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On the Abhyankar’s question for affine plane curves with one place at infinity

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1 Introduction

Let $C$ be an irreducible algebraic curve in complex affine plane $\mathbb{C}^2$. We say that $C$ has one place at infinity, if the closure of $C$ intersects with the $\infty$-line in $\mathbb{P}^2$ at only one point $P$ and $C$ is locally irreducible at that point $P$.

The problem of finding the canonical models of curves with one place at infinity under the polynomial transformations of the coordinates of $\mathbb{C}^2$ has been studied by many mathematicians since Suzuki [10] and Abhyankar–Moh [2] proved independently that the canonical model of $C$ is a line when $C$ is non-singular and simply connected.

Sathaye [8] introduce the Abhyankar’s question for curves with one place at infinity and Sathaye–Stenerson [9] suggested a candidate of counter example for this question. However, they could not give the answer to the question since the root computation for a huge polynomial system was required.

We found a counter example for the Abhyankar’s question using computer algebra system. In this report, we give the details.
2 Preliminaries

Let $C$ be a curve with one place at infinity defined by a polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbb{C}^2$. Assume that $\deg_x f = m$, $\deg_y f = n$ and $d = \gcd(m, n)$. The dual graph corresponding to the minimal resolution of the singularity of $C$ at infinity is the following [11]:

![Dual Graph](image)

**DEFINITION 1** ($\delta$-sequence) Let $f$ be the defining polynomial of a curve $C$ with one place at infinity. Let $\delta_k$ ($0 \leq k \leq h$) be the order of the pole of $f$ on $E_{j_k}$ in the above dual graph. We shall call the sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$ the $\delta$-sequence of $C$ (or of $f$).

We have the following fact since $\deg_x f = m$ and $\deg_y f = n$.

**Fact 1** $\delta_0 = n$, $\delta_1 = m$

We set $L_k$ for each $k$ ($1 \leq k \leq h$) like the following figure:

![Diagram with $L_k$](image)

**DEFINITION 2** ($(p, q)$-sequence) Now, we assume that the weights of $L_k$ is of the following form:
We define the natural numbers \( p_k, a_k, q_k, b_k \) satisfying

\[
(p_k, a_k) = 1, \ (q_k, b_k) = 1, \ 0 < a_k < p_k, \ 0 < b_k < q_k,
\]

\[
\frac{p_k}{a_k} = m_1 - \frac{1}{m_2} - \frac{1}{m_3} - \ldots - \frac{1}{m_r} \quad \text{and} \quad \frac{q_k}{b_k} = n_1 - \frac{1}{n_2} - \frac{1}{n_3} - \ldots - \frac{1}{n_s}.
\]

We shall call the sequence \( \{(p_1, q_1), (p_2, q_2), \ldots, (p_h, q_h)\} \) the \((p, q)\)-sequence of \( C \) (or of \( f \)).

There are the following Abhyankar–Moh’s semigroup theorem and its converse theorem by Sathaye–Stenerson as results for \( \delta \)-sequence. We set \( \mathbb{N} = \{ n \in \mathbb{Z} \mid n \geq 0 \} \) and \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \).

**Theorem 1 (Abhyankar-Moh [1, 3, 4])** Let \( C \) be an affine plane curve with one place at infinity. Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) be the \( \delta \)-sequence of \( C \) and \( \{(p_1, q_1), \ldots, (p_h, q_h)\} \) be the \((p, q)\)-sequence of \( C \). We set \( d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\} \) \((1 \leq k \leq h+1)\). We have then,

(i) \( q_k = d_k/d_{k+1}, \) \( d_{h+1} = 1 \) \((1 \leq k \leq h)\),

(ii) \( d_{k+1}p_k = \begin{cases} \delta_1 & (k = 1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \leq k \leq h) \end{cases} \)

(iii) \( q_kd_k \in \mathbb{N}\delta_0 + \mathbb{N}\delta_1 + \cdots + \mathbb{N}\delta_{k-1} \) \((1 \leq k \leq h)\).

