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Numerical Methods with Fourth Order Accuracy for Two-Point Boundary Value Problems

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1 Introduction

We consider the two-point boundary value problem for the semilinear ODE

$$-\frac{d}{dx}(p(x)\frac{du}{dx})+f(x,u)=0\ ,\quad a\le x\le b\ , \tag{1.1}$$

subject to separated boundary conditions

$$B_1(u)=\alpha_1u(a)-\alpha_2u'(a)=0\ ,\quad \alpha_1\ge 0\ ,\alpha_2\ge 0\ ,\ (\alpha_1,\alpha_2)\ne(0,0), \tag{1.2}$$

$$B_2(u)=\beta_1u(b)+\beta_2u'(b)=0\ ,\quad \beta_1\ge 0\ ,\beta_2\ge 0\ ,\ (\beta_1,\beta_2)\ne(0,0). \tag{1.3}$$

We assume that $p\in C^1[a,b], p(x)>0$ in $[a,b], f\in C([a,b]\times\mathbf{R})$ and $\frac{\partial f}{\partial u}$ exists, is continuous and nonnegative in $[a,b]\times\mathbf{R}$.

We put

$$a=x_0<x_1<\cdots<x_{n+1}=b\ ,$$
$$x_{i+\frac12}=\frac12(x_i+x_{i+1})\ ,\ x_{i+\frac14}=\frac12(x_i+x_{i+\frac12})\ ,\ x_{i+\frac34}=\frac12(x_{i+\frac12}+x_{i+1}),$$
$$h_{i+1}=x_{i+1}-x_i\ \ i=0,1,2,\cdots,n\ ,\ \ h=\max_i h_i.$$

As is shown in Yamamoto [3], the Green function $G(x,\xi)$ for the operator $L=-\frac{d}{dx}(p\frac{d}{dx}[\,])$ on $\mathcal{D}=\{u\in C^2[a,b]|B_1(u)=B_2(u)=0\}$ exists if and only if $\alpha_1+\beta_1>0$. It is then shown there that the problem (1.1)-(1.3) has a unique solution in $\mathcal{D}$. It is also shown that in the Shortley -Weller approximation

$$HA^{(sw)}\mathbf{U}+F(\mathbf{U})=\mathbf{0}, \tag{1.4}$$

the Green matrix $[A^{(sw)}]^{-1}=(g_{ij}^{(sw)})$ approximates the Green function $G(x,\xi)$, where

$$H=\mathrm{diag}(w_0^{-1},w_1^{-1},\cdots,w_{n+1}^{-1}),$$
$$w_j=\begin{cases}\frac{h_1}{2}\ (j=0),\\ \frac{h_j+h_{j+1}}{2}\ (1\le j\le n),\\ \frac{h_{n+1}}{2}\ (j=n+1),\end{cases}$$
$$\mathbf{U}=(U_0,U_1,\cdots,U_{n+1})^t,$$

and

$$F(\mathbf{U})=(f(x_0,U_0),f(x_1,U_1),\cdots f(x_{n+1},U_{n+1}))^t.$$

It follows from (1.4) that

$$U+[A^{(sw)}]^{-1}H^{-1}F(\mathbf{U})=\mathbf{0}. \tag{1.5}$$

The i-th relation of (1.5)

$$U_i + \sum_{j=0}^{n+1} g_{ij}^{(sw)} w_j f(x_j, U_j) = 0,$$

is an approximation of the equation

$$u(x_i) + \int_a^b G(x_i, \xi) f(\xi, u(\xi)) d\xi = 0,$$

by the trapezoidal rule. Furthermore, in [3] , a tridiagonal matrix A with $A^{-1} = (G(x_i, x_j))$ is determined under the assumption $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$ without loss of generality, and a new discretized formula

$$HA\mathbf{U} + F(\mathbf{U}) = \mathbf{0}, \tag{1.6}$$

is derived. It is also shown that both (1.5) and (1.6) have the second order accuracy for any nodes.

In this paper, on the basis of these results, we present two numerical methods with $O(h^4)$ accuracy. Numerical examples are also given.

