Absolute Differential Calculus and its Applications to Projective Differential Geometry of Hypersurfaces in four Dimensional Space.

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ABSTRACT

In this paper, generalize the absolute differential calculus due to M. René Lagrange and apply it to projective differential geometry of hypersurfaces in the four dimensional space.

Non-developable surfaces in the three dimensional space can be classified projectively as follows.

(a) Ruled surfaces.

1 Surfaces of the second degree for which Darboux curves are indeterminate.

2 Ruled surfaces having only one family of generating lines upon which there is only one family of Darboux curves.

(b) Curved surfaces upon which there are three distinct familes Darboux curves.

From a similar point of view I classify hypersurfaces in the four dimensional space into eleven classes and investigate the properties of lhe hypersurfaces in each class.

CHAPTER I

FUNDAMENTAL QUANTITIES

1 Consider a hypersurface in a $n+1$ dimensional space defined by the equations

$$x_i = x_i(u_1, u_2, \ldots\ldots\ldots, u_n),$$
$$(i = 0, 1, \ldots\ldots\ldots, n+1),$$

1 Ann. Toulouse. **14**, 1 (1922).

where the x's are analytic functions of u's subject to the condition that the rank of the matrix

$$\left\| \begin{matrix} x_0 & \frac{\partial x_0}{\partial u_1} & \cdots\cdots & \frac{\partial x_0}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n+1} & \frac{\partial x_{n+1}}{\partial u_1} & \cdots\cdots & \frac{\partial x_{n+1}}{\partial u_n} \end{matrix} \right\|$$

is $n+1$.

Put

$$(1) \quad h_{ij} = \left| \begin{matrix} x_0 & \frac{\partial x_0}{\partial u_1} & \cdots\cdots & \frac{\partial x_0}{\partial u_n} & \frac{\partial^2 x_0}{\partial u_i \partial u_j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n+1} & \frac{\partial x_{n+1}}{\partial u_1} & \cdots\cdots & \frac{\partial x_{n+1}}{\partial u_n} & \frac{\partial^2 x_{n+1}}{\partial u_i \partial u_j} \end{matrix} \right|,$$

$$(2) \quad h = | h_{ij} |.$$

Hereafter, we shall denote such a determinant as that on the right side of (1) by

$$\left| x \ \frac{\partial x}{\partial u_1} \cdots\cdots \frac{\partial x}{\partial u_n} \ \frac{\partial^2 x}{\partial u_i \partial u_j} \right|.$$

Let X_i' ($i=0, 1, \ldots\ldots, n+1$) be the cofactors of the elements in the last column in the determinant h_{ij}. Then they are the hyperplane coordinates of the hyperplane tangent to the hypersurface at the point x and we have

$$(xX') = x_0 X_0' + \cdots\cdots + x_{n+1} X'_{n+1} = 0,$$

$$\left(\frac{\partial x}{\partial u_i} X' \right) = 0,$$

$$\left(\frac{\partial^2 x}{\partial u_i \partial u_j} X' \right) = h_{ij}$$

and accordingly,

$$\left(x \ \frac{\partial X'}{\partial u_i} \right) = 0,$$

$$-\left(\frac{\partial x}{\partial u_i} \ \frac{\partial X'}{\partial u_j} \right) = -\left(\frac{\partial x}{\partial u_j} \ \frac{\partial X'}{\partial u_i} \right) = \left(x \ \frac{\partial^2 X'}{\partial u_i \partial u_j} \right) = h_{ij}.$$

By the assumption at least one of $X_0', \ldots\ldots, X'_{n+1}$ does not vanish identically. Assume $X_0' \neq 0$. Then we have from the above equations

$$\begin{vmatrix} x_1 & \frac{\partial x_1}{\partial u_1} & \cdots\cdots & \frac{\partial x_1}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n+1} & \frac{\partial x_{n+1}}{\partial u_1} & \cdots\cdots & \frac{\partial x_{n+1}}{\partial u_n} \end{vmatrix} \begin{vmatrix} X_1' & \frac{\partial X_1'}{\partial u_1} & \cdots\cdots & \frac{\partial X_1'}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ X'_{n+1} & \frac{\partial X'_{n+1}}{\partial u_1} & \cdots\cdots & \frac{\partial X'_{n+1}}{\partial u_n} \end{vmatrix}$$

$$= (-1)^{n+1} x_0 X_0' h$$

But the function x_0 does not vanish identically, unless the given hypersurface is a hyperplane.

