# Some Theorems on the Orders of Transcendental Integral Functions of Two Independent Variables.

By

**Ryô Yasuda.**

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ABSTRACT

This paper consists of two parts. The first part is an extension of the theorem given in my previous paper[1] and the main theorem runs as follows: Let

$$F(z, z') \equiv \sum_{q=0}^{\infty} e^{Q_q(z)} h^{(q)}(z) z'^q \equiv \sum_{q=0}^{\infty} e^{Q_q(z)} \left( \sum_{p=0}^{\infty} A_p^{(q)} z^p \right) z'^q$$

be a transcendental integral function of $z$ and $z'$ such that

(1) $F(z, z')$ is of finite total order $\lambda$ in $z$ and $z'$, and the transcendental integral function

$$f(z') \equiv \sum_{q=0}^{\infty} c_q z'^q, \quad \text{where } c_q \text{ is the maximum coefficient of } e^{Q_q(z)} h^{(q)}(z),$$

is of apparent order $\mu$ $(>0)$,

(2) the canonical products $h^{(q)}(z)$ $(q=0, 1, 2, \ldots\ldots)$ are uniformly increasing functions (for $q=0, 1, 2, \ldots\ldots$) at most of $\nu^{th}$ order.

Then $F(z, z')$ will be of apparent order $\mu$ in $z'$ for any finite $z$ except the points of a punctual set $S_2$, for which $F(z, z')$ will be of order lower than $\mu$. Moreover, the set $S_2$ has no limiting point at finiteness and the order of convergency of the points in $S_2$ is at most equal to $\nu$.

In the second part, I considered the system of transcendental integral functions

$$F_l(z, z') \equiv \sum_{q=0}^{\infty} e^{Q_q(z)} h_l^{(q)}(z) z'^q \equiv \sum_{q=0}^{\infty} e^{Q_q(z)} \left( \sum_{p=0}^{l} A_p^{(q)} z^p \right) z'^q \quad (l = 1, 2, 3, \cdots)$$

under an additional condition,

(3) the multiplicity of every zero point of each of $h^{(q)}(z)$ $(q=0, 1, 2, \ldots\ldots)$ at the origin is less than a certain finite constant $s$.

Let $E$ be the set of the repeated limiting points of the zero points of $h_l^{(q)}(z)$ $(q=0,$

1 These Memoirs, 7, 345 (1924).

$1, 2, \ldots\ldots; l=1, 2, \ldots\ldots$) first taking $\lim l=\infty$ and then $\lim q=\infty$. Also let $E'$ be the set of the repeated limiting points of the same zero points, first taking $\lim q=\infty$ and then $\lim l=\infty$. Then we have $E=E'$. We have also the following theorems:

For any positive value $R_l \left(\left\{\frac{M-1}{e(\nu+\varepsilon)}\right\}^{\frac{1}{\nu+\epsilon}} \leqq R' < \left\{\frac{M}{e(\nu+\varepsilon)}\right\}^{\frac{1}{\nu+\epsilon}}\right)$ there corresponds a positive integer $L$ such that any $F_l(z, z')$ $(l \geqq L)$ is of apparent order $\mu$ in $z'$ for any $z$ $(|z| < R')$ except the points, at most $M+1$ in number, for which $F_l(z, z')$ is of apparent order lower than $\mu$.

If there be three positive values $G$, $K$ (however great) and $k\left(>\frac{1}{\mu}\right)$ such that to each $q_m(q_m \geqq G)$ there corresponds a value of $l_n$, and we have

$$\left| b_{l_n}^{(q_m)} - b_{l_n} \right| \cdot K^{q_m} \geqq \eta_n,$$

$$\frac{-\log \left| e^{Q_{q_m}(b_{l_n})} \cdot h_{l_n}^{(q_m)}(b_{l_n}) \right|}{q_m \log q_m} \geqq k,$$

where $b_{l_n}$ is a limiting point of the zero points $z=b_{l_n}^{(q_m)}$ of $h_{l_n}^{(q_m)}(z)$ for $\lim q_m=\infty$, then $F(z, z')$ will be of apparent order lower than $\mu$ in $z'$ for $z=\beta=\lim\limits_{l_n=\infty} b_{l_n}$.

Some similar theorems may also hold.

Jule Sire proved the following theorem.[1]

*Theorem. If* $F(u, v) \equiv \sum\limits_{n=0}^{\infty} a_n(u) v^n$ *be a transcendental integral function of finite total order* $\lambda$ *in* $u$ *and* $v$ *and if the transcendental integral function* $f(v) \equiv \sum\limits_{n=0}^{\infty} c_n v^n$, *where* $c_n$ *is the maximum coefficient of* $a_n(u)$, *be of apparent order* $\mu$ $(>0)$, *then* $F(u, v)$ *will be a transcendental integral function of apparent order* $\mu$ *in* $v$ *for any finite value of* $u$ *except the points of a punctual set* (un ensemble ponctuel) $M$ *for which* $F(u, v)$ *is of order lower than* $\mu$.

Among the definitions he gave in his papers, the following are specially important in later discussions.

Definition 1. Let $P_{q_1}(z)$, $P_{q_2}(z), \ldots\ldots$, $P_{q_m}(z), \ldots\ldots$, where $P_{q_m}(z) \equiv \sum\limits_{i=1}^{\varphi(q_m)} (z-a_{q_m i})$, be a series of polynomials such that all the zero points of each polynomial are in a circle $C_{R_0}$, center the origin and radius equal to $R_0$, and $\varphi(q_m) \leqq B q_m$ where $B$ is a constant. Let $S$ be the set

1 Rendiconti del Circolo Matematico di Palermo, **31**, 1—91, (1911).

of zero points of $P_{q_m}(z)$ $(m=1, 2, \ldots\ldots)$. We say that $z=a$ is a *regular limiting point* of $S$ when for any prescrived positive value $\varepsilon$, there corresponds a positive integer $Q(\varepsilon)$ such that each $P_{q_m}(z)$ $(q_m > Q(\varepsilon))$ has at least one zero point in the circle $|z-a| \leqq \varepsilon$.[1]

Definition 2. We say that $z=a$ is a *point of less increase* (un point de moindre croissance) of the series $P_{q_m}(z)$ $(m=1, 2, 3, \ldots\ldots)$, if

$$\underline{\lim_{q_m=\infty}} \frac{-\log |P_{q_m}(a)|}{q_m \log q_m} = k > 0.$$ [2]

Let $S_1$ be the set of the regular limiting points of $S$. If $z=b$ be not a point of the set $S_1$, we have evidently

$$\underline{\lim_{q_m=\infty}} \frac{-\log |P_{q_m}(b)|}{q_m \log q_m} = 0.$$

Accordingly if $S_2$ be the set of the points of less increase at finiteness, we have

I. $S_2 \leqq S_1$.

In this paper, under some additional condition, we shall first investigate the distribution of the points of less increase of a transcendental integral function $F(z, z') \equiv \sum_{p, q=0}^{\infty} e^{Q_q(z)} A_p^{(q)} z^p z'^q$ in the $z'$-plane and then find some properties concerning the order of $F_l(z, z') \equiv \sum_{p=0}^{l} \sum_{q=0}^{\infty} e^{Q_q(z)} A_p^{(q)} z^p z'^q$ $(l=1, 2, 3, \ldots\ldots)$ in $z'$.

1. *Lemma (Lindelöf's theorem). If $r_1, r_2, r_3, \ldots\ldots (r_1 \leqq r_2 \leqq r_3 \leqq \ldots\ldots)$ be the absolute values of the zero points of a transcendental integral function $f(z)$ which has no zero point at the origin, then* $\frac{1}{r_n} < \frac{\sqrt[n]{M(r)}}{r}$ *for any $r$, where $M(r)$ is the maximum value of $|f(z)|$ for $|z|=r$.*

The proof[3] is given under the assumption that no two $r_i$'s are equal. But it will be similarly proved with a slight modification when any number of $r_i$'s are equal.