**Theorem 2 (Sathaye–Stenerson [9])** Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) \((h \geq 1)\) be the sequence of \( h+1 \) natural numbers. We set \( d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\} \) \((1 \leq k \leq h+1)\) and \( q_k = d_k/d_{k+1} \) \((1 \leq k \leq h)\). Furthermore, suppose that the following conditions are satisfied:
(1) \( \delta_0 < \delta_1 \),
(2) \( q_k \geq 2 \) \((1 \leq k \leq h)\),
(3) \( d_{h+1} = 1 \),
(4) \( \delta_k < q_{k-1} \delta_{k-1} \) \((2 \leq k \leq h)\),
(5) \( q_k \delta_k \in \mathbb{N} \delta_0 + \mathbb{N} \delta_1 + \cdots + \mathbb{N} \delta_{k-1} \) \((1 \leq k \leq h)\).

Then, there exists a curve with one place at infinity of the \( \delta \)-sequence \( \{\delta_0, \delta_1, \ldots, \delta_h\} \).

Suzuki [11] gave an algebraic-geometric proof of the above two theorems by the consideration of the resolution graph at infinity. Further, Suzuki gave an algorithm for mutual conversion of a dual graph and a \( \delta \)-sequence.

### 3 Construction of defining polynomials of curves

We shall assume that \( f(x, y) \) is monic in \( y \). We define approximate roots by Abhyankar’s definition.

**DEFINITION 3 (approximate roots)** Let \( f(x, y) \) be the defining polynomial, monic in \( y \), of a curve with one place at infinity. Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) be the \( \delta \)-sequence of \( f \). We set \( n = \deg_y f \), \( d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\} \) and \( n_k = n/d_k \) \((1 \leq k \leq h+1)\). Then, for each \( k \) \((1 \leq k \leq h+1)\), a pair of polynomials \((g_k(x, y), \psi_k(x, y))\) satisfying the following conditions is uniquely determined:

- (i) \( g_k \) is monic in \( y \) and \( \deg_y g_k = n_k \),
- (ii) \( \deg_y \psi_k < n - n_k \),
- (iii) \( f = g_k^{d_k} + \psi_k \).

We call this \( g_k \) the \( k \)-th approximate root of \( f \).

We can easily get the following fact from the definition of approximate roots.

**Fact 2** We have

\[
\begin{align*}
  g_1 &= y + \sum_{j=0}^{\lfloor p/q \rfloor} c_k x^k, \\
  g_{h+1} &= f
\end{align*}
\]

where \( c_k \in \mathbb{C} \), \( p = \deg_x f/d \), \( q = \deg_y f/d \), \( d = \gcd\{\deg_x f, \deg_y f\} \) and \( \lfloor p/q \rfloor \) is the maximal integer \( \ell \) such that \( \ell \leq p/q \).
**DEFINITION 4** (Abhyankar-Moh’s condition) We shall call the conditions (1) – (5) concerning \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) in Theorem 2 Abhyankar-Moh’s condition.

The following theorem gives normal forms of defining polynomials of curves with one place at infinity and the method of construction of their defining polynomials.

**Theorem 3** ([5]) Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} (h \geq 1) \) be a sequence of natural numbers satisfying Abhyankar-Moh’s condition (see **DEFINITION 4**). Set \( d_k = \gcd \{\delta_0, \delta_1, \ldots, \delta_{k-1}\} (1 \leq k \leq h + 1) \) and \( q_k = d_k/d_{k+1} (1 \leq k \leq h) \).