Therefore, we know that the necessary and sufficient condition that the manifoldness of the tangent hyperplanes of the hypersurface may be n is that $h \neq 0$.

we shall assume that the manifoldness of the tangent hyperplanes of the given hypersurface is n and that h is different from zero in the domain of u's which we consider.

2 Put

$$X_i = \frac{X_i'}{h^{\frac{1}{n+2}}}, \quad H_{ij} = \frac{h_{ij}}{h^{\frac{1}{n+2}}}.$$

Then

$$H = |\, H_{ij} \,| = h^{\frac{2}{n+2}},$$

$$(3) \quad (xX) = \left(\frac{\partial x}{\partial u_i} X\right) = \left(x \frac{\partial X}{\partial u_i}\right) = 0,$$

$$(4) \quad \left(\frac{\partial^2 x}{\partial u_i \partial u_j} X\right) = -\left(\frac{\partial x}{\partial u_i} \frac{\partial X}{\partial u_j}\right) = -\left(\frac{\partial x}{\partial u_j} \frac{\partial X}{\partial u_i}\right) = \left(x \frac{\partial^2 X}{\partial u_i \partial u_j}\right) = H_{ij}.$$

In virtue of the equations (3), the ratios of the cofactors of the elements of the last column in the determinant

$$\left| X \frac{\partial X}{\partial u_1} \cdots\cdots \frac{\partial X}{\partial u_n} \; \frac{\partial^2 X}{\partial u_i \partial u_j} \right|$$

are equal to $x_0 : x_1 : \cdots\cdots : x_{n+1}$.

But we have

$$\begin{vmatrix} x_1 & \frac{\partial x_1}{\partial u_1} & \cdots\cdots & \frac{\partial x_1}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n+1} & \frac{\partial x_{n+1}}{\partial u_1} & \cdots\cdots & \frac{\partial x_{n+1}}{\partial u_n} \end{vmatrix} \begin{vmatrix} X_1 & \frac{\partial X_1}{\partial u_1} & \cdots\cdots & \frac{\partial X_1}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ X_{n+1} & \frac{\partial X_{n+1}}{\partial u_1} & \cdots\cdots & \frac{\partial X_{n+1}}{\partial u_n} \end{vmatrix}$$

$$= (-1)^{n+1} x_0 X_0 H$$

Therefore, the said cofactors are equal to

$$(-1)^{n+1}x_i \sqrt{H} \quad (i=0, 1, \ldots\ldots, n+1)$$

respectively, and accordingly

$$(5)\quad \frac{1}{\sqrt{H}} \left| X \frac{\partial X}{\partial u_1} \cdots\cdots \frac{\partial X}{\partial u_n} \frac{\partial^2 X}{\partial u_i \partial u_j} \right|$$
$$=(-1)^{n+1}\left(x \frac{\partial^2 X}{\partial u_i \partial u_j}\right)=(-1)^{n+1} H_{ij}.$$

Put

$$(6)\quad \varphi = \frac{1}{\sqrt{H}} \left| x \frac{\partial x}{\partial u_1} \cdots\cdots \frac{\partial x}{\partial u_n} d^2x \right|$$
$$=\sum_{\lambda,\mu=1}^{n} H_{\lambda\mu}\, du_\lambda\, du_\mu,$$

$$(7)\quad \psi = \frac{3}{2} d\left\{\frac{1}{\sqrt{H}} \left| x \frac{\partial x}{\partial u_1} \cdots\cdots \frac{\partial x}{\partial u_n} d^2x \right|\right\}$$
$$-\frac{1}{\sqrt{H}} \left| x \frac{\partial x}{\partial u_1} \cdots\cdots \frac{\partial x}{\partial u_n} d^3 x \right|$$
$$=\sum_{\lambda,\mu,\nu=1}^{n} K_{\lambda\mu\nu}\, du_\lambda\, du_\mu\, du_\nu, \qquad (K_{ijl}=K_{jil}=K_{jli})$$

Hereafter, we shall omit the symbol of the summatitn Σ and denote the indices with respect to which the summation shall be made from 1 to n by greek letters $\alpha, \beta, \gamma, \lambda, \mu, \nu, \sigma, \tau, \rho$, etc.