2 Numerical methods with fourth order accuracy

In this section, we propose two methods with $O(h^4)$ accuracy for solving (1.1)-(1.3). The first one (Method A) is faster than the usual finite difference method and applies to the case where f is linear with respect to u, while the other one (Method B) applies to the case where f is nonlinear.

2.1 Method A

Let $f = q(x)u - r(x)$ with $q, r \in C[a, b]$. Then the method consists of the following steps.

STEP A1 We use the fourth-order Runge-Kutta method ($\frac{1}{6}$ formula) to solve the initial value probrem

$$y_1' = y_2, \tag{2.1}$$

$$y_2' = \frac{1}{p(x)}(q(x)y_1 - p'(x)y_2), \tag{2.2}$$

$$y_1(a) = \alpha_2 \ , \ y_2(a) = \alpha_1, \tag{2.3}$$

at $x_0, x_{\frac{1}{2}}, x_1, \cdots, x_n, x_{n+\frac{1}{2}}, x_{n+1}$ with step sizes $\frac{h_1}{2}, \frac{h_1}{2}, \frac{h_2}{2}, \frac{h_2}{2}, \cdots, \frac{h_{n+1}}{2}, \frac{h_{n+1}}{2}$. Observe that functional values of the right-hand sides of (2.1) and (2.2) at the nodes $x_0, x_{\frac{1}{2}}, x_1, \cdots, x_n, x_{n+\frac{1}{2}}, x_{n+1}$ and auxiliary nodes $x_{\frac{1}{4}}, x_{\frac{3}{4}}, \cdots, x_{n+\frac{1}{4}}, x_{n+\frac{3}{4}}$ are necessary throughout the computation. We denote the numerical solution by $\mathbf{Y}_i = (Y_i^{(1)}, Y_i^{(2)})$, $i = 0, \frac{1}{2}, 1, \cdots, n, n+\frac{1}{2}, n+1$.

STEP A2 We use the fourth-order Runge-Kutta method to solve the same system (2.1)-(2.2) with initial conditions

$$y_1(b) = \beta_2 \ , \ y_2(b) = -\beta_1,$$

at $x_{n+1}, x_{n+\frac{1}{2}}, x_n, \cdots, x_1, x_{\frac{1}{2}}, x_0$ with step sizes $-\frac{h_{n+1}}{2}, -\frac{h_{n+1}}{2}, \cdots, -\frac{h_1}{2}, -\frac{h_1}{2}$ (i.e., in the inverse direction). Denote the results by $\bar{\mathbf{Y}}_i = (\bar{Y}_i^{(1)}, \bar{Y}_i^{(2)})$, $i = n+1, n+\frac{1}{2}, n, \cdots, 1, \frac{1}{2}, 0$.

STEP A3 Let

$$\Delta = -p(b) \begin{vmatrix} Y_{n+1}^{(1)} & \beta_2 \\ Y_{n+1}^{(2)} & -\beta_1 \end{vmatrix},$$

and put

$$\tilde{g}_{ij} = \begin{cases} \frac{Y_i^{(1)} \bar{Y}_j^{(1)}}{\Delta} & (i \leq j), \\ \frac{\bar{Y}_i^{(1)} Y_j^{(1)}}{\Delta} & (i \geq j), \quad i, j = 0, \frac{1}{2}, 1, \cdots, n, n+\frac{1}{2}, n+1. \end{cases}$$

STEP A4 We put

$$\varphi_{ij} = \tilde{g}_{ij} r(x_j) = \tilde{g}_{ij} r_j, \tag{2.4}$$

and compute

$$U_i^A = \sum_{j=0}^{n} \frac{h_{j+1}}{6} (\varphi_{ij} + 4\varphi_{ij+\frac{1}{2}} + \varphi_{ij+1}) \ , \ i = 0, 1, 2, \cdots, n+1, \tag{2.5}$$

2.2 Method B

This method applies to the case where f is not linear.

