\ j,\ k = 1,\ 2, \dots\dots, n).$

Next, suppose that the condition (2) are satisfied.

Denote by a the hyperplane tangent to the given hypersurface at A as well as its hyperplane coordinates multiplied by a common factor properly chosen and put

$$a_i = \frac{\partial a}{\partial w_i}, \quad (i = 1,\ 2, \dots\dots, n),$$

$$a_{n+1} = \frac{\overline{\partial a_\sigma}}{\partial w_\sigma}.$$

Then $a, a_1, \dots\dots, a_{n+1}$ also satisfy the system of the fundamental equations (2) in article 11.[1] But

1 Jôyô Kanitani, *loc. cit.*

$$(t_j, t_j^{(1)} \ldots\ldots, t_j^{(n+1)}) \ (j=0, 1, \ldots\ldots, n+1)$$

are $n+2$ independent systems of the solutions of the system of fundamental equations

Therefore, if

$$(T_o, T_1, \ldots\ldots, T_{n+1})$$

be the hyperplane coordinates of a, we have

$$(3) \quad \begin{cases} T_o = C_{oo} t_o + C_{o1} t_1 + \ldots\ldots + C_{on+1} t_{n+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ T_{n+1} = C_{n+1,\,o} t_o + \ldots\ldots\ldots + C_{n+1,\,n+1} t_{n+1} \end{cases}$$

From (3) we have

$$0 = \sum_{\sigma=0}^{n+1} t_\sigma T_\sigma = \sum_{\sigma,\,\tau=0}^{n+1} C_{\sigma\tau} t_\sigma t_\tau.$$

Therefore, the given surface is a quadric.

24. On a hypersurface of the second degree, choose a projective line element so that all the $(n-1)$-flats of the congruence subjected to it are on a fixed hyperplane. Then we have

$$\frac{\partial w_{ho}}{\partial w_i} = 0, \quad k_{ijl} = 0.$$

$$\begin{pmatrix} i, j, l = 1, 2, \ldots\ldots, n \\ h = 1, 2, \ldots\ldots, n+1 \end{pmatrix}$$

From (23), (24) in article **13** and these equations we have

$$\tau_{ijhl} = 0, \quad (i \neq j \neq k \neq l)$$

$$\tau_{ijki} = 0, \quad (i \neq j \neq k)$$

$$\tau_{ijji} = \frac{\partial w_{n+1\,i}}{\partial w_i}, \quad (i \neq j)$$

$$\frac{\partial w_{n+1\,i}}{\partial w_i} = \frac{1}{n} \frac{\partial w_{n+1\,\sigma}}{\partial w_\sigma} = \theta.$$

Therefore, the fundamental equations become

$$dA = dw_\lambda A_\lambda,$$

$$dA_i = \tau_{i\lambda\rho} dw_\rho A_\lambda + dw_i A_{n+1}, \quad (i = 1, 2, \ldots\ldots, n)$$

$$dA_{n+1} = \theta dA.$$

From the last equation we have

$$0 = \frac{\partial}{\partial u_j}\left(\frac{\partial A_{n+1}}{\partial u_i}\right) - \frac{\partial}{\partial u_i}\left(\frac{\partial A_{n+1}}{\partial u_j}\right) = \frac{\partial \theta}{\partial u_j}\frac{\partial A}{\partial u_i} - \frac{\partial \theta}{\partial u_i}\frac{\partial A}{\partial u_j}$$

and accordingly,

$$\frac{\partial\theta}{\partial u_i}=0. \quad (i=1,\ 2,\ldots\ldots,\ n)$$

Therefore, θ is constant.

Put

$$\theta=k$$

Then, by the theorem of R. Lagrange,[1] we can choose the curvilinear coordinates so that

$$dw_1^2+\ldots\ldots\ldots\ldots+dw_n^2$$
$$=\frac{1}{2}\frac{du_1^2+\ldots\ldots+du_n^2}{u_1^2+\ldots\ldots+u_n^2-k}$$

25. Suppose that a hypersurface S is represented upon another hypersurface S_1 in such a manner that the asymptotic curves are in correspondence. Choose the curvilinear coordinates in such a manner that the corresponding points on S and S_1 correspond to the same values of them. Then

$$\mu h_{ij}^{(1)}=h_{ij},\ (i,\ j=1,\ 2,\ldots\ldots,\ n),$$

where $h_{ij}^{(1)}$ denotes the quantities on S_1.

If

$$ds^2=gh_{\sigma\tau}\,du_\sigma\,du_\tau$$

is an invariant form,

$$ds_1^2=\mu g\ h_{\sigma\tau}^{(1)}\,du_\sigma\,du_\tau$$

is also an invariant form, and we have

$$ds^2=ds_1^2.$$

Therefore, *a necessary and sufficient condition that a hypersurface S may be represented to a hypersurface of the second degree in such a manner that the asymptotic curves are in correspondence is that the form φ may be reduced to the form*

$$\rho(du_1^2+\ldots\ldots\ldots\ldots+du_n^2)$$

where ρ is a function of u's.

This condition may be written as follows.[2]

$$\tau_{ijkl}=0,\ (i\neq j\neq k\neq l)$$

$$\tau_{ijil}=\frac{1}{n-2}\tau_{\lambda j\cdot\lambda l},\ (i\neq j\neq l).$$

$$\tau_{ijij}=\frac{1}{n-1}\,(\tau_{i\lambda i\lambda}+\tau_{j\lambda j\lambda})-\frac{1}{(n-1)(n-2)}\tau_{\lambda\mu\lambda\mu}.$$

1 *Loc. cit.*

2 R. Lagrange, *loc. cit.*

$$\frac{1}{n-1}\frac{\overline{\partial}\tau_{\lambda i\lambda j}}{\partial w_i}=\frac{1}{n-1}\frac{\overline{\partial}\tau_{\lambda i\lambda j}}{\partial w_j}-\frac{1}{2(n-1)(n-2)}\frac{\overline{\partial}\tau_{\lambda\mu\lambda\mu}}{\partial w_j}. \quad (i \neq j).$$

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