1. We define \( g_k (0 \leq k \leq h + 1) \) as follows:

\[
\begin{align*}
g_0 &= x, \\
g_1 &= y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbb{C}, \; p = \delta_1/d_2, \; q = \delta_0/d_2, \\
g_{i+1} &= g_i^q + a_{\alpha_0\alpha_1\cdots\alpha_{i-1}} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_{i-1}^{\alpha_{i-1}} \\
&\quad + \sum_{(a_0, a_1, \ldots, a_i) \in \Lambda_i} c_{a_0a_1\cdots a_i} g_0^{a_0} g_1^{a_1} \cdots g_i^{a_i},
\end{align*}
\]

where \((\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1})\) is the sequence of \( i \) non-negative integers satisfying

\[
\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i, \quad \bar{\alpha}_j < q_j (0 < j < i)
\]

and

\[
\Lambda_i = \left\{ (\alpha_0, \alpha_1, \ldots, \alpha_i) \in \mathbb{N}^{i+1} \mid \alpha_j < q_j (0 < j < i), \alpha_i < q_i - 1, \sum_{j=0}^{i} \alpha_j \delta_j < q_i \delta_i \right\}.
\]

Then, \( g_0, g_1, \ldots, g_h \) are approximate roots of \( f(=g_{h+1}) \), and \( f \) is the defining polynomial, monic in \( y \), of a curve with one place at infinity of the \( \delta \)-sequence \( \{\delta_0, \delta_1, \ldots, \delta_h\} \).

2. The defining polynomial \( f \), monic in \( y \), of a curve with one place at infinity of the \( \delta \)-sequence \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) is obtained by the procedure of (1), and the values of parameters \( \{a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}}\}_{1 \leq i \leq h} \) and \( \{c_{a_0a_1\cdots a_i}\}_{0 \leq i \leq h} \) are uniquely determined for \( f \).
4 Abhyankar's Question

DEFINITION 5 (planar semigroup) Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} (h \geq 1) \) be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) is said to be a planar semigroup.

DEFINITION 6 (polynomial curve) Let \( C \) be an algebraic curve defined by \( f(x, y) = 0 \), where \( f(x, y) \) is an irreducible polynomial in \( \mathbb{C}[x, y] \). We call \( C \) a polynomial curve, if \( C \) has a parametrisation \( x = x(t), y = y(t) \), where \( x(t) \) and \( y(t) \) are polynomials in \( \mathbb{C}[t] \).

Abhyankar's Question: Let \( \Omega \) be a planar semigroup. Is there a polynomial curve with \( \delta \)-sequence generating \( \Omega \) ?

Moh [6] showed that there is no polynomial curve with \( \delta \)-sequence \( \{6, 8, 3\} \). But there is a polynomial curve \((x, y) = (t^3, t^8)\) with \( \delta \)-sequence \( \{3, 8\} \) which generates the same semigroup as above. Sathaye–Stenerson [9] proved that the semigroup generated by \( \{6, 22, 17\} \) has no other \( \delta \)-sequence generating the same semigroup, and proposed the following conjecture for this question.

Sathaye–Stenerson's Conjecture: There is no polynomial curve having the \( \delta \)-sequence \( \{6, 22, 17\} \).

By Theorem 3, the defining polynomial of the curve with one place at infinity of the \( \delta \)-sequence \( \{6, 22, 17\} \) as follows:
\[
f = (g_2^2 + a_{2,1}x^2g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 \\
+ c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0}
\]
where
\[
g_1 = y + c_3x^3 + c_2x^2 + c_1x + c_0, \\
g_2 = (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\
+ (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\
+ (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}.
\]

Since \( C \) has one place at infinity and genus zero if and only if \( C \) has polynomial parametrization (Abhyankar), \( \{6, 22, 17\} \) is a counter example if it can be shown that the above type curve does not include a polynomial curve.
5 Approach by using a computer algebra system

We assume that \( C \) is a polynomial curve and has the \( \delta \)-sequence \( \{6, 22, 17\} \). Therefore \( C \) has the following polynomial parametrization:

\[
\begin{align*}
    x &= t^6 + a_1 t^5 + a_2 t^4 + a_3 t^3 + a_4 t^2 + a_5 t + a_6 \\
    y &= t^{22} + b_1 t^{21} + b_2 t^{20} + b_3 t^{19} + \cdots + b_{21} t + b_{22}
\end{align*}
\]

It follows that \( \deg g_2(x(t), y(t)) = 17 \) from the form of \( f \) and \( g_2 \) in the previous section. We can get the polynomial system \( I \) with 11 variables and 17 polynomials after eliminating variables from the coefficients of all terms of \( t \)-degree more than 18 in \( g_2(x(t), y(t)) \).