STEP B0 Find $U = (U_0, U_{\frac{1}{4}}, U_{\frac{1}{2}}, U_{\frac{3}{4}}, U_1, \cdots, U_n, U_{n+\frac{1}{4}}, U_{n+\frac{1}{2}}, U_{n+\frac{3}{4}}, U_{n+1})^t$ by solving (1.1)-(1.3) at $x_0, x_{\frac{1}{4}}, x_{\frac{1}{2}}, x_{\frac{3}{4}}, x_1, \cdots, x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}$ with the use of (1.5) and (1.6). Let $u^{(0)}(x)$ be the cubic spline function which is uniquely determined by the conditions

$$\text{(i)} \quad u_0(x_j) = U_j \ , \ j = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \cdots, n, n+\frac{1}{4}, n+\frac{1}{2}, n+\frac{3}{4}, n+1,$$

$$\text{(ii)} \quad u_0'(x_0) = \frac{\alpha_1}{\alpha_2} U_0 \ , \ u_0'(x_{n+1}) = -\frac{\beta_1}{\beta_2} U_{n+1}.$$

STEP B1 Replace (2.2) by

$$y_2' = \frac{1}{p(x)} \{ f_u(x, u^{(0)}(x)) y_1 - p'(x) y_2 \},$$

and execute **STEP's A1-A3** as **STEP's B1-B3**.

STEP B4 Replace (2.4) by

$$\varphi_{ij} = g_{ij} \{ f_u(x_j, U_j) U_j - f(x_j, U_j) \} \ , j = 0, \frac{1}{2}, 1, \cdots, n+\frac{1}{2}, n+1,$$

and compute the right-hand side of (2.5). We denote it by U_i^B , $i = 0, 1, 2, \cdots, n+1$.

Then it is expected that the numerical results $\{U_i^A\}$ and $\{U_i^B\}$, $i = 0, 1, 2, \cdots, n+1$ have the fourth order accuracy. This is true and will be proved in the next section.

Remark 2.1 If f is linear $(f = q(x)u - r(x))$, then $f_u(x, u)u - f(x, u) = r(x)$. Therefore, we may consider that Method A is a special case of Method B.

3 Fourth order accuracy of the methods

In this section, we shall prove $O(h^4)$ accuracy of Methods A and B.

Theorem 3.1 *Let $f(x,u) = q(x)u - r(x)$, $q \in C[a,b]$ and $r \in C^1[a,b]$. Then*

$$U_i^A - u_i = O(h^4)\ ,\ i = 0,1,2,\cdots,n.$$

Proof Let $\tilde{G}(x,\xi)$ be the Green function for $\tilde{L}u = -\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)u$ on $\mathcal{D}$. Then, by **STEP's A1** and **A2**, $\{Y_i^{(1)}\}$ and $\{\bar{Y}_i^{(1)}\}$ have the fourth order accuracy. Hence, by **STEP A3**,

$$\tilde{g}_{ij} = \tilde{G}(x_i,x_j) + O(h^4)\ ,\ i,j = 0,\frac{1}{2},1,\cdots,n,n+\frac{1}{2},n+1, \tag{3.1}$$

and, putting $\tilde{G}_{ij} = \tilde{G}(x_i,x_j)$, we have

$$\begin{aligned}
u_i &= \int_a^b \tilde{G}(x_i,\xi)r(\xi)d\xi,\\
&= \sum_{j=0}^{n}\int_{x_j}^{x_{j+1}} \tilde{G}(x_i,\xi)r(\xi)d\xi,\\
&= \sum_{j=0}^{n}\frac{h_{j+1}}{6}[\tilde{G}_{ij}r_j + 4\tilde{G}_{ij+\frac{1}{2}}r_{j+\frac{1}{2}} + \tilde{G}_{ij+1}r_{j+1}] + O(h^4),\\
&= \sum_{j=0}^{n}\frac{h_{j+1}}{6}[\varphi_{ij} + 4\varphi_{ij+\frac{1}{2}} + \varphi_{ij+1}] + O(h^4),\\
&= U_i^A + O(h^4)\ ,\ i = 0,1,2,\cdots,n+1,
\end{aligned}$$

where we have applied the Simpson rule to each integral on $[x_j,x_{j+1}]$ by noting $\tilde{G}(x_i,\xi)r(\xi) \in C^{1,1}[x_j,x_{j+1}]$. Q.E.D.

