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<th>Title</th>
<th>Numerical Methods with Fourth Order Accuracy for Two-Point Boundary Value Problems (Feasibility of Theoretical Arguments of Mathematical Analysis on Computer)</th>
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</thead>
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<tr>
<td>Author(s)</td>
<td>Aguchi, Seiji; Yamamoto, Tetsuro</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2004, 1381: 11-20</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-06</td>
</tr>
<tr>
<td>URL</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Numerical Methods with Fourth Order Accuracy for Two-Point Boundary Value Problems

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1 Introduction

We consider the two-point boundary value problem for the semilinear ODE
\[ -\frac{d}{dx}(p(x) \frac{du}{dx}) + f(x, u) = 0, \quad a \leq x \leq b, \quad (1.1) \]
subject to separated boundary conditions
\[ B_1(u) = \alpha_1 u(a) - \alpha_2 u'(a) = 0, \quad \alpha_1 \geq 0, \alpha_2 \geq 0, \quad (\alpha_1, \alpha_2) \neq (0, 0), \quad (1.2) \]
\[ B_2(u) = \beta_1 u(b) + \beta_2 u'(b) = 0, \quad \beta_1 \geq 0, \beta_2 \geq 0, \quad (\beta_1, \beta_2) \neq (0, 0). \quad (1.3) \]

We assume that \( p \in C^1[a, b], p(x) > 0 \) in \([a, b]\), \( f \in C([a, b] \times \mathbb{R}) \) and \( \frac{\partial f}{\partial u} \) exists, is continuous and nonnegative in \([a, b] \times \mathbb{R}\).

We put
\[ a = x_0 < x_1 < \cdots < x_{n+1} = b, \]
\[ x_{i+1} = \frac{1}{2}(x_i + x_{i+1}), \quad x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+\frac{1}{2}}), \quad x_{i+\frac{3}{4}} = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i+1}), \]
\[ x_{i+\frac{3}{4}} = \frac{1}{2}(x_i + x_{i+1}), \quad h_{i+1} = x_{i+1} - x_i, \quad i = 0, 1, 2, \ldots, n, \quad h = \max_i h_i. \]

As is shown in Yamamoto [3], the Green function \( G(x, \xi) \) for the operator \( L = -\frac{d}{dx}(p(x) \frac{d}{dx}) \) on \( D = \{ u \in C^2[a, b] | B_1(u) = B_2(u) = 0 \} \) exists if and only if \( \alpha_1 + \beta_1 > 0 \).

It is then shown there that the problem (1.1)-(1.3) has a unique solution in \( D \). It is also shown that in the Shortley-Weller approximation
\[ HA^{(sw)}U + F(U) = 0, \quad (1.4) \]
the Green matrix \( [A^{(sw)}]^{-1} = (g_{ij}^{(sw)}) \) approximates the Green function \( G(x, \xi) \), where
\[
H = \text{diag}(w_0^{-1}, w_1^{-1}, \ldots, w_{n+1}^{-1}), \quad \]
\[
w_j = \begin{cases} \frac{h}{2} & (j = 0), \\ \frac{h_j + h_{j+1}}{2} & (1 \leq j \leq n), \\ \frac{h_{n+1}}{2} & (j = n + 1), \end{cases} \quad \]
\[ U = (U_0, U_1, \ldots, U_{n+1})^t, \]
and
\[ F(U) = (f(x_0, U_0), f(x_1, U_1), \ldots, f(x_{n+1}, U_{n+1}))^t. \]

It follows from (1.4) that
\[ U + [A^{(sw)}]^{-1}H^{-1}F(U) = 0. \quad (1.5) \]
The i-th relation of (1.5)
\[ U_i + \sum_{j=0}^{n+1} g_{ij}^{(sw)} w_j f(x_j, U_j) = 0, \]
is an approximation of the equation
\[ u(x_i) + \int_a^b G(x_i, \xi)f(\xi, u(\xi))d\xi = 0, \]
by the trapezoidal rule. Furthermore, in [3], a tridiagonal matrix \(A^{-1} = (G(x_i, x_j))\) is determined under the assumption \(\alpha_1\alpha_2\beta_1\beta_2 \neq 0\) without loss of generality, and a new discretized formula
\[ HAU + F(U) = 0, \] (1.6)
is derived. It is also shown that both (1.5) and (1.6) have the second order accuracy for any nodes.