*Theorem. Let*

$$F(z, z') \equiv \sum_{q=0}^{\infty} f^{(q)}(z) z'^q \equiv \sum_{p, q=0}^{\infty} A_p^{(q)} e^{Q_q(z)} z^p z'^q \equiv \sum_{q=0}^{\infty} e^{Q_q(z)} h^{(q)}(z) z'^q$$

*be a transcendental integral function of $z$ and $z'$ such that*

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1 Circolo Matematico, *loc. cit.* p. 33.

2 Circolo Matematico, *loc. cit.* p. 3.

3 Borel, Functions méromorphes, p. 105.

(1) *$F(z, z')$ satisfies the conditions of J. Sire's theorem above cited,*

(2) *the canonical product $h^{(q)}(z)$ $(q=0, 1, 2, \ldots\ldots)$ of the primary factors of the zero points of $f^{(q)}(z)$ are uniformly increasing functions*[1] *(for $q=0, 1, 2, \ldots\ldots$) at most of $\nu^{th}$ order.*

*Then $F(z, z')$ will be of apparent order $\mu$ in $z'$ for any finite $z$ except the points of a punctual set $S_2$, for which $F(z, z')$ will be of order lower than $\mu$. Moreover, the set $S_2$ has no limiting point at finiteness and the order of convergency of the points in $S_2$ is at most equal to $\nu$.*

In virtue of the condition (1), the first part of the theorem follows at once from the theorem of J. Sire. For the second part, put

$$h^{(q)}(z) \equiv z^{s_q} g^{(q)}(z) \quad (q=0, 1, 2, \ldots\ldots)$$

where $s_q$ is a positive integer or a zero, and $g^{(q)}(z)$ has no zero point at the origin. By the condition (2), for any prescribed positive value $\varepsilon$, there corresponds a positive value $R$, independent of $q$, such that

$$\left| h^{(q)}(z) \right| \leq e^{r^{\nu+\varepsilon}} \quad \text{for} \quad |z| = r \geqq R.$$

we have therefore

$$\left| g^{(q)}(z) \right| \leq \frac{e^{r^{\nu+\varepsilon}}}{r^{s_q}} \leq e^{r^{\nu+\varepsilon}} \quad \text{for} \quad r \geqq R \geqq 1.$$

Supposing that $r_1^{(q)}, r_2^{(q)}, r_3^{(q)}, \ldots\ldots$ be the moduli of zero points of $g^{(q)}(z)$, arranged in order of magnitude, we have by the lemma

$$\frac{1}{r_n^{(q)}} < \frac{\sqrt[n]{M^{(q)}(r)}}{r} \leq \frac{\left(e^{r^{\nu+\varepsilon}}\right)^{\frac{1}{n}}}{r} \quad \text{for} \quad r \geqq R \geqq 1,$$

where $M^{(q)}(r)$ is the maximum value of $g^{(q)}(z)$ for $|z| = r$. For any fixed value of $n$, $\dfrac{\sqrt[n]{e^{r^{\sigma}}}}{r}$ is minimum when $r = \left(\dfrac{n}{\sigma}\right)^{\frac{1}{\sigma}}$. Accordingly

$$\frac{1}{r_n^{(q)}} < \frac{e^{\frac{1}{\nu+\varepsilon}}}{\left(\dfrac{n}{\nu+\varepsilon}\right)^{\frac{1}{\nu+\varepsilon}}} \equiv A\left(\frac{1}{n}\right)^{\frac{1}{\nu+\varepsilon}} \quad \text{for} \quad n \geqq (\nu+\varepsilon) R^{\nu+\varepsilon},$$

1 These Momoirs, **6**, 253 (1923).

where

$$A=\left\{e(\nu+\varepsilon)\right\}^{\frac{1}{\nu+\epsilon}},$$

i.e.

$$r_n^{(q)} > \frac{1}{A} n^{\frac{1}{\nu+\epsilon}} \quad \text{for } n \geqq (\nu+\varepsilon)R^{\nu+\epsilon} \quad (q=0, 1, 2, \ldots\ldots).$$

Thus we have :

II. *There exist at most* $n-1$ *zero points of each* $g^{(q)}(z)$ $(q=0, 1, 2, \ldots\ldots)$ *in the circle* $C_n$, *center the origin and radius equal to* $\frac{n^{\frac{1}{\nu+\epsilon}}}{A}$ (*which is previously taken to be* $\geqq R$).

Given any prescribed positive value $\delta$, let $q_1, q_2, q_3, \ldots\ldots (q_1 < q_2 < q_3 < \ldots\ldots)$ be the values of $q$ which satisfy

$$\left| \frac{-\log | c_{q_m} |}{q_m \log q_m} - \frac{1}{\mu} \right| \leqq \delta,$$

where $c_{q_m}$ is the maximum coefficient[1] of $f^{(q_m)}(z)$. Also let $P^{(q_m)}(z)$ be the polynomial whose zero points are those of $f^{(q_m)}(z)$ in the circle $C_n$ and whose coefficient of the highest degree is 1. Then the number of zero points of $P^{(q_m)}(z)$ is, by II, at most equal to $s_{q_m}+n-1$. There exist however at most $n$ *distinct* zero points of $P^{(q_m)}(z)$. We have therefore at most $n$ regular limiting points and by I at most $n$ points of less increase of $P^{(q_m)}(z)$ $(m=1, 2, \ldots\ldots)$. Consider a circle $C'_n$, concentric to $C_n$, of radius $\frac{n^{\frac{1}{\nu+\epsilon}}}{A} - \eta$ where $\eta$ is any positive number. J. Sire proved[2] that the points in the circle $C'_n$ in the $z$-plane, at which the order of $F(z, z')$ is less than $\mu$, are the points, and the only points, of less increase of $P^{(q_m)}(z)$ $(m=1, 2, \ldots\ldots)$. We may therefore conclude that there can not exist more than $n$ points of $S_2$ in the circle $C'_n$, $n$ and $\eta$ being arbitrary, we may conclude that the set of points $S_2$ has no limiting point at finiteness.

Next, let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots$ be the points of $S_2$, arranged in order of magnitude. Since there exist at most $n$ points of $S_2$ in the circle $C'_n$, we have

$$\left| \alpha_{n+1} \right| > \frac{n^{\frac{1}{\nu+\epsilon}}}{A} - \eta \quad \text{for} \quad \frac{n^{\frac{1}{\nu+\epsilon}}}{A} \geqq R.$$

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1 Circolo Matematico. *loc. cit.*

2 Circolo Matematico. *loc. cit.* pp. 70—83.

As $\varepsilon$ and $\eta$ are arbitrary positive values we may conclude that the order of convergency of the points of $S_2$ is at most equal to $\nu$.

2. M. Edm. Maillet proved the theorem :[1]

*Let $G(z)$ be a canonical product of primary factors of the order $\rho$. For any prescribed positive values $\varepsilon$ and $\eta$ $(\eta\leqq 1)$, there corresponds a positive value $R_1$ such that $|G(z)|>e^{-r^{\rho+\varepsilon}}$ for any finite point $z$ $(|z|\geqq R_1)$ in the exterior of all the circles $C_n$, centers the zero points $a_n$ $(n=1, 2, 3,\ldots\ldots)$ and radii equal to $\eta$.*

In his proof, he reduced the case $\rho\geqq 1$ to the one $\rho<1$. As $G(z)$ is a transcendental integral function of $\rho^{th}$ $(\rho<1)$ order, we have by the lemma in art. 1.

$$\frac{1}{r_n}<\frac{\sqrt[n]{M(r)}}{r}$$

An easy calculation leads to the result that

$$r_n>n^{\frac{1}{\sigma}} \qquad \text{for} \quad n\geqq N(\varepsilon)$$

where $\sigma=\rho+\frac{\varepsilon}{2}<1$. For a point $z$ in the exterior of all the circles $C_n$ $(n=1, 2, 3,\ldots\ldots)$, we have $|z-a_n|>\eta$. Let $n_1$, $n_2$ and the modulus $r$ of $z$ be such that

$$r_{n_1}\leqq 2r\leqq r_{n_1}+1$$

$$n_2^{\frac{\sigma}{1}}\leqq 2r\leqq (n_2+1)^{\frac{1}{\sigma}},$$

where $n_2\geqq N(\varepsilon)$. Then as $r_{n_2}+2>(n_2+1)^{\frac{1}{\sigma}}>2r\geqq r_{n_1}$ we have $n_2\geqq n_1$. Put

$$G(z)\equiv G_1(z)\cdot G_2(z)\cdot G_3(z),$$

where

$$G_1(z)\equiv \prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right),$$

$$G_2(z)\equiv \prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right),$$

$$G_3(z)\equiv \prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right).$$

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1 Jordan's Journal de Mathématiques, **8**, 339 (1902).