Theorem 3.2 *Let f_u satisfy a uniform Lipschitz condition with respect to u with Lipschitz constant K in a bounded domain of $[a,b]\times\mathbf{R}$ which includes the solution curve $(x,u(x))$, $x \in [a,b]$ and let*

$$f(x,u(x)) - f_u(x,u(x))u(x) \in C^{1,1}[a,b],$$

Then

$$U_i^B - u_i = O(h^4)\ ,\ i = 0,1,2,\cdots,n+1.$$

Proof Let

$$\Phi(v) = -\frac{d}{dx}(p(x)\frac{dv}{dx}) + f(x,v)\ ,\ v \in \mathcal{D}.$$

Then

$$\Phi'(v) = -\frac{d}{dx}(p(x)\frac{d[\,]}{dx}) + f_u(x,v)[\,].$$

Hence, if we put $\eta = [\Phi'(u^{(0)})]^{-1}\Phi(v)$, or $\Phi'(u^{(0)})\eta = \Phi(v)$, then

$$-\frac{d}{dx}(p(x)\frac{d\eta}{dx}) + f_u(x,u^{(0)})\eta = -\frac{d}{dx}(p(x)\frac{dv}{dx}) + f(x,v),$$

which implies

$$-\frac{d}{dx}(p(x)\frac{d(\eta - v)}{dx}) + f_u(x, u^{(0)})(\eta - v) = f(x, v) - f_u(x, u^{(0)})v.$$

It follows from this that

$$\eta(x) - v(x) = \int_a^b \tilde{G}(x, \xi)[f(\xi, v(\xi)) - f_u(\xi, u^{(0)}(\xi))v(\xi)]d\xi,$$

and

$$\begin{aligned}\eta(x) &= [\Phi'(u^{(0)})]^{-1}\Phi(v), \\ &= v(x) + \int_a^b \tilde{G}(x, \xi)[f(\xi, v(\xi)) - f_u(\xi, u^{(0)}(\xi))v(\xi)]d\xi. \qquad (3.2)\end{aligned}$$

Similarly, we obtain

$$[\Phi'(u^{(0)})]^{-1}\{\Phi'(v) - \Phi'(w)\} = \int_a^b \tilde{G}(x, \xi)[f_u(\xi, v(\xi)) - f_u(\xi, w(\xi))][\ \ ]d\xi.$$

We now consider the first step of Newton's method starting from $u^{(0)}(x)$ and put

$$u^{(1)}(x) = u^{(0)}(x) - [\Phi'(u^{(0)})]^{-1}\Phi(u^{(0)}). \qquad (3.3)$$

If $u = u(x)$ stands for the exact solution of (1.1)-(1.3) whose existence is guaranteed by Theorem 3.1 in [3], then

$$\begin{aligned}&u^{(1)}(x) - u(x) \\ &= u^{(0)}(x) - u(x) - [\Phi'(u^{(0)})]^{-1}\Phi(u^{(0)}), \\ &= -[\Phi'(u^{(0)})]^{-1}\{\Phi(u^{(0)}) + \Phi'(u^{(0)})(u - u^{(0)})\}, \\ &= \int_0^1 [\Phi'(u^{(0)})]^{-1}\{\Phi'(u^{(0)} + \theta(u - u^{(0)})) - \Phi'(u^{(0)})\}(u - u^{(0)})d\theta, \\ &= \int_0^1 \int_a^b \tilde{G}(x, \xi)[f_u(\xi, u^{(0)} + \theta(u - u^{(0)})) - f_u(\xi, u^{(0)})](u - u^{(0)})d\xi d\theta, \qquad (3.4)\end{aligned}$$

since

$$0 = \Phi(u) = \Phi(u^{(0)}) + \int_0^1 \Phi'(u^{(0)} + \theta(u - u^{(0)}))(u - u^{(0)})d\theta.$$

We thus obtain from (3.4)

$$\begin{aligned}\|u^{(1)} - u\|_\infty &\leq K \max_{a\leq x\leq b} \int_a^b \tilde{G}(x, \xi)\|u - u^{(0)}\|_\infty^2 d\xi \int_0^1 \theta d\theta, \\ &= \frac{K}{2} \max_{a\leq x\leq b} \int_a^b \tilde{G}(x, \xi)d\xi \|u - u^{(0)}\|_\infty^2. \qquad (3.5)\end{aligned}$$