In this paper, on the basis of these results, we present two numerical methods with \(O(h^4)\) accuracy. Numerical examples are also given.

## 2 Numerical methods with fourth order accuracy

In this section, we propose two methods with \(O(h^4)\) accuracy for solving (1.1)-(1.3). The first one (Method A) is faster than the usual finite difference method and applies to the case where \(f\) is linear with respect to \(u\), while the other one (Method B) applies to the case where \(f\) is nonlinear.

### 2.1 Method A

Let \(f = q(x)u - r(x)\) with \(q, r \in C[a, b]\). Then the method consists of the following steps.

**STEP A1** We use the fourth-order Runge-Kutta method \(\left(\frac{1}{6}\right)\ formula\) to solve the initial value problem
\[
\begin{align*}
y'_1 &= y_2, \\
y'_2 &= \frac{1}{p(x)}(q(x)y_1 - p'(x)y_2), \\
y_1(a) &= \alpha_2, \quad y_2(a) = \alpha_1,
\end{align*}
\] (2.1) (2.2) (2.3)

at \(x_0, x_{\frac{1}{2}}, x_1, \ldots, x_n, x_{n+\frac{1}{2}}, x_{n+1}\) with step sizes \(\frac{h}{2}, \frac{h}{2}, \frac{h}{2}, \ldots, \frac{h_{n+1}}{2}, \frac{h_{n+1}}{2}\). Observe that functional values of the right-hand sides of (2.1) and (2.2) at the nodes \(x_0, x_{\frac{1}{2}}, x_1, \ldots, x_n, x_{n+\frac{1}{2}}, x_{n+1}\) and auxiliary nodes \(x_{\frac{1}{4}}, x_{\frac{3}{4}}, \ldots, x_{n+\frac{3}{4}}, x_{n+\frac{1}{2}}\) are necessary throughout the computation. We denote the numerical solution by \(Y_i = (Y_1^{(1)}, Y_2^{(2)}), \ i = 0, \frac{1}{2}, 1, \ldots, n, n + \frac{1}{2}, n + 1\).
STEP A2 We use the fourth-order Runge-Kutta method to solve the same system (2.1)-(2.2) with initial conditions

\[ y_1(b) = \beta_2, \quad y_2(b) = -\beta_1, \]

at \( x_{n+1}, x_{n+\frac{1}{2}}, x_n, \cdots, x_1, x_{\frac{1}{2}}, x_0 \) with step sizes \(-\frac{h_{n+1}}{2}, -\frac{h_n}{2}, \cdots, -\frac{h_2}{2}, -\frac{h_1}{2}\) (i.e., in the inverse direction). Denote the results by \( \mathbf{Y}_i = (\mathbf{Y}_i^{(1)}, \mathbf{Y}_i^{(2)}) \), \( i = n + 1, n + \frac{1}{2}, n \), \( \cdots, 1, \frac{1}{2}, 0 \).

STEP A3 Let

\[ \Delta = -p(b) \begin{vmatrix} Y_{n+1}^{(1)} & \beta_2 \\ Y_{n+1}^{(2)} & -\beta_1 \end{vmatrix}, \]

and put

\[ \tilde{g}_{ij} = \begin{cases} \frac{y_i^{(1)}y_j^{(1)}}{\Delta} & \text{if } i \leq j, \\ \frac{y_i^{(1)}y_j^{(1)}}{\Delta} & \text{if } i \geq j, \end{cases} \]

\[ i, j = 0, \frac{1}{2}, 1, \cdots, n, n + \frac{1}{2}, n + 1. \]

STEP A4 We put

\[ \varphi_{ij} = \tilde{g}_{ij} r(x_j) = \tilde{g}_{ij} r_j, \]

and compute

\[ U_i^A = \sum_{j=0}^{n} \frac{h_{j+1}}{6} (\varphi_{ij} + 4\varphi_{ij+\frac{1}{2}} + \varphi_{ij+1}), \quad i = 0, 1, 2, \cdots, n + 1, \]

(2.4)

(2.5)

2.2 Method B

This method applies to the case where \( f \) is not linear.