By a simple calculation, we have

$$| G_1(z) | > \left(\frac{\eta}{2r}\right)^{n_1},$$

$$| G_2(z) | > \left(\frac{1}{2}\right)^{n_2-n_1},$$

$$| G_3(z) | > e^{-\frac{\sigma}{1-\sigma}(2r)^{\sigma}}$$

Hence

$$| G(z) | \equiv | G_1(z) | \cdot | G_2(z) | \cdot | G_3(z) | > e^{-r^{\sigma+\frac{\epsilon}{2}}} = e^{-r^{\rho+\epsilon}}$$

for $|z| = r \geqq R_1$,
where

$$R_1^{\frac{\epsilon}{2}} \geqq 2^{\sigma}\left(\log\frac{R}{\eta} + \log 2 + \frac{\sigma}{1-\sigma}\right).$$

We may similarly prove the theorem :

III. *For any prescribed positive values* $\varepsilon$, $\eta$ *and* $R_2(\geqq R)$, *there corresponds a positive value* $K$ *such that the canonical product* $G_4(z)$ *of the primary factors of the zero points of* $G(z)$, *whose moduli are greater than* $R_2+\eta$, *satisfies*

$$| G_4(z) | \geqq e^{-KR_2^{\rho+\epsilon}}$$

*for any point* $z$ $(|z| \leqq R_2)$.

As $h^{(q)}(z)$ $(q=0, 1, 2, \ldots\ldots)$ are uniformly increasing, we have similarly as in art. 1

$$r_n^{(q)} > n^{\frac{1}{\nu+\epsilon}} \text{ for } n \geqq N(\varepsilon) = (\nu+\varepsilon)R^{\nu+\epsilon} \quad (q=0, 1, 2, \ldots\ldots).$$

Hence, corresponding to the theorem of M. Edm. Maillet, we have

IV. *For any assigned values of* $q$ *and for any prescribed positive values* $\varepsilon$ *and* $\eta$ $(\leqq 1)$, *there corresponds a positive value* $R_1$, *independent of* $q$, *such that*

$$| h^{(q)}(z) | > e^{-r^{\nu+\epsilon}}$$

*for any finite point* $z$ $(|z| \geqq R_1)$ *in the exterior of all the circles* $C_n^{(q)}$ $(n=1, 2, 3, \ldots\ldots)$, *center the zero points* $a_n^{(q)}$ *of* $h^{(q)}(z)$ *and radii equal to* $\eta$.

Similarly, corresponding to III, we have

V. *For any prescribed positive values* $\varepsilon$, $\eta$ *and* $R_2(\geqq R)$, *there corresponds a positive value* $K$ *such that the canonical product* $G_4^{(q)}(z)$ *of* *the primary factors of the zero points of $h^{(q)}(z)$, whose moduli are greater than $R_2+\eta$, satisfies*

$$|G_1^{(q)}(z)| \geqq e^{-KR_2^{\nu+\epsilon}} \qquad (q=0,\ 1,\ 2,\ldots\ldots)$$

*for any point $z$ ($|z|\leqq R_2$).*

Suppose that

$$h^{(q)}(z)\equiv z^{s_q}\, g^{(q)}(z)\equiv z^{s_q}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n^{(q)}}\right)e^{\frac{z}{a_n^{(q)}}+\frac{1}{2}\left(\frac{z}{a_n^{(q)}}\right)^2+\ldots+\frac{1}{p_q}\left(\frac{z}{a_n^{(q)}}\right)^{p_q}}$$

where $p_q\leqq p\leqq\nu$ for $q=0,\ 1,\ 2,\ldots\ldots$

In virtue of II, we may assign a positive integer $M$ such that the number $M_q$ of zero points of $g^{(q)}(z)$ $(q=0,\ 1,\ 2,\ldots\ldots)$ in the circle $|z|=R_2+\eta$ is at most equal to $M$. By the lemma of art. 1, we have

$$\frac{R_2}{r_1^{(q)}}< e^{R_2^{\nu+\epsilon}}$$

Hence we have

$$\left|\prod_{n=1}^{M_q} e^{\frac{z}{a_n^{(q)}}+\frac{1}{2}\left(\frac{z}{a_n^{(q)}}\right)^2+\ldots+\frac{1}{p_q}\left(\frac{z}{a_n^{(q)}}\right)^{p_q}}\right|> e^{-M\left(e^{R_2^{\nu+\epsilon}}+\ldots+\frac{1}{p}e^{pR_2^{\nu+\epsilon}}\right)}.$$

Suppose that $q_1, q_2, q_3,\ldots\ldots$ have the same meaning as in art. 1. Then we may similarly prove as in the treatise of J. Sire[1] that

$$\lim_{q_m=\infty}\frac{-\log\left|\frac{1}{c_{q_m}}e^{Q_{q_m}(z)}\cdot e^{\sum_{n=1}^{M_{q_m}}\left(\frac{z}{a_n^{(q_m)}}+\ldots+\frac{1}{p_{q_m}}\left(\frac{z}{a_n^{(q_m)}}\right)^{p_{q_m}}\right)}\cdot G_1^{(q_m)}(z)\right|}{q_m\log q_m}=0$$

uniformly for $|z|\leqq R_2$. But

$$\lim_{q_m=\infty}\frac{-\log|G_1^{(q_m)}(z)|}{q_m\log q_m}\leqq\lim_{q_m=\infty}\frac{-\log\left|e^{-KR_2^{\nu+\epsilon}}\right|}{q_m\log q_m}=0,$$

and

$$\lim_{q_m=\infty}\frac{-\log\left|e^{\sum_{1}^{M_{q_m}}\left(\frac{z}{a_n^{(q_m)}}+\ldots\ldots+\frac{1}{p_{q_m}}\left(\frac{z}{a_n^{(q_m)}}\right)^{p_{q_m}}\right)}\right|}{q_m\log q_m}$$

1 Circolo Matematico. *loc. cit.* pp. 76—81.

$$\leqq \lim_{q_m=\infty} \frac{-\log \left| e^{-M'\left(e^{R_2^{\nu+\epsilon}}+\cdots\cdots+\frac{1}{p}e^{pR_2}\right)^{\nu+\epsilon}} \right|}{q_m \log q_m}=0.$$

Similarly we have

$$\lim_{q_m=\infty} \frac{-\log \mid G_4^{(q_m)}(z) \mid}{q_m \log q_m} \geqq 0$$

and

$$\lim_{q_m=\infty} \frac{-\log \left| e^{\sum_1^{M_{q_m}}\left(\frac{z}{a_n^{(q_m)}}+\frac{1}{2}\left(\frac{z}{a_n^{(q_m)}}\right)^2+\cdots\cdots+\frac{1}{p_{q_m}}\left(\frac{z}{a_n^{(q_m)}}\right)^{p_{q_m}}\right)} \right|}{q_m \log q_m} \geqq 0.$$

We have therefore

$$\lim_{q_m=\infty} \frac{-\log \left| \dfrac{e^{Q_{q_m}(z)}}{c_{q_m}} \right|}{q_m \log q_m}=0$$

uniformly for all $z$ ($|z| \leqq R_2$) and from which it follows that

$$\underline{\lim}_{q=\infty} \frac{-\log |e^{Q_q(z)}|}{q \log q} \leqq \frac{1}{\mu} = \underline{\lim}_{q=\infty} \frac{-\log | c_q |}{q \log q}.$$