From the equations

$$(d^2x\, X)=-(dx\, dX)=(x\, d^2X)=\varphi$$

we have

$$(d^3x\ X) + (d^2x\ dX)$$
$$=-(d^2x\ dX) - (dx\ d^2X)$$
$$=(dx\ d^2X) + (x\ d^3X) = d\varphi$$

and accordingly

$$(d^2x\ dX) = \psi - \tfrac{1}{2}\, d\varphi,$$
$$(dx\ d^2X) = -\psi - \tfrac{1}{2}\, d\varphi,$$
$$(x\ d^3X) = \psi + \tfrac{3}{2}\, d\varphi.$$

From the last equation we have

$$(8)\quad \frac{3}{2} d\left\{\frac{1}{\sqrt{H}} \left| X \frac{\partial X}{\partial u_1} \cdots\cdots \frac{\partial X}{\partial u_n} d^2X \right|\right\}$$
$$-\frac{1}{\sqrt{H}} \left| X \frac{\partial X}{\partial u_1} \cdots\cdots \frac{\partial X}{\partial u_n} d^3X \right|$$
$$= (-1)^n \psi.$$

CHAPTER II

COVARIANT DERIVATIVES AND CONTRAVARIANT DERIVATIVES.

3 Let

$$dw_i = a_1^i\, du_1 + \ldots\ldots + a_n^i\, du_n \quad (i=1,\ldots\ldots, n)$$

be n linearly independent Pfaffian expressions and denote by b_j^i the cofactor of a_j^i in the determinant $|a_j^i|$ divided by the value of this determinant. Then we have

(1) $a_\sigma^i\, b_\sigma^j = a_i^\sigma\, b_j^\sigma = \varepsilon_{ij}$, ($\varepsilon_{ii}=1$, $\varepsilon_{ij}=0$, if $i \neq j$)

(2) $du_j = b_j^\sigma\, dw_\sigma$.

If $f(u_1,\ldots\ldots, u_n)$ be a function of u's, we have

$$df = \frac{\partial f}{\partial u_\sigma} du_\sigma = \frac{\partial f}{\partial u_\sigma} b_\sigma^\lambda\, dw_\lambda .$$

If we put

(3) $$\frac{\partial f}{\partial w_i} = \frac{\partial f}{\partial u_\sigma} b_\sigma^i,$$

we have

$$df = \frac{\partial f}{\partial w_\lambda} dw_\lambda,$$

(4) $$\frac{\partial f}{\partial u_j} = \frac{\partial f}{\partial w_\sigma} a_j^\sigma.$$

Any Pfaffian expression may be put in the form

$$d\Omega = \lambda_1\, dw_1 + \ldots\ldots\ldots\ldots + \lambda_n\, dw_n.$$

We shall also denote the coefficient λ_i of dw_i in this expression by $\dfrac{\partial \Omega}{\partial w_i}$.

Then we have

$$d\Omega = \frac{\partial \Omega}{\partial w_\sigma} dw_\sigma.$$

If we put

(5) $g_{ij} = H_{\sigma\tau}\, b_\sigma^i\, b_\tau^j$,

(6) $k_{ijl} = K_{\sigma\tau\rho}\, b_\sigma^i\, b_\tau^j\, b_\rho^l$,

we have

$$\varphi = g_{\lambda\mu}\, dw_\lambda\, dw_\mu,$$

$$\psi = k_{\lambda\mu\nu}\, dw_\lambda\, dw_\mu\, dw_\nu,$$

$$g=|g_{ij}|=Hb^2,$$

where

$$b=|b^i_j|,$$

and accordingly

(7) $\sqrt{g}=b\sqrt{H}$,

if we choose the sign of $\sqrt{g}$ properly.