STEP B0 Find \( U = (U_0, U_\frac{1}{2}, U_1, U_\frac{3}{4}, U_1, U_n, U_{n+\frac{1}{2}} U_{n+\frac{3}{4}}, U_{n+1}) \) by solving (1.1)-(1.3) at \( x_0, x_{\frac{1}{2}}, x_1, x_{\frac{3}{4}}, x_1, \cdots, x_n, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1} \) with the use of (1.5) and (1.6). Let \( u^{(0)}(x) \) be the cubic spline function which is uniquely determined by the conditions

(i) \( u_0(x_j) = U_j, \quad j = 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 1, \cdots, n, n + \frac{1}{4}, n + \frac{1}{2}, n + \frac{3}{4}, n + 1, \)

(ii) \( u_0(x_0) = \frac{\alpha_1}{\alpha_2} U_0, \quad u_0(x_{n+1}) = -\frac{\beta_1}{\beta_2} U_{n+1}. \)

STEP B1 Replace (2.2) by

\[ y'_2 = \frac{1}{p(x)} \{ f_u(x, u^{(0)}(x)) y_1 - p(x) y_2 \}, \]

and execute STEP's A1-A3 as STEP's B1-B3.

STEP B4 Replace (2.4) by

\[ \varphi_{ij} = g_{ij} \{ f_u(x_i, U_j) U_j - f(x_i, U_j) \}, \quad j = 0, \frac{1}{2}, 1, \cdots, n + \frac{1}{2}, n + 1, \]

and compute the right-hand side of (2.5). We denote it by \( U_i^B, \quad i = 0, 1, 2, \cdots, n + 1. \)

Then it is expected that the numerical results \( \{ U_i^A \} \) and \( \{ U_i^B \}, \quad i = 0, 1, 2, \cdots, n + 1 \) have the fourth order accuracy. This is true and will be proved in the next section.

Remark 2.1 If \( f \) is linear \( (f = q(x) u - r(x)) \), then \( f_u(x, u) u - f(x, u) = r(x) \).

Therefore, we may consider that Method A is a special case of Method B.
3 Fourth order accuracy of the methods

In this section, we shall prove $O(h^4)$ accuracy of Methods A and B.

**Theorem 3.1** Let $f(x, u) = q(x)u - r(x)$, $q \in C[a, b]$ and $r \in C^1[a, b]$. Then

$$U_i^A - u_i = O(h^4), \quad i = 0, 1, 2, \ldots, n.$$  

**Proof** Let $\tilde{G}(x, \xi)$ be the Green function for $\tilde{L}u = -\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)u$ on $D$. Then, by STEP's A1 and A2, $\{Y_i^{(1)}\}$ and $\{\bar{Y}_i^{(1)}\}$ have the fourth order accuracy. Hence, by STEP A3,

$$\tilde{g}_{ij} = \tilde{G}(x_i, x_j) + O(h^4), \quad i, j = 0, \frac{1}{2}, 1, \cdots, n, n + \frac{1}{2}, n + 1,$$

and, putting $\tilde{G}_{ij} = \tilde{G}(x_i, x_j)$, we have

$$u_i = \int_a^b \tilde{G}(x_i, \xi)r(\xi)d\xi,$$

$$= \sum_{j=0}^n \int_{x_j}^{x_{j+1}} \tilde{G}(x_i, \xi)r(\xi)d\xi,$$

$$= \sum_{j=0}^n \frac{h_{j+1}}{6} [\tilde{G}_{ij}r_j + 4\tilde{G}_{ij+\frac{1}{2}}r_{j+\frac{1}{2}} + \tilde{G}_{ij+1}r_{j+1}] + O(h^4),$$

$$= \sum_{j=0}^n \frac{h_{j+1}}{6} [\varphi_{ij} + 4\varphi_{ij+\frac{1}{2}} + \varphi_{ij+1}] + O(h^4),$$

$$= U_i^A + O(h^4), \quad i = 0, 1, 2, \cdots, n + 1,$$

where we have applied the Simpson rule to each integral on $[x_j, x_{j+1}]$ by noting $\tilde{G}(x_i, \xi)r(\xi) \in C^{1,1}[x_j, x_{j+1}]$. Q.E.D.