We shall now prove that

$$\underline{\lim}_{q=\infty} \frac{-\log |e^{Q_q(z)}|}{q \log q} \geqq \frac{1}{\mu}.$$

If $c_q'$ be the maximum coefficient[1] in the expansion of $e^{Q_q(z)}$, we have

$$| c_q' | \leqq | c_q | \leqq | c_q' | \, e^{R_2^{\nu+\epsilon}},$$

i. e.

$$\lim_{q=\infty} \frac{-\log \left| \dfrac{c_q}{c_q'} \right|}{q \log q}=0.$$

If $M_q(r)$ be the maximum value of $| e^{Q_q(z)} |$ for $| z | = r$, we have, as J. Sire[2] treated,

$$\lim_{q=\infty} \frac{-\log \left| \dfrac{c_q'}{M_q(r)} \right|}{q \log q}=0.$$

From $| e^{Q_q(z)} | \leqq M_q(r)$, it follows that

1 Circolo Matematico. *loc. cit.*

2 Circolo Matematico. *loc. cit.* pp. 15—17.

$$\frac{-\log|e^{Q_q(z)}|}{q\log q} \geqq \frac{-\log M_q(r)}{q\log q}$$

and hence

$$\varliminf_{q=\infty} \frac{-\log|e^{Q_q(z)}|}{q\log q} \geqq \varliminf_{q=\infty} \frac{-\log|c_q|}{q\log q} = \frac{1}{\mu}.$$

We have therefore

$$\text{VI.} \quad \varliminf_{q=\infty} \frac{-\log|e^{Q_q(z)}|}{q\log q} = \frac{1}{\mu}$$

*uniformly for all* $z$ ($|z| \leqq R_2$), *where* $R_2$ *is any positive value.*

3. We shall hereafter assume that

(3) $s_q$ ($q=0, 1, 2, \ldots\ldots$) are limited and $\leqq s$.

Suppose that $|z| < R_1$, where $R_1$ is the determined value in IV. Consider a circle $C$, centre the origin and radius equal to $2R_1$. Then in virtue of II and (3), there exists a positive integer $M$ such that the zero points of $h^{(q)}(z)$ ($q=0, 1, 2, \ldots\ldots$) in the circle $C$ can not exceed $M+s$ in number. For any $z$ ($|z| < R_1$) in the exterior of all the circles $C_n^{(q)}$ ($n=1, 2, \ldots\ldots$), centers the zero points $a_n^{(q)}$ of $h^{(q)}(z)$ and radii equal to $\eta$, we have

$$\left| h^{(q)}(z) \right| = \left| z^{s_q} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n^{(q)}}\right) e^{\frac{z}{a_n^{(q)}} + \ldots\ldots + \frac{1}{p_q}\left(\frac{z}{a_n^{(q)}}\right)^{p_q}} \right|,$$

$$|z|^{s_q} \geqq \eta^{s_q} \geqq \eta^s,$$

$$\left| 1 - \frac{z}{a_n^{(q)}} \right| \geqq \frac{\eta}{2R_1}$$

for $a_n^{(q)}$ in the circle $C$. The canonical product of the primary factors of the zero points of $h^{(q)}(z)$ in the exterior of the circle $C$ is absolutely less than $e^{-K \cdot R_1^{\nu+\varepsilon}}$ (by V). As I did in the preceding article, we have

$$\left| \prod e^{\frac{z}{a_n^{(q)}} + \frac{1}{2}\left(\frac{z}{a_n^{(q)}}\right)^2 + \ldots\ldots + \frac{1}{p_q}\left(\frac{z}{a_n^{(q)}}\right)^{p_q}} \right|$$

$$> e^{-M\left(e^{R_1^{\nu+\epsilon}} + \ldots\ldots + \frac{1}{p} e^{pR_1^{\nu+\epsilon}}\right)}$$

where the product $\Pi$ is taken for all $a_n^{(q)}$ in the circle $C$. Consequently

$$\left| h^{(q)}(z) \right| > \eta^{s} \left(\frac{\eta}{2R_1}\right)^{M} \cdot e^{-KR_1^{\nu+\varepsilon} - M\left(e^{R_1^{\nu+\epsilon}} + \ldots\ldots + \frac{1}{p} e^{pR_1^{\nu+\epsilon}}\right)} \geqq D > 0,$$

where $D$ is independent of $q$. Thus we have:

VII. *For any assigned value of $q$ and for any prescribed positive value $\eta$, there corresponds a positive value $D$, independent of $q$, such that*

$$| h^{(q)}(z) | > D$$

*for any point $z$ ($|z| < R_1$) in the exterior of all the circles $C_n^{(q)}$ ($n = 1, 2, 3, \ldots\ldots$).*

VIII. *For any prescribed positive values $\eta$ (however small) and $R_2$ (however great), there exists a positive integer $L$ (independent of $q$) such that to each zero point of one of*

$$h^{(q)}(z) \equiv A_o^{(q)} + A_1^{(q)} z + \ldots\ldots + A_n^{(q)} z^n + \ldots\ldots$$

*and*

$$h_l^{(q)}(z) \equiv A_o^{(q)} + A_1^{(q)} z + \ldots\ldots + A_l^{(q)} z^l \qquad (l \geqq L)$$

*whose modulus is less than $R_2$, there corresponds at least one zero point of the other and the distance between these zero points is less than $\eta$.*

Determine a positive integer $M$ as to satisfy

$$\frac{(M-1)^{\frac{1}{\nu+\epsilon}}}{A} < R_2 \leqq \frac{M^{\frac{1}{\nu+\epsilon}}}{A},$$

then each $g^{(q)}(z)$ ($q = 0, 1, 2, \ldots\ldots$) has at most $M-1$ zero points of moduli inferior to $R_2$. Hence $h^{(q)}(z)$ has at most $M$ distinct zero Points of moduli inferior to $R_2$. Let $\eta'$ be a positive value not greater than

$\dfrac{\eta}{2(M+1)}$ and $\dfrac{(M+1)^{\frac{1}{\nu+\epsilon}} - M^{\frac{1}{\nu+\epsilon}}}{2(M+1)A}$. In virtue of IV, there exists a finite positive value $R_1(\eta')$, independent of $q$, such that

$$| h^{(q)}(z) | > e^{-r^{\nu+\epsilon}}$$

for any finite point $z$ ($|z| \geqq R_1(\eta')$) in the exterior of all the circles $C_n^{(q)}$ ($n = 1, 2, \ldots\ldots$). Put $\dfrac{(M+1)^{\frac{1}{\nu+\epsilon}}}{A} = R_3$. If $2R_3 > R_1(\eta')$, we have

$$| h^{(q)}(z) | > e^{-(2R_3)^{\nu+\epsilon}}$$

for any point $z$ $(2R_3 \geqq |z| \geqq R_1(\eta'))$ in the exterior of all the circles $C_n^{(q)}$ $(n=1, 2, \ldots\ldots)$. For $|z| < R_1(\eta')$, we have, by VII, a positive number $D$, independent of $q$, such that

$$| h^{(q)}(z) | > D$$

for any $z$ $(|z| < R_1(\eta'))$ in the exterior of all the circles $C_n^{(q)}(n=1, 2, \ldots\ldots)$. Let $D'$ be the smaller of $e^{-(2R_3)^{\nu+\epsilon}}$ and $D$. In case of $2R_3 \leqq R_1(\eta')$, we have $D'=D$. Put

$$h^{(q)}(z) \equiv h_l^{(q)}(z) + \varphi_l^{(q)}(z).$$

From $R_3 > R$, we have

$$| h^{(q)}(z) | \leqq e^{(2R_3)^{\nu+\epsilon}} \quad \text{for } |z| \leqq 2R_3,$$