Furthermore, we have from (3.2) and (3.3)

$$\begin{aligned}u^{(1)}(x) &= -\int_a^b \tilde{G}(x, \xi)[f(\xi, u^{(0)}(\xi)) - f_u(\xi, u^{(0)}(\xi))u^{(0)}(\xi)]d\xi, \\ &= \int_a^b \tilde{G}(x, \xi)[f_u(\xi, u^{(0)}(\xi))u^{(0)}(\xi) - f(\xi, u^{(0)}(\xi))]d\xi,\end{aligned}$$

and

$$
\begin{aligned}
u_i^{(1)} &\equiv u^{(1)}(x_i) \\
&= \sum_{j=0}^{n} \frac{h_{j+1}}{6}[\varphi_{ij} + 4\varphi_{ij+\frac{1}{2}} + \varphi_{ij+1}] + O(h^4), \\
&= U_i^B + O(h^4) \ , \ i = 0, 1, 2, \cdots, n+1.
\end{aligned}
$$

Hence we have

$$U_i^B - u_i = u_i^{(1)} - u_i + O(h^4),$$

and, from (3.5)

$$
\begin{aligned}
|U_i^B - u_i| &\leq |u_i^{(1)} - u_i| + O(h^4) \\
&\leq \|u^{(1)} - u\|_\infty + O(h^4) \\
&\leq (\frac{K}{2} \max_{a\leq x\leq b} \int_a^b \tilde{G}(x,\xi)d\xi)\|u - u^{(0)}\|_\infty^2 + O(h^4). \qquad (3.6)
\end{aligned}
$$

Let $w(x) = u(x) - u^{(0)}(x)$ and $l(x)$ be the piecewise linear function such that

$$l(x_j) = w(x_j) \ , \ j = 0, \frac{1}{2}, 1, \cdots, n, n + \frac{1}{2}, n+1. \qquad (3.7)$$

Then, in each open interval $(x_j, x_{j+\frac{1}{2}})$, $j = 0, \frac{1}{2}, 1, \cdots, n, n+\frac{1}{2}, n+1$, we have

$$w(x) - l(x) = \frac{1}{2}w''(x_j + \theta(x_{j+1} - x_j))(x - x_j)(x - x_{j+\frac{1}{2}}) \ , \ x \in (x_j, x_{j+\frac{1}{2}}),$$

where $0 < \theta < 1$. Hence

$$|w(x)| \leq |l(x)| + \frac{1}{8}(\sup_{(x_j, x_{j+\frac{1}{2}})} |w''(\xi)|)h^2 = |l(x)| + O(h^2) \ , \ x_j < x < x_{j+\frac{1}{2}},$$

Observe here that by (3.7) this holds also for $x = x_j$ and $x_{j+\frac{1}{2}}$. Therefore,

$$\|w\|_\infty \leq \|l\|_\infty + O(h^2) = O(h^2),$$

since l is piecewise linear and

$$\|l\|_\infty = \max_j |l(x_j)| = \max_j |u(x_j) - U_j| = O(h^2),$$

where $j = 0, \frac{1}{2}, 1, \cdots, n, n+\frac{1}{2}, n+1$.

Consequently we obtain from (3.6)

$$|U_i^B - u_i| \leq O(\|w\|_\infty^2) + O(h^4) = O(h^4),$$

which proves Theorem 3.2. Q.E.D.

Remark 3.1 Let $f_0 = f(x, 0)$. Then it can be shown (c.f.[3]) that

$$\|u\|_\infty \leq (\max_{a\leq x\leq b} \int_a^b G(x,\xi)d\xi)\|f_0\|_\infty \equiv M(say),$$

where $G(x,\xi)$ is the Green function for $L = -\frac{d}{dx}(p\frac{d[\,]}{dx})$ on $\mathcal{D}$. Hence, as a bounded domain in Theorem 3.2, we may take

$$\Omega = [a, b] \times [-M, M].$$

4 Numerical examples

In this section we give two examples which show $O(h^4)$ accuracy of our methods.