**Theorem 3.2** Let $f_u$ satisfy a uniform Lipschitz condition with respect to $u$ with Lipschitz constant $K$ in a bounded domain of $[a, b] \times \mathbb{R}$ which includes the solution curve $(x, u(x))$, $x \in [a, b]$ and let

$$f(x, u(x)) - f_u(x, u(x))u(x) \in C^{1,1}[a, b],$$

Then

$$U_i^B - u_i = O(h^4), \quad i = 0, 1, 2, \cdots, n + 1.$$  

**Proof** Let

$$\Phi(v) = -\frac{d}{dx}(p(x)\frac{dv}{dx}) + f(x, v), \quad v \in D.$$  

Then

$$\Phi'(v) = -\frac{d}{dx}(p(x)\frac{dv}{dx}) + f_u(x, v)[v].$$

Hence, if we put $\eta = [\Phi'(u^{(0)})]^{-1}\Phi(v)$, or $\Phi'(u^{(0)})\eta = \Phi(v)$, then

$$-\frac{d}{dx}(p(x)\frac{d\eta}{dx}) + f_u(x, u^{(0)})\eta = -\frac{d}{dx}(p(x)\frac{dv}{dx}) + f(x, v),$$

and, putting $\tilde{G}_{ij} = \tilde{G}(x_i, x_j)$, we have

$$u_i = \int_a^b \tilde{G}(x_i, \xi)r(\xi)d\xi,$$

$$= \sum_{j=0}^n \int_{x_j}^{x_{j+1}} \tilde{G}(x_i, \xi)r(\xi)d\xi,$$

$$= \sum_{j=0}^n \frac{h_{j+1}}{6} [\tilde{G}_{ij}r_j + 4\tilde{G}_{ij+\frac{1}{2}}r_{j+\frac{1}{2}} + \tilde{G}_{ij+1}r_{j+1}] + O(h^4),$$

$$= \sum_{j=0}^n \frac{h_{j+1}}{6} [\varphi_{ij} + 4\varphi_{ij+\frac{1}{2}} + \varphi_{ij+1}] + O(h^4),$$

$$= U_i^A + O(h^4), \quad i = 0, 1, 2, \cdots, n + 1,$$
which implies
\[
-\frac{d}{dx}(p(x)\frac{d(\eta - v)}{dx}) + f_u(x, u^{(0)})(\eta - v) = f(x, v) - f_u(x, u^{(0)})v.
\]

It follows from this that
\[
\eta(x) - v(x) = \int_a^b \tilde{G}(x, \xi)[f(\xi, v(\xi)) - f_u(\xi, u^{(0)}(\xi))v(\xi)]d\xi,
\]
and
\[
\eta(x) = \left[\Phi'(u^{(0)})\right]^{-1}\Phi(v),
\]
\[
= v(x) + \int_a^b \tilde{G}(x, \xi)[f(\xi, v(\xi)) - f_u(\xi, u^{(0)}(\xi))v(\xi)]d\xi. \tag{3.2}
\]

Similarly, we obtain
\[
\left[\Phi'(u^{(0)})\right]^{-1}\left\{\Phi'(u) - \Phi'(w)\right\} = \int_a^b \tilde{G}(x, \xi)[f_u(\xi, v(\xi)) - f_u(\xi, w(\xi))]d\xi.
\]

We now consider the first step of Newton's method starting from \(u^{(0)}(x)\) and put
\[
u^{(1)}(x) = u^{(0)}(x) - \left[\Phi'(u^{(0)})\right]^{-1}\Phi(u^{(0)}). \tag{3.3}
\]