and hence

$$\left| \varphi_l^{(q)}(z) \right| \leqq e^{(2R_3)^{\nu+\epsilon}} \cdot \sum_{l+1}^{\infty} \left( \frac{R_3}{2R_3} \right)^n = 2 \cdot \left( \frac{1}{2} \right)^{l+1} \cdot e^{(2R_3)^{\nu+\epsilon}},$$

$$\left| \frac{d\varphi_l^{(q)}(z)}{dz} \right| \leqq \left( \frac{1}{2} \right)^l \cdot \frac{l+2}{R_3} \cdot e^{(2R_3)^{\nu+\epsilon}}$$

for $|z| \leqq R_3$. Hence for any prescribed positive value $\varepsilon'$, there corresponds a finite positive integer $L$ such that

$$| \varphi_l^{(q)}(z) | < \varepsilon' D' \qquad (q=0, 1, 2, \ldots\ldots)$$

and

$$\left| \frac{d\varphi_l^{(q)}(z)}{dz} \right| < \varepsilon' D' \quad (q=0, 1, 2, \ldots\ldots)$$

for $|z| \leqq R_3$ and $l \geqq L$. We may also assign a positive value $G$ such that

$$| h^{(q)}(z) | < G \qquad (q=0, 1, 2, \ldots\ldots)$$

and

$$\left| \frac{dh^{(q)}(z)}{dz} \right| < G \qquad (q=0, 1, 2, \ldots\ldots)$$

for $|z| \leqq R_3$, and from which, it follows that

$$| h_l^{(q)}(z) | \leqq | h^{(q)}(z) | + | \varphi_l^{(q)}(z) | < G + \varepsilon' D \quad (q=0, 1, 2, \ldots\ldots)$$

and

$$\left| \frac{dh_l^{(q)}(z)}{dz} \right| \leqq \left| \frac{dh^{(q)}(z)}{dz} \right| + \left| \frac{d\varphi_l^{(q)}(z)}{dz} \right| < G + \varepsilon' D \quad (q=0, 1, 2, \ldots\ldots)$$

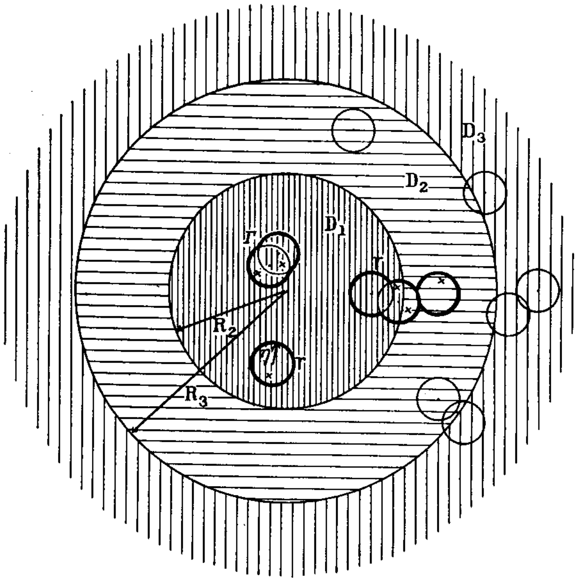
for $|z| \leq R_3$.

Now we divide the whole plane into three parts (see figure):

i) $D_1$: a domain composed of all interior points of the circle, center the origin and radius equal to $R_2$,

ii) $D_2$: a domain bounded by two concentric circles, centers the origin and radii equal to $R_2$ and $R_3$ respectively,

iii) $D_3$: a domain composed of all the points which belong to neither $D_1$ nor $D_2$.




• zero point of $h^{(q)}(z)$
× zero point of $h_l^{(q)}(z)$

We consider the circles $C_n^{(q)}$ $(n=1, 2, \ldots\ldots)$, whose centers are the zero points $a_n^{(q)}$ of $h^{(q)}(z)$ and the radii equal to $\eta'$. As

$$\eta' \leq \frac{(M+1)^{\frac{1}{\nu+\epsilon}} - M^{\frac{1}{\nu+\epsilon}}}{2(M+1)A} \leq \frac{R_3 - R_2}{2(M+1)}$$

all the circles $C_n^{(q)}$ whose centers $a_n^{(q)}$ $(n=1, 2, \ldots\ldots)$ are in $D_1$ lie wholly outside all the circles $C_n^{(q)}$ whose centers are in $D_3$; and if any one of the former intersects with a circle $C_n^{(q)}$ whose center is either in $D_1$ or in $D_2$, and which intersects with another one and so on, the system of these intersecting circles lies wholly outside all the circles whose centers are in $D_3$. Accordingly if a system of intersecting circles contains at least one circle whose center lies in $D_1$, the system contains at most $M+1$ circles. Let $a_{n'}^{(q)}$ be any zero point of $h^{(q)}(z)$ such that $|a_{n'}^{(q)}|<R_2$. Now we take a closed curve $\Gamma$ as follows: If $C_{n'}^{(q)}$ lies wholly outside all other circles $C_n^{(q)}$, let $\Gamma$ be $C_{n'}^{(q)}$. If $C_{n'}^{(q)}$ intersects with another $C_n^{(q)}$, let $\Gamma$ be the closed curve composed of the circular arcs of the intersecting circles and containing all the interior points of at least one of these circles in the interior. Then the arc length of $\Gamma$ is at most equal to $2(M+1)\pi\eta'$. On the curve $\Gamma$,

$$|h^{(q)}(z)|\geqq D'$$
$$|h_l^{(q)}(z)|>D'-\varepsilon'D'=(1-\varepsilon')D'$$

for $l\geqq L$. Hence

$$\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \frac{\dfrac{dh^{(q)}(z)}{dz}}{h^{(q)}(z)}dz$$

and

$$\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \frac{\dfrac{dh_l^{(q)}(z)}{dz}}{dh_l^{(q)}(z)}dz$$

represent the numbers of zero points of $h^{(q)}(z)$ and $h_l^{(q)}(z)$ respectively in the interior of $\Gamma$. We have

$$\left|\frac{\dfrac{dh^{(q)}(z)}{dz}}{h^{(q)}(z)}-\frac{\dfrac{dh_l^{(q)}(z)}{dz}}{h_l^{(q)}(z)}\right|=\left|\frac{h_l^{(q)}(z)\,\dfrac{d\varphi_l^{(q)}(z)}{dz}-\varphi_l^{(q)}(z)\,\dfrac{dh_l^{(q)}(z)}{dz}}{h^{(q)}(z)\,h_l^{(q)}(z)}\right|$$
$$<\frac{2\varepsilon'D'\,(G+\varepsilon'D')}{(1-\varepsilon')\,D'^2}=\frac{2\varepsilon'\,(G+\varepsilon'D')}{(1-\varepsilon')\,D'}$$

Take $\varepsilon'$ so small that

$$\frac{2\varepsilon'\,(G+\varepsilon'D')}{(1-\varepsilon')\,D'}\leqq\frac{1}{(M+1)\,\eta'},$$

then we have

$$\left|\frac{1}{2\pi\sqrt{-1}}\int_\Gamma\left(\frac{\dfrac{dh^{(q)}(z)}{dz}}{h^{(q)}(z)}-\frac{\dfrac{dh_l^{(q)}(z)}{dz}}{h_l^{(q)}(z)}\right)dz\right|$$
$$<\frac{1}{2\pi}\cdot\frac{2\varepsilon'\,(G+\varepsilon'D')}{(1-\varepsilon')\,D'}\cdot 2(M+1)\,\pi\eta'\leqq 1.$$

Thus the number of zero points of $h^{(q)}(z)$ in $\Gamma$ is equal to that of $h_l^{(q)}(z)$

$(l \geqq L)$ in $\Gamma$. But any two interior points of $\Gamma$ are at a distance less than

$$2(M+1)\eta' \leqq \eta,$$

and hence, to each zero point $a_n^{(q)}$ $(|a_n^{(q)}| < R_2)$ of $h^{(q)}(z)$ $(q=0, 1, 2, \ldots\ldots)$, there corresponds at least one zero point of $h_l^{(q)}(z)$ $(l \geqq L)$ and their distance is less than $\eta$. Next, we consider the curves $\Gamma$'s corresponding to all the zero points of $h^{(q)}(z)$ in $D_1$. If there be a zero point $z=a$ of $h_l^{(q)}(z)$ $(l \geqq L)$ in the exterior of all $\Gamma$'s and in the domain $D_1$, we have

$$h_l^{(q)}(a) = h^{(q)}(a) - \varphi_l^{(q)}(a) = 0,$$

which is absurd, since

$$|h^{(q)}(a)| > D'$$

and

$$|\varphi_l^{(q)}(a)| < \varepsilon' D'.$$

We may therefore conclude that to each zero point $z=a$ $(|a| < R_2)$ of $h_l^{(q)}(z)$ $(l \geqq L;\ q=0, 1, 2, \ldots\ldots)$, there corresponds at least one zero point of $h^{(q)}(z)$ and their distance is inferior to $\eta$.