Example 4.1 (for Method A)

$$-(p(x)u')' + q(x)u - r(x) = 0, \ 0 \le x \le 1,$$
$$\begin{cases} u(0) - \frac{e}{e-1}u'(0) = 0, \\ u(1) + u'(1) = 0, \end{cases}$$
$$\alpha_1 = 1, \quad \alpha_2 = \frac{e}{e-1}, \quad \beta_1 = 1, \quad \beta_2 = 1,$$
$$p(x) = e^{1-x}, \quad q(x) = e^{1-x},$$
$$r(x) = (2+x)e^{1-x} + 1 - x.$$

The exact solution is $u(x) = x(1 - e^{x-1}) + 1$.

Example 4.2 (for Method B)

$$-(p(x)u')' + e^u - r(x) = 0, \ 0 \le x \le 1,$$
$$\begin{cases} u(0) - u'(0) = 0, \\ 2u(1) + u'(1) = 0, \end{cases}$$
$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \beta_1 = 2, \quad \beta_2 = 1,$$
$$p(x) = e^{1-x}, \quad r(x) = -e^{1-x}(3x^2 - 6x - 1) + e^{x(1-x)(1+x)+1}.$$

The exact solution is $u(x) = x(1-x)(1+x) + 1$.

We used random partitions which are generated by the following rule: Given a positive integer $m \ge 3$, mesh sizes h_i, $i = 1, 2, \cdots$ are generated as uniform random numbers in $[(\frac{1}{2^m})^3, \frac{1}{2^m}]$. If $s_{n-1} \equiv 1 - \sum_{i=1}^{n-1} h_i > \frac{1}{2^m}$ and $s_n \le \frac{1}{2^m}$ for some n, then the process to generate the random partitions is completed by putting $h_{n+1} = s_n$. We took $m = 3, 4, \cdots, 8, 9$.

For each example, our methods were tested five times respectively on random nodes generated as above, for each m.

Putting

$$\mathbf{U}^A = (U_0^A, U_1^A, \cdots, U_{n+1}^A)^t,$$

$$\mathbf{U}^B = (U_0^B, U_1^B, \cdots, U_{n+1}^B)^t,$$

and

$$\mathbf{u} = (u_0, u_1, \cdots, u_{n+1})^t,$$

we show the results of computation for Examples 4.2 and 4.2 in Tables 4.1 and 4.2, respectively. The tables show that the methods have $O(h^4)$ accuracy.

m	trials	n	h	$\lVert \mathbf{U}^A - \mathbf{u} \rVert_\infty$	$\lVert \mathbf{U}^A - \mathbf{u} \rVert_\infty / h^4$
	1	17	1.223895e-01	3.384464e-07	1.508388e-03
	2	18	1.153867e-01	3.924097e-07	2.213690e-03
3	3	15	1.240584e-01	5.862044e-07	2.474823e-03
	4	16	1.225179e-01	4.551847e-07	2.020179e-03
	5	12	1.244729e-01	4.939088e-07	2.057537e-03
	1	27	6.041188e-02	3.322788e-08	2.494671e-03
	2	30	6.239891e-02	3.020547e-08	1.992406e-03
4	3	35	5.886830e-02	2.126016e-08	1.770274e-03
	4	34	5.898872e-02	1.964134e-08	1.622165e-03
	5	30	6.195111e-02	3.448446e-08	2.341138e-03
	1	58	3.109850e-02	2.201986e-09	2.354272e-03
	2	68	3.119227e-02	1.425668e-09	1.506020e-03
5	3	67	3.018979e-02	1.437301e-09	1.730244e-03
	4	61	3.118545e-02	1.897978e-09	2.006705e-03
	5	74	3.122495e-02	1.522347e-09	1.601425e-03
	1	118	1.554630e-02	1.237253e-10	2.118117e-03
	2	125	1.562335e-02	1.081961e-10	1.815996e-03
6	3	139	1.562373e-02	1.103393e-10	1.851787e-03
	4	137	1.556528e-02	9.734236e-11	1.658344e-03
	5	126	1.553671e-02	1.256542e-10	2.156458e-03
	1	252	7.780265e-03	6.883161e-12	1.878496e-03
	2	249	7.810531e-03	7.614132e-12	2.045965e-03
7	3	260	7.806598e-03	6.836753e-12	1.840783e-03
	4	252	7.809653e-03	7.008172e-12	1.883986e-03
	5	268	7.800795e-03	6.432632e-12	1.737134e-03
	1	528	3.872770e-03	4.183320e-13	1.859664e-03
	2	514	3.903242e-03	4.345413e-13	1.872101e-03
8	3	511	3.904853e-03	4.638512e-13	1.995077e-03
	4	519	3.868232e-03	3.872458e-13	1.729564e-03
	5	518	3.905439e-03	4.751755e-13	2.042558e-03
	1	1017	1.949030e-03	3.108624e-14	2.154239e-03
	2	1034	1.950155e-03	3.108624e-14	2.149274e-03
9	3	1015	1.952595e-03	3.708145e-14	2.550985e-03
	4	1039	1.953121e-03	3.308465e-14	2.273576e-03
	5	1029	1.951585e-03	2.886580e-14	1.989912e-03