\( \{6, 22, 17\} \) is a counter example of Abhyankar’s question if \( I \) does not have a root. For such a huge polynomial system it is suitable to compute the Gröbner basis of the ideal. However, it was impossible to compute the Gröbner basis of \( I \) even if using a computer with 8GB memory.

We classified \( \delta \)-sequences with genus \( \leq 50 \) into groups which generate the same semigroup. Furthermore, we listed \( \delta \)-sequences with the following three properties: (i) There is no other \( \delta \)-sequence which generates the same semigroup. (ii) The number of generators is 3. (iii) \( k \)-number \( \geq -1 \). Then, we obtained \( \{6, 15, 4\}, \{4, 14, 9\}, \{6, 15, 7\}, \{6, 21, 4\}, \cdots \). The Gröbner basis computations for the polynomial systems corresponding to these \( \delta \)-sequences showed that \( \{6, 21, 4\} \) was a counter example of Abhyankar’s question.

The defining polynomial of the curve with one place at infinity of the \( \delta \)-sequence \( \{6, 21, 4\} \) as follows:

\[
f = g_2^3 + a_{2,0}x^2 + c_{1,0,1}xg_2 + c_{1,0,0}x + c_{0,0,1}g_2 + c_{0,0,0}
\]

where

\[
\begin{align*}
    g_2 &= g_1^2 + a_7 x^7 + c_{6,0} x^6 + c_{5,0} x^5 + c_{4,0} x^4 + c_{3,0} x^3 \\
         &\quad + c_{2,0} x^2 + c_{1,0} x + c_{0,0} \\
    g_1 &= y + c_3 x^3 + c_2 x^2 + c_1 x + c_0
\end{align*}
\]

Let the following be the polynomial parametrization of the polynomial curve with \( \delta \)-sequence \( \{6, 21, 4\} \):

\[
\begin{align*}
    x &= t^6 + a_1 t^5 + a_2 t^4 + a_3 t^3 + a_4 t^2 + a_5 t + a_6 \\
    y &= t^{21} + b_1 t^{20} + b_2 t^{19} + b_3 t^{18} + \cdots + b_{20} t + b_{21}
\end{align*}
\]
By the same operation as the case of \{6, 22, 17\} we can get the polynomial system $J$ with 7 variables $\{a_2, a_3, a_4, a_5, a_6, b_{12}, b_{18}\}$ and 13 polynomials from $\deg_t g_2(x(t), y(t)) = 4$.

We used the total degree reverse lexicographic ordering (DRL) with $a_2 > a_3 > a_4 > a_5 > a_6 > b_{12} > b_{18}$ to the Gröbner basis computation. CPU time for the computation is 3 hours 40 minutes and the required memory is 850MB. The computer is a PC AthlonMP 2200+ with 4GB memory. The computer algebra system is Risa/Asir [7] on FreeBSD 4.7.

The obtained Gröbner basis $G$ of $J$ was not \{1\}. However, the normal form of the coefficient $p$ of the term with $t$-degree $= 4$ in $g_2(x(t), y(t))$ with respect to $G$ is 0. This shows that $p \in J$. Thus, we get $\deg_t g_2(x(t), y(t)) < 4$. Since this is contradictory for $\deg_t g_2(x(t), y(t)) = 4$, there is no polynomial curve with $\delta$-sequence \{6, 21, 4\}.

References