4. Let the set of the limiting points of the zero points $\{b_1^{(q)}, b_2^{(q)}, \ldots\ldots, b_l^{(q)}\}$ of $h_l^{(q)}(z)$ for $q=0, 1, 2, \ldots\ldots$ ($l$: fixed) be $E_l$, and let the set of the limiting points of $\{b_1^{(q)}, b_2^{(q)}, \ldots\ldots, b_l^{(q)}\}$ for $l=1, 2, 3, \ldots\ldots$ ($q$: fixed) be $E^{(q)}$. Then $E^{(q)}$ is, by VIII, the set of the zero points of $h^{(q)}(z)$, and hence it contains only a finite number of points in any finite part of the plane. Thus $E^{(q)}$ has no limiting point at finiteness. Express the points of $E_l$ by $b_l$'s and those of $E^{(q)}$ by $b^{(q)}$'s. Let $E$ be the set of the limiting points of $E^{(q)}$ $(q=0, 1, 2, \ldots\ldots)$ in any finite part of the plane. Also let $E'$ be the set of the limiting points of $E_l$ $(l=1, 2, 3, \ldots\ldots)$ in any finite part of the plane such that for any point $\beta$ of $E'$ there exists a point $z=b_{l_n}$ in $E_{l_n}$ and $\lim_{n=\infty} b_{l_n} = \beta$, and conversely all the limiting points of $b_{l_n}$ $(n=1, 2, \ldots\ldots;\ b_{l_n}$: limited) are the points of $E'$.

Now we shall prove

IX. $E \equiv E'$.

For let $\beta$ be a point of $E$ and let $b^{(q_1)}, b^{(q_2)}, \ldots\ldots$ be the points such that $\lim_{n=\infty} b^{(q_n)} = \beta$. For any prescribed positive value $\eta$ there corresponds, by VIII, a positive value $L$ such that each $h_l^{(q_n)}(z)$ $(q=0, 1, 2, \ldots\ldots;\ l \geqq L)$ has at least one zero point $b_l^{(q_n)}$ and

$$|b_l^{(q_n)} - b^{(q_n)}| < \eta.$$

Taking limit $n=\infty$, we have

$$|b_l - \beta| \leqq \eta. \qquad \text{Q. E. D.}$$

Conversely, let $\beta'$ be a point of $E'$ and let $\lim_{n=\infty} b_{l_n}=\beta'$. If $\beta'$ be not a point of $E$, there should be a positive number $\eta$ such that $|\beta'-\beta|>\eta$, where $\beta$ is any point of $E$. As $\lim_{n=\infty} b_{n}=\beta'$, there exists a positive integer $N$ such that

$$\left| b_{l_n}-\beta' \right|<\frac{\eta}{3} \quad \text{for } n\geqq N.$$

Since $b_{l_n}$ is a point of $E_{l_n}$, there exist $b_{l_n}^{(q_n m)}$ $(m=1, 2, \ldots\ldots)$ and a positive integer $M(n)$ such that

$$\left| b_{l_n}^{(q_n, m)}-b_{l_n} \right|<\frac{\eta}{3} \quad \text{for } m\geqq M(n).$$

There exists, by VIII, a positive integer $L$ such that for $l_n\geqq L$

$$\left| b_{l_n}^{(q)}-b^{(q)} \right|<\frac{\eta}{3} \quad (q=0, 1, 2,\ldots\ldots).$$

Consequently, we have

$$|b^{(q_n, m)}-\beta'|\leqq|b_{l_n}-\beta'|+|b_{l_n}^{(q_n,m)}-b_{l_n}|+|b_{l_n}^{(q_n,m)}-b^{(q_n m)}|<\eta,$$

which holds for infinitely many $b^{(q_n, m)}$'s. Hence there would exist at least one limiting point $\beta$ of these $b^{(q_n, m)}$'s such that

$$|\beta-\beta'|\leqq\eta,$$

which is against the assumption.

Let $\beta$ be a point of $E$ and let $b^{(q_m)}$ $(q_1<q_2<q_3<\ldots\ldots)$ be the points whose limiting point is $\beta$. Considering a sequence of numbers $\eta_n$ $(\eta_1>\eta_2>\eta_3>\ldots\ldots;\ \lim_{n=\infty}\eta_n=0)$, let $l_n$ be the least integer which corresponds to $\eta_n$. Then we have, by VIII,

$$|b_{l_n}^{(q_m)}-b^{(q_m)}|<\eta_n \quad (m=1, 2, 3,\ldots\ldots;\ n=n_1, n_1+1,\ldots\ldots).$$

X. *If there exists a sequence of finite positive integers $G_n$ $(n=1, 2,\ldots\ldots)$ such that*

$$|b_{l_n}^{(q_m)}-b_{l_n}|\leqq\eta_n$$

*for $q_m\geqq G_n$, then $\beta$ will be a regular limiting point of $b^{(q_m)}$ $(m=1, 2,\ldots\ldots)$.*

Since

$$|b_{l_n}^{(q_m)}-b^{(q_m)}|<\eta_n \quad (m=1, 2,\ldots\ldots)$$

we have

$$|b_{l_n}-\beta|\leqq\eta_n.$$

Consequently,

$$| b'^{(q_m)}-\beta | \leqq | b^{(q_m)}-b_{l_n}^{(q_m)} | + | b_{l_n}^{(q_m)}-b_{l_n} | + | b_{l_n}-\beta | \leqq 3\eta_n$$

for $q_m \geqq G_m$. Q. E. D.

As a special case of this theorem, we have

XI. *If* $b_{l_n}$ $(n=1, 2, \ldots\ldots)$ *be a regular limiting point of* $b_{l_n}^{(q_m)}$ $(m=1, 2, \ldots\ldots)$, *then the limiting point* $\beta$ *of* $b_{l_n}$ $(n=1, 2, \ldots\ldots)$ *will be also a regular limiting point of the limiting points* $b'^{(q_m)}$ $(m=1, 2, \ldots\ldots)$ *of the points* $b_{l_n}^{(q_m)}$ $(n=1, 2, \ldots\ldots)$.

XII. *If there be three positive values* $G$, $K$ (*however great*) *and* $k$ (*however small*) *such that to each* $q_m$ $(q_m \geqq G)$, *there corresponds a value of* $l_n$ *and we have*

$$| b_{l_n}^{(q_m)}-b_{l_n} | \cdot K^{q_m} \geqq \eta_n,$$

$$| b_{l_n}^{(q_m)}-b_{l_n} | \leqq e^{-k\, q_m \log q_m}$$

*then*

$$\underset{q_m=\infty}{\underline{lim}} \frac{-\log | P^{(q_m)}(\beta) |}{q_m \log q_m} \geqq k,$$

*where* $P^{(q_m)}(z)$ *is, as in art.* 1, *a polynomial whose coefficient of the highest degree is* 1 *and whose zero points are those of* $h^{(q_m)}(z)$ *in the circle* $C$, *center the origin and radius* $R'$ $(>| \beta |)$.