If \(u = u(x)\) stands for the exact solution of (1.1)-(1.3) whose existence is guaranteed by Theorem 3.1 in [3], then
\[
u^{(1)}(x) = u^{(0)}(x) - u(x)
\]
\[
= -\left[\Phi'(u^{(0)})\right]^{-1}\{\Phi(u^{(0)}) + \Phi'(u^{(0)})(u - u^{(0)})\},
\]
\[
= \int_0^1 \left[\Phi'(u^{(0)})\right]^{-1}\{\Phi(u^{(0)} + \theta(u - u^{(0)})) - \Phi'(u^{(0)})\}(u - u^{(0)})d\theta,
\]
\[
= \int_0^1 \int_a^b \tilde{G}(x, \xi)[f_u(\xi, u^{(0)} + \theta(u - u^{(0)})) - f_u(\xi, u^{(0)}))]u - u^{(0)})d\xi d\theta, \tag{3.4}
\]

since
\[
0 = \Phi(u) = \Phi(u^{(0)}) + \int_0^1 \Phi'(u^{(0)} + \theta(u - u^{(0)}))(u - u^{(0)})d\theta.
\]

We thus obtain from (3.4)
\[
\|u^{(1)} - u\|_{\infty} \leq K \max_{a \leq x \leq b} \int_a^b \tilde{G}(x, \xi)\|u - u^{(0)}\|_{\infty}^2d\xi \int_0^1 \theta d\theta,
\]
\[
= \frac{K}{2} \max_{a \leq x \leq b} \int_a^b \tilde{G}(x, \xi)d\xi\|u - u^{(0)}\|_{\infty}^2. \tag{3.5}
\]

Furthermore, we have from (3.2) and (3.3)
\[
u^{(1)}(x) = -\int_a^b \tilde{G}(x, \xi)[f(\xi, u^{(0)}(\xi)) - f_u(\xi, u^{(0)}(\xi))u^{(0)}(\xi)]d\xi,
\]
\[
= \int_a^b \tilde{G}(x, \xi)[f_u(\xi, u^{(0)}(\xi))u^{(0)}(\xi) - f(\xi, u^{(0)}(\xi))]d\xi,
and
\[ u_i^{(1)} \equiv u^{(1)}(x_i) = \sum_{J^{=0}}^{n} \frac{h_{j+1}}{6} [\varphi_{ij} + 4\varphi_{ij+\frac{1}{2}} + \varphi_{i_{J}+1}] + O(h^4), \]
\[ = U_i^B + O(h^4), \quad i = 0, 1, 2, \ldots, n + 1. \]

Hence we have
\[ U_i^B - u_i = u_i^{(1)} - u_i + O(h^4), \]
and, from (3.5)
\[ |U_i^B - u_i| \leq |u_i^{(1)} - u_i| + O(h^4) \leq ||u^{(1)} - u||_{\infty} + O(h^4). \quad (3.6) \]

Let \( w(x) = u(x) - u^{(0)}(x) \) and \( l(x) \) be the piecewise linear function such that
\[ l(x_j) = w(x_j), \quad j = 0, \frac{1}{2}, 1, \ldots, n + \frac{1}{2}, n + 1. \quad (3.7) \]

Then, in each open interval \((x_j, x_{j+\frac{1}{2}})\), \( j = 0, \frac{1}{2}, 1, \ldots, n + \frac{1}{2}, n + 1 \), we have
\[ w(x) - l(x) = \frac{1}{2} w''(x_j + \theta(x_{j+1} - x_j))(x-x_j)(x-x_{j+\frac{1}{2}}), \quad x \in (x_j, x_{j+\frac{1}{2}}), \]
where \( 0 < \theta < 1 \). Hence
\[ |w(x)| \leq |l(x)| + \frac{1}{8} (\sup_{x_{j}(,x_{j+\frac{1}{2}})}|w''(\xi)|)h^2 = |l(x)| + O(h^2), \quad x_j < x < x_{j+\frac{1}{2}}, \]
Observe here that by (3.7) this holds also for \( x = x_j \) and \( x_{j+\frac{1}{2}} \). Therefore,
\[ ||w||_{\infty} \leq ||l||_{\infty} + O(h^2) = O(h^2), \]
since \( l \) is piecewise linear and
\[ ||l||_{\infty} = \max_j |l(x_j)| = \max_j |u(x_j) - U_j| = O(h^2), \]
where \( j = 0, \frac{1}{2}, 1, \ldots, n + \frac{1}{2}, n + 1. \)