Table 4.1:

m	trials	n	h	$\|\mathbf{U}^B - \mathbf{u}\|_\infty$	$\|\mathbf{U}^B - \mathbf{u}\|_\infty / h^4$
3	1	19	1.225948e-01	1.527966e-06	6.764348e-03
	2	20	1.113242e-01	9.555517e-07	6.221513e-03
	3	20	1.241483e-01	2.497541e-06	1.051355e-02
	4	22	1.005830e-01	8.758147e-07	8.556843e-03
	5	16	1.174889e-01	8.894018e-07	4.667786e-03
4	1	29	6.072545e-02	6.891545e-08	5.067966e-03
	2	34	6.241280e-02	6.118753e-08	4.032443e-03
	3	28	6.012013e-02	8.934540e-08	6.838997e-03
	4	32	6.205068e-02	5.740185e-08	3.872040e-03
	5	33	5.901221e-02	8.834457e-08	7.284714e-03
5	1	64	3.121642e-02	4.699362e-09	4.948877e-03
	2	62	3.112109e-02	4.946169e-09	5.272900e-03
	3	62	3.034005e-02	4.795598e-09	5.659492e-03
	4	67	3.123324e-02	4.248158e-09	4.464087e-03
	5	64	3.069758e-02	5.682930e-09	6.399646e-03
6	1	125	1.557187e-02	3.366383e-10	5.725334e-03
	2	118	1.557797e-02	3.854466e-10	6.545173e-03
	3	125	1.561955e-02	3.182044e-10	5.346041e-03
	4	137	1.558305e-02	3.736484e-10	6.336562e-03
	5	123	1.561532e-02	2.369784e-10	3.985708e-03
7	1	249	7.812281e-03	2.143907e-11	5.755653e-03
	2	262	7.741920e-03	1.826073e-11	5.083037e-03
	3	259	7.809713e-03	1.646083e-11	4.424982e-03
	4	242	7.759725e-03	1.986722e-11	5.479637e-03
	5	253	7.800625e-03	1.939160e-11	5.237161e-03
8	1	515	3.899097e-03	1.201927e-12	5.200227e-03
	2	531	3.898602e-03	1.065370e-12	4.611740e-03
	3	512	3.900684e-03	1.268319e-12	5.478544e-03
	4	532	3.894496e-03	1.127320e-12	4.900524e-03
	5	508	3.903123e-03	1.092459e-12	4.707132e-03
9	1	1032	1.952887e-03	7.815970e-14	5.373715e-03
	2	1023	1.950455e-03	8.237855e-14	5.692074e-03
	3	1014	1.951790e-03	7.394085e-14	5.095096e-03
	4	1008	1.952195e-03	7.638334e-14	5.259033e-03
	5	1032	1.950088e-03	7.482903e-14	5.174323e-03

Table 4.2:

References

[1] T.Yamamoto, Harmonic relations between Green's functions and Green's matrices for boundary value problems (in RIMS Kokyuroku 1169,RIMS,Kyoto University,2000), 15-26.

[2] T.Yamamoto, Harmonic relations between Green's functions and Green's matrices for boundary value problems II (in RIMS Kokyuroku 1286,RIMS, Kyoto University, 2002),27-33.

[3] T.Yamamoto, Harmonic relations between Green's functions and Green's matrices for boundary value problems (in this issue, RIMS Kokyuroku, 2004)