In virtue of II and (3), the number of zero points of $h^{(q_m)}(z)$ in the circle $C$ is not greater than a certain finite positive integer $N$. Accordingly

$$\begin{aligned} | P^{(q_m)}(\beta) | &< | b'^{(q_m)}-\beta | \cdot (2R')^{N-1} \\ &\leqq \{ | b'^{(q_m)}-b_l^{(q_m)} | + | b_l^{(q_m)}-b_{l_n} | + | b_{l_n}-\beta | \}(2R')^{N-1} \\ &< \left(2\eta_n + \left| b_{l_n}^{(q_m)}-b_{l_n} \right| \right) (2R')^{N-1} \\ &\leqq \left(1+2K^{q_m}\right) \cdot e^{-k\, q_m \log q_m} (2R')^{N-1}. \end{aligned}$$

We have, therefore,

$$\underset{q_m=\infty}{\underline{lim}} \frac{-\log | P^{(q_m)}(\beta) |}{q_m \log q_m} \geqq \underset{q_m=\infty}{\underline{lim}} \frac{-\log \{ (2R')^{N-1} (1+2K^{q_m}) \}}{q_m \log q_m} + k$$
$$\geqq k.$$

As a special case of this theorem, we have

XIII. *If there be three positive values,* $L$, $G$ (*however great*) *and* $k$ (*however small*) *such that*

$$\left| b_{l_n}^{(q_m)}-b_{l_n} \right| \leqq e^{-k\, q_m \log q_m}$$

*for any* $q_m$, $l_n$ $(q_m \geqq G,\ l_n \geqq L)$ *then*

$$\underset{q_m=\infty}{\underline{lim}} \frac{-\log|P^{(q_m)}(\beta)|}{q_m \log q_m} \geqq k,$$

*where* $P^{(q_m)}(z)$ *has the same meaning as in* XII.

5. We now consider a system of transcendental integral functions

$$F_l(z, z') \equiv \sum_{q=0}^{\infty} e^{Q_q(z)} (A_0^{(q)} + A_1^{(q)} z + \ldots\ldots + A_l^{(q)} z^l) z'^q \quad (l = 1, 2, \ldots\ldots).$$

Since by VI

$$\underset{q=\infty}{\underline{lim}} \frac{-\log|e^{Q_q(z)}|}{q \log q} = \frac{1}{\mu}$$

for any finite $z$, $F_l(z, z')$ will be of apparent order $\mu$ in $z'$ for any finite $z$ except the points for which

$$\underset{q=\infty}{\underline{lim}} \frac{-\log|h_l^{(q)}(z)|}{q \log q} > 0.$$

There exist, however, at most $l$ finite points in the z-plane, for which $F_l(z, z')$ is of apparent order lower than $\mu$. For otherwise $F_l(z, z')$ would be of apparent order lower than $\mu$ for any finite $z$. We may therefore conclude:

XIV. *$F_l(z, z')$ is of apparent order $\mu$ in $z'$ for any finite $z$ except the points, at most $l$ in number, for which $F_l(z, z')$ is of order lower than $\mu$.*

Let $M$ be any positive integer $> 1 + (AR)^{\nu+\epsilon}$, where $\epsilon$, $R$ and $A$ have the same meaning as on p. 165. Put

$$\frac{(M-1)^{\frac{1}{\nu+\epsilon}}}{A} = R_1, \quad \frac{M^{\frac{1}{\nu+\epsilon}}}{A} = R_2, \quad \frac{(M+1)^{\frac{1}{\nu+\epsilon}}}{A} = R_3,$$

and let $\eta$ be a positive value less than 1 and $\frac{R_2 - R_1}{3}$. For these values of $\eta$ and $R_2$, there corresponds, by VIII, a positive integer $L_1$ (independent of $q$) such that to each zero point of one of

$$h^{(q)}(z) \equiv A_0^{(q)} + A_1^{(q)} z + \ldots\ldots + A_n^{(q)} z^n + \ldots\ldots$$

and

$$h_l^{(q)}(z) \equiv A_0^{(q)} + A_1^{(q)} z + \ldots\ldots + A_l^{(q)} z^l \quad (l \geqq L_1),$$

whose modulus is less than $R_2$, there corresponds at least one zero point$^{(q)}$ of the other and the distance between these zero points is less than $\eta$. From the assigned theorem, we may infer that the number $N_l^{(q)}$ of the zero points of $h_l^{(q)}(z)$ $(l \geqq L_1)$, whose moduli are less than $R_1 + 2\eta$, is at most equal to the number of those points of $h^{(q)}(z)$, whose moduli are less than $R_2$, i. e.

$$N_l^{(q)} \leqq s+M.$$

We may similarly assign a positive integer $L_2$ (independent of $q$) such that the zero points of $h_l^{(q)}(z)$ $(l \geqq L_2)$, whose moduli are less than $R_2+1$, are at most equal to $s+M'$ in number, where $M'$ is the least positive integer $>\left\{A(R_2+1)\right\}^{\nu+\epsilon}-1$. We have evidently

$$L_2 \geqq L_1.$$

Put

$$P_l^{(q)}(z) \equiv \prod_{n=1}^{N_l^{(q)}} (z-b_{l,n}^{(q)})$$

and

$$h_l^{(q)}(z) \equiv P_l^{(q)}(z) \cdot \bar{h}_l^{(q)}(z),$$

where $b_{l,n}^{(q)}$ $(n=1, 2, \ldots\ldots; \ |b_{l,1}^{(q)}| \leqq |b_{l,2}^{(q)}| \leqq \ldots\ldots)$ are the zero points of $h_l^{(q)}(z)$. Then

$$|P_l^{(q)}(z)| \geqq \eta^{N_l^{(q)}} \geqq \eta^{s+M} \qquad (l \geqq L_1)$$

for $|z|=R_2$. We have

$$\left| h_l^{(q)}(z) \right| \leqq e^{(R_2+\eta)^{\nu+\epsilon}} \cdot \left(\frac{R_2}{R_2+\eta}\right)^{l+1} \cdot \frac{R_2+\eta}{\eta} < e^{(R_2+\eta)^{\nu+\epsilon}} \cdot \frac{R_2+\eta}{\eta}$$

for $|z|=R_2$, and

$$\left| \bar{h}_l^{(q)}(z) \right| \equiv \left| \frac{h_l^{(q)}(z)}{P_l^{(q)}(z)} \right| < e^{R_2+\eta^{\nu+\epsilon}} \cdot \frac{R_2+\eta}{\eta^{s+M+1}}.$$

Since $\bar{h}_l^{(q)}(z)$ is a polynomial, the maximum value of $|\bar{h}_l^{(q)}(z)|$ for $|z| \leqq R_1+\eta$ is not greater than that of $|\bar{h}_l^{(q)}(z)|$ for $|z|=R_2$; and we have

$$\left| \bar{h}_l^{(q)}(z) \right| < e^{(R_2+\eta)^{\nu+\epsilon}} \cdot \frac{R_2+\eta}{\eta^{s+M+1}}$$

for $|z| \leqq R_1+\eta$. Since the zero points of $h_l^{(q)}(z)$ $(l \geqq L_2)$, whose moduli are less than $R_2+1$, are not greater than $s+M'$ in number,

$$\left| \bar{h}_l^{(q)}(z) \right| \geqq \eta^{s+M'}$$

for $|z| \leqq R_1+\eta$. We have therefore

$$\eta^{s+M'} \leqq \left| \bar{h}_l^{(q)}(z) \right| < e^{(R_2+\eta)^{\nu+\epsilon}} \cdot \frac{R_2+\eta}{\eta^{s+M+1}} \qquad (l \geqq L_2 \geqq L_1)$$

for $|z|=R_1+\eta$.