Consequently we obtain from (3.6)
\[ |U_i^B - u_i| \leq O(||w||_{\infty}^2) + O(h^4) = O(h^4), \]
which proves Theorem 3.2. Q.E.D.

Remark 3.1 Let \( f_0 = f(x, 0) \). Then it can be shown (c.f.[3]) that
\[ ||u||_{\infty} \leq (\max_{a \leq x \leq b} \int_a^b G(x, \xi)d\xi) ||f_0||_{\infty} \equiv M \text{(say)}, \]
where \( G(x, \xi) \) is the Green function for \( L = -\frac{d}{dx} (p \frac{d}{dx}) \) on \( D \). Hence, as a bounded domain in Theorem 3.2, we may take
\[ \Omega = [a, b] \times [-M, M]. \]
4 Numerical examples

In this section we give two examples which show $O(h^4)$ accuracy of our methods.

Example 4.1 (for Method A)

\[-(p(x)u')' + q(x)u - r(x) = 0, \quad 0 \leq x \leq 1,
\]
\[\begin{align*}
    u(0) - \frac{e}{e-1}u'(0) &= 0, \\
    u(1) + u'(1) &= 0,
\end{align*}\]
\[\alpha_1 = 1, \quad \alpha_2 = \frac{e}{e-1}, \quad \beta_1 = 1, \quad \beta_2 = 1,
\]
\[p(x) = e^{1-x}, \quad q(x) = e^{1-x}, \quad r(x) = (2 + x)e^{1-x} + 1 - x.
\]

The exact solution is $u(x) = x(1 - e^{x-1}) + 1$.

Example 4.2 (for Method B)

\[-(p(x)u')' + e^u - r(x) = 0, \quad 0 \leq x \leq 1,
\]
\[\begin{align*}
    u(0) - u'(0) &= 0, \\
    2u(1) + u'(1) &= 0,
\end{align*}\]
\[\alpha_1 = 1, \quad \alpha_2 = 1, \quad \beta_1 = 2, \quad \beta_2 = 1,
\]
\[p(x) = e^{1-x}, \quad r(x) = -e^{1-x}(3x^2 - 6x - 1) + e^{x(1-x)(1+x)+1}.
\]

The exact solution is $u(x) = x(1 - x)(1 + x) + 1$.

We used random partitions which are generated by the following rule: Given a positive integer $m \geq 3$, mesh sizes $h_i$, $i = 1, 2, \ldots$ are generated as uniform random numbers in $[(\frac{1}{2^m})^3, \frac{1}{2^m}]$. If $s_{n-1} \equiv 1 - \sum_{i=1}^{n-1} h_i > \frac{1}{2^m}$ and $s_n \leq \frac{1}{2^m}$ for some $n$, then the process to generate the random partitions is completed by putting $h_{n+1} = s_n$. We took $m = 3, 4, \ldots, 8, 9$.

For each example, our methods were tested five times respectively on random nodes generated as above, for each $m$.

Putting
\[
U^A = (U_0^A, U_1^A, \ldots, U_{n+1}^A)^t,
\]
\[
U^B = (U_0^B, U_1^B, \ldots, U_{n+1}^B)^t,
\]
and
\[
u = (u_0, u_1, \ldots, u_{n+1})^t,
\]
we show the results of computation for Examples 4.2 and 4.2 in Tables 4.1 and 4.2, respectively. The tables show that the methods have $O(h^4)$ accuracy.
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