4 We have

$$(\delta,\ d)\ w_i=\delta d w_i-d\delta w_i$$
$$=\left(\frac{\partial a^i_\sigma}{\partial u_\tau}-\frac{\partial a^i_\tau}{\partial u_\sigma}\right)du_\sigma\ \partial u_\tau$$
$$=\left(\frac{\partial a^i_\sigma}{\partial u_\tau}-\frac{\partial a^i_\tau}{\partial u_\sigma}\right)b^\lambda_\sigma\ b^\mu_\tau\ dw_\lambda\ \delta w_\mu.$$

If we put

(8) $a_{jki}=b^j_\sigma\ b^k_\tau\left(\frac{\partial a^i_\sigma}{\partial u_\tau}-\frac{\partial a^i_\tau}{\partial u_\sigma}\right),$

we have

$$(\delta,\ d)w_i=a_{\sigma\tau i}\ dw_\sigma\ \delta w_\tau.$$

In virtue of (1) and (3), a_{jki} may be reduced to the form

(9) $a_{jki}=a^i_\sigma\left(\frac{\partial b^k_\sigma}{\partial w_j}-\frac{\partial b^j_\sigma}{\partial w_k}\right).$

From (9) we have

(10) $a_{jki}+a_{kji}=0.$

Let us put

$$\begin{bmatrix} i\ j \\ k \end{bmatrix}=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial w_i}+\frac{\partial g_{ik}}{\partial w_j}-\frac{\partial g_{ij}}{\partial w_k}-g_{i\lambda}\ a_{jk\lambda}-g_{j\lambda}\ a_{ik\lambda}-g_{k\lambda}\ a_{ij\lambda}\right)$$

From (10) and (11) we have

(12) $\begin{bmatrix} i\ j \\ k \end{bmatrix}-\begin{bmatrix} j\ i \\ k \end{bmatrix}=-g_{k\lambda}\ a_{ij\lambda},$

(13) $\begin{bmatrix} i\ k \\ j \end{bmatrix}-\begin{bmatrix} j\ k \\ i \end{bmatrix}=\frac{\partial g_{ij}}{\partial w_k}.$

Denote by g^{ij} the cofactor of g_{ij} in the determinant $|g_{ij}|$ divided by g and put

(14) $\left\{\begin{matrix} i\ j \\ l \end{matrix}\right\}=g^{l\sigma}\begin{bmatrix} i\ j \\ \sigma \end{bmatrix}.$

Then we have

(15) $\begin{bmatrix} i\ j \\ k \end{bmatrix} = g_{k\sigma} \begin{Bmatrix} i\ j \\ \sigma \end{Bmatrix},$

(16) $\begin{Bmatrix} i\ j \\ l \end{Bmatrix} - \begin{Bmatrix} j\ i \\ l \end{Bmatrix} = -a_{ijl},$

(17) $\dfrac{\partial \log \sqrt{g}}{\partial w_i} = \begin{Bmatrix} \sigma\ i \\ \sigma \end{Bmatrix},$

(18) $\dfrac{\partial g_{ij}}{\partial w_k} = -g^{i\sigma} \begin{Bmatrix} \sigma\ k \\ j \end{Bmatrix} - g^{j\sigma} \begin{Bmatrix} \sigma\ k \\ i \end{Bmatrix}.$

5 Now we shall consider the transformation of the Pfaffian expressions to which we refer.

Let

$$dw_i' = p_1^i\, dw_1 + \ldots\ldots + p_n^i\, dw_n \quad (=1, \ldots\ldots, n)$$

be n linearly independent Pfaffian expressions and denote by q_j^i the cofactor of p_j^i in the determinant $|q_j^i|$ divided by the value of this determinant. Then we have

$$dw_j = q_j^\sigma\, dw_\sigma',$$

$$\frac{\partial f}{\partial w_i'} = \frac{\partial f}{\partial w_\sigma} q_\sigma^i,$$

$$g'_{ij} = g_{\sigma\tau}\, q_\sigma^i\, q_\tau^j,$$

(21) $k'_{ijl} = k_{\sigma\tau\rho}\, q_\sigma^i\, q_\tau^j\, q_\rho^l,$

and accordingly

(22) $\dfrac{\partial f}{\partial w_j} = \dfrac{\partial f}{\partial w'_\lambda}\, p_j^\lambda,$

(23) $g_{lm} = g'_{\sigma\tau}\, p_l^\sigma\, p_m^\tau,$

(24) $g'^{ij} = g^{\sigma\tau}\, p_\lambda^i\, p_\mu^j,$

(25) $g^{lm} = g'^{\sigma\tau}\, q_i^\sigma\, q_j^\tau.$

From the equation

$$\begin{aligned}(\delta, d) w_i &= a_{\sigma\tau i}\, dw_\sigma\, \delta w_\tau = a_{\sigma\tau i}\, q_\sigma^\lambda\, q_\tau^\mu\, dw'_\lambda\, \delta w'_\mu \\ &= \delta\left(q_i^\lambda\, dw'_\lambda\right) - d\left(q_i^\mu\, \delta w'_\mu\right) \\ &= \left(\frac{\partial q_i^\lambda}{\partial w'_\mu} - \frac{\partial q_i^\mu}{\partial w'_\lambda} + q_i^\nu\, a'_{\lambda\mu\nu}\right) dw'_\lambda\, \delta w'_\mu\end{aligned}$$

we have

(26) $$\frac{\partial q_i^j}{\partial w'_k} - \frac{\partial q_i^k}{\partial w_j'} + a'_{jk\nu}\, q_i^\nu = a_{\sigma\tau i}\, q_\sigma^j\, q_\tau^k.$$

In virtue of (26), we have

$$\begin{bmatrix} i\ j \\ k \end{bmatrix}' = \begin{bmatrix} \sigma\ \tau \\ \rho \end{bmatrix} q_\sigma^i\, q_\tau^j\, q_\rho^k + g_{\sigma\rho}\, q_\rho^k\, \frac{\partial q_\sigma^i}{\partial w_j'}$$

and accordingly

(27) $$\frac{\partial q_l^i}{\partial w_j'} + \begin{Bmatrix} \sigma\ \tau \\ l \end{Bmatrix} q_\sigma^i\, q_\tau^i = \begin{Bmatrix} i\ j \\ \lambda \end{Bmatrix}' q_l^\lambda.$$

From (27) we have

(28) $$\frac{\partial p_l^i}{\partial w_j} + \begin{Bmatrix} \sigma\ \tau \\ l \end{Bmatrix}' p_i^\sigma\, p_j^\tau = \begin{Bmatrix} i\ j \\ \rho \end{Bmatrix} p_\rho^l.$$

6 Consider a m-ple system of quantities (functions, Pfaffian expressions etc.)

$$X_{r_1\ldots\ldots r_m}, \qquad (r_1, \ldots\ldots, r_m = 1 \ldots\ldots n)$$

We shall call the expressions

$$\frac{\overline{\partial} X_{r_1\ldots\ldots r_m}}{\partial w_h} = \frac{\partial X_{r_1\ldots\ldots r_m}}{\partial w_h} - \begin{Bmatrix} r_\sigma\ h \\ \tau \end{Bmatrix} X_{r_1\ldots\ldots r_{\sigma-1}\,\tau\, r_{\sigma+1}\ldots\ldots r_m}$$

and

$$\overline{d} X_{r_1\ldots\ldots r_m} = \frac{\overline{\partial} X_{r_1\ldots\ldots r_m}}{\partial w_\lambda}\, dw_\lambda$$

$$= dX_{r_1\ldots\ldots r_m} - \begin{Bmatrix} r_\sigma\ \lambda \\ \tau \end{Bmatrix} X_{r_1\ldots\ldots r_{\sigma-1}\,\tau\, r_{\sigma+1}\ldots\ldots r_m}\, dw_\lambda$$

the covariant partial derivative and the differential of $X_{r_1\ldots\ldots r_m}$ respectively.

Next, let us put

$$\frac{\overline{\partial} \dot{X}_{r_1\ldots\ldots \dot{r}_m}}{\partial w_h} = \frac{\partial X_{r_1\ldots\ldots r_m}}{\partial w_h} + \begin{Bmatrix} \tau\ \lambda \\ r_\sigma \end{Bmatrix} X_{r\ldots\ldots r_{\sigma-1}\,\tau\, r_{\sigma+1}\ldots\ldots r_m}$$

and call the expressions

$$\frac{\overline{\partial} \dot{X}_{r_1\ldots\ldots \dot{r}_m}}{\partial \dot{w}_h} = g^{h\nu}\, \frac{\overline{\partial} \dot{X}_{r_1\ldots\ldots \dot{r}_m}}{\partial \dot{w}_h}$$

and

$$\overline{\partial} X_{r_1\ldots\ldots r_m} = g_{\lambda\mu}\, \frac{\overline{\partial} \dot{X}_{r_1\ldots\ldots \dot{r}_m}}{\partial \dot{w}_\lambda}\, dw_\mu = \frac{\overline{\partial} X_{r\ldots\ldots r_m}}{\partial w_\nu}\, dw_\nu$$

$$= dX_{r_1\ldots\ldots r_m} + \begin{Bmatrix} \tau\ \nu \\ r_\sigma \end{Bmatrix} X_{r_1\ldots\ldots r_{\sigma-1}\,\tau\, r_{\sigma+1}\ldots\ldots r_m}\, dw_\nu$$

the contravariant partial derivative and the differential of $X_{r_1\ldots\ldots r_m}$ respectively.

Finally, we shall call the expression

$$\overline{d}X_{r_1\ldots\ldots r_m\, \dot{s}\ldots\ldots \dot{s}_p} = dX_{r_1\ldots\ldots r_m\, s\ldots\ldots s_p}$$
$$-\left\{\begin{matrix} r_\sigma & \lambda \\ & \tau \end{matrix}\right\} X_{r_1\ldots\ldots r_{\sigma-1}\tau\, r_{\sigma+1}\ldots\ldots r_m\, s\ldots\ldots s_p}\, dw_\lambda$$
$$+\left\{\begin{matrix} \tau & \lambda \\ & s_\sigma \end{matrix}\right\} X_{r_1\ldots\ldots r_m\, s_1\ldots\ldots s_{\sigma-1}\,\tau\, s_{\sigma+1}\ldots\ldots s_p}\, dw_\lambda$$

the mixed differential of $X_{r_1\ldots\ldots r_m\, s_1\ldots\ldots p}$.

Then we have

$$\overline{d}g_{ij}=0,$$
$$\overline{d}g^{ij}=0,$$
$$\overline{\partial}(dw_i)-\overline{d}(\partial w_i)=0,$$
$$\frac{\overline{\partial}}{\partial w_j}\left(\frac{\overline{\partial f}}{\partial w_i}\right)-\frac{\overline{\partial}}{\partial w_i}\left(\frac{\partial f}{\partial w_j}\right)=0.$$

For covariant differentials the following formulas hold: —

1° If

$$Z_{r_1\ldots\ldots r} = X_{r_1\ldots\ldots r_m} + Y_{r_1\ldots\ldots r_m},$$

then

$$\overline{d}Z_{r_1\ldots\ldots r_m} = \overline{d}X_{r_1\ldots\ldots r_m} + \overline{d}Y_{r_1\ldots\ldots r_m}.$$

2° If

$$Z_{r_1\ldots\ldots m\, s_1\ldots\ldots s_p} = X_{r_1\ldots\ldots r_m}\, Y_{s_1\ldots\ldots s_p},$$

then

$$\overline{d}Z_{r_1\ldots\ldots r_m\, s_1\ldots\ldots s_p} = Y_{s_1\ldots\ldots s_p}\,\overline{d}X_{r_1\ldots\ldots r_m}$$
$$+X_{r_1\ldots\ldots m}\,\overline{d}Y_{s_1\ldots\ldots s_p}.$$

3° If

$$Z_{r_1\ldots\ldots r_m\, s_1\ldots\ldots p} = X_{r_1\ldots\ldots r_m\, \sigma_1\ldots\ldots \sigma_q}\, Y_{s\ldots\ldots s_p\, \sigma_1\ldots\ldots \sigma_q},$$

then

$$\overline{d}Z_{r_1\ldots\ldots r_m\, s_1\ldots\ldots s_p} = Y_{s_1\ldots\ldots s_p\, \sigma\ldots\ldots \sigma_q}\,\overline{d}X_{r_1\ldots\ldots r_m\, \sigma\ldots\ldots \sigma_p}$$
$$+X_{r\ldots\ldots r_m\, \sigma\ldots\ldots \sigma_p}\,\overline{d}Y_{s\ldots\ldots s_p\, \dot{\sigma}\ldots\ldots \dot{\sigma}_p}$$

Similar formulas also hold for covariant and mixed differe tials. As a special case of 3° if

$$Z=X_{\sigma_1\ldots\ldots \sigma_q}\, Y_{\sigma_1\ldots\ldots \sigma_q},$$

we have

$$dZ=Y_{\sigma_1\ldots\ldots \sigma_q}\,\overline{d}X_{\sigma_1\ldots\ldots \sigma_q}+X_{\sigma_1\ldots\ldots \sigma_q}\,\overline{d}Y_{\dot{\sigma}_1\ldots\ldots \dot{\sigma}_q}$$

If, by the transformation of Pfaffian expressions