$$\frac{-\log(\eta^{s+M'})}{q\log q} \geqq \frac{-\log|\bar{h}_l^{(q)}(z)|}{q\log q} > \frac{-\log\left\{e^{(R_2+\eta)^{\nu+\epsilon}}\cdot(R_2+\eta)/\eta^{s+M+1}\right\}}{q\log q}.$$

and

$$\lim_{q=\infty}\frac{-\log|\bar{h}_l^{(q)}(z)|}{q\log q}=0 \quad (q=0,\ 1,\ 2,\ldots\ldots;\ l\geqq L_2)$$

uniformly for $|z|\leqq R_1+\eta$. Hence we have

XV. *For any positive value* $R'$ $(=R'+\eta)$ *we may assign a positive integer* $L$ *such that*

$$\lim_{q=\infty}\frac{-\log|\bar{h}_l^{(q)}(z)|}{q\log q}=0 \quad (q=0,\ 1,\ 2,\ldots\ldots;\ l\geqq L)$$

*uniformly for* $|z|\leqq R'$.

$$f_l^{(q)}(z)\equiv e^{Q_q(z)}\cdot h_l^{(q)}(z)\equiv P_l^{(q)}(z)\cdot e^{Q_q(z)}\cdot\bar{h}_l^{(q)}(z),$$

where $P_l^{(q)}(z)$ has the meaning as on p. 197. Then we have, by VI and XV,

XVI. $$\lim_{q=\infty}\frac{-\log|e^{Q_q(z)}\cdot\bar{h}_l^{(q)}(z)|}{q\log q}=\frac{1}{\mu} \quad (l\geqq L)$$

*uniformly for* $|z|\leqq R'$.

Since $P_l^{(q)}(z)$ $(q=0,\ 1,\ 2,\ldots\ldots;\ l\geqq L)$ has at most $M+1$ distinct roots, we may conclude, as in art. 1, that

XVII. *For any positive value* $R'$ $\left(\left(\frac{(M+1)}{A}\right)^{\frac{1}{\nu+\epsilon}}\leqq R'<\left(\frac{M}{A}\right)^{\frac{1}{\nu+\epsilon}}\right)$

*there corresponds a positive integer* $L$ *such that the points of less increase of* $P_l^{(q)}(z)$ $(q=0,\ 1,\ 2,\ldots\ldots;\ l$: *fixed* $\geqq L)$ *whose moduli are less than* $R'$, *are at most* $M+1$ *in number*[1].

Thus we may conclude as before

XVIII. *For any positive value* $R'$ $\left(\left(\frac{(M+1)}{A}\right)^{\frac{1}{\nu+\epsilon}}\leqq R'<\left(\frac{M}{A}\right)^{\frac{1}{\nu+\epsilon}}\right)$

*there corresponds a positive integer* $L$ *such that any* $F_l(z, z')$ $(l\geqq L)$ *is of apparent order* $\mu$ *in* $z'$ *for any* $z$ $(|z|<R')$ *except the points, at most* $M+1$ *in number, for which* $F_l(z, z')$ *is of apparent order lower than* $\mu$.

---

1 The theorem holds not only for $q=0,\ 1,\ 2,\ldots\ldots$, but also for $q=q_1,\ q_2,\ q_3,\ldots\ldots$

Since

$$\lim_{q=\infty} \frac{-\log|e^{Q_q(z)}\cdot \bar{h}_l^{(q)}(z)|}{q\log q}=\frac{1}{\mu} \qquad (l\geqq L)$$

uniformly for $|z|\leqq R'$, we may assign, as in art. 1, the values of $q$, $q_1$, $q_2,\ldots\ldots$, $q_m,\ldots\ldots$, which correspond to $\delta$ ($\delta>0$) and

$$\left|\frac{-\log|e^{Q_{q_m}(z)}\cdot \bar{h}_l^{(q_m)}(z)|}{q_m\log q_m}-\frac{1}{\mu}\right|\leqq\delta \quad (l\geqq L)$$

uniformly for $|z|\leqq R'$ (i. e. $q_m$ ($m=1, 2,\ldots\ldots$) is independent of $z$ ($|z|\leqq R'$)). Put

$$F_l(z, z')\equiv\sum_{m=1}^{\infty} f_l^{(q_m)}(z)\, z'^{q_m}+H(z, z') \qquad (l\geqq L).$$

Then $H(z, z')$ is of apparent order at most $\dfrac{\mu}{1+2\mu\delta}$ in $z'$ for any $z$ ($|z|\leqq R'$). Hence as the order of $F_l(z, z')$ in $z'$ is concerned, we have only to consider

$$\sum_{m=1}^{\infty} f_l^{(q_m)}(z)\, z'^{q_m} \qquad (l\geqq L).$$

Let $l_n$ ($L\leqq l_1<l_2<\ldots\ldots$), $b_{l_n}^{(q_m)}$[1], $b_{l_n}$, and $\beta$ have the same meaning as in art. 4, then corresponding to XII, we have

XIX. *If there be three positive values $G$, $K$ (however great) and $k\left(>\dfrac{1}{\mu}\right)$ such that to each $q_m$ ($q_m\geqq G$), there corresponds a value of $l_n$, and we have*

$$\left|b_{l_n}^{(q_m)}-b_{l_n}\right|\cdot K^{q_m}\geqq\eta_n,$$

$$\frac{-\log|f_{l_n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}\geqq k,$$

*then $F(z, z')$ will be of apparent order lower than $\mu$ in $z'$ for $z=\beta=\lim\limits_{l_n=\infty} b_{l_n}$.*

We may assume, without loss of generality, that $\delta<k-\dfrac{1}{\mu}$.

For otherwise, we take a sub-sequence of $\{q_m\}$, which corresponds to $\delta\left(<k-\dfrac{1}{\mu}\right)$.

---

1 $b_{l_n}^{(q_m)}$ is one of the zeropoints of $f_{l_n}^{(q_m)}(z)$ which are at the shortest distance from the point $b_{l_n}$.

$$\frac{-\log|f_{l_n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}=\frac{-\log|P_{l_n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}+\frac{-\log|e^{Q_{q_m}^{(b_{l_n})}}\cdot \bar{h}_{l_n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}.$$

$$\frac{-\log|e^{Q_{q_m}^{(b_{l_n})}}\cdot \bar{h}_{n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}\leqq\frac{1}{\mu}+\delta,$$

for $|b_{l_n}|\leqq R'$, where $R'>|\beta|$. Hence we have

$$\frac{-\log|P_{l_n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}\geqq k-\left(\frac{1}{\mu}+\delta\right).$$

Since $P_{l_n}^{(q_m)}(z)$ has at most $M+s$ roots $\left(R'<\frac{M^{\frac{1}{\nu+\epsilon}}}{A}\right)$, we have

$$\frac{-\log|b_{l_n}^{(q_m)}-b_{l_n}|^{M+s}}{q_m\log q_m}\geqq\frac{-\log|P_{l_n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}.$$

We have therefore

$$\frac{-\log|b_{l_n}^{(q_m)}-b_{l_n}|}{q_m\log q_m}\geqq\frac{k-\frac{1}{\mu}-\delta}{M+s},$$

or

$$|b_{l_n}^{(q_m)}-b_{l_n}|\leqq e^{-\frac{k-\frac{1}{\mu}-\delta}{M+s}q_m\log q_m},$$

and by XIII

$$\varliminf_{q_m=\infty}\frac{-\log|P^{(q_m)}(\beta)|}{q_m\log q_m}\geqq\frac{k-\frac{1}{\mu}-\delta}{M+s},$$

which shows that $z=\beta$ is a point of less increase of $P^{(q_m)}(z)$ $(m=1,2,\ldots\ldots)$. The same reasoning as on p. 165 leads to the results.

As a special case of XIX, we have

XX. *If* $\varliminf_{q_m=\infty}\frac{-\log|f_{l_n}^{(q_m)}(b_{l_n})|}{q_m\log q_m}=k_{l_n}$ *uniformly for* $l_n$ $(n=1,2,\ldots\ldots)$ *and if*

$$\varliminf_{l_n=\infty}k_{l_n}=k>0,$$

*then* $F(z,z')$ *will be of apparent order lower than* $\mu$ *in* $z'$ *for* $z=\beta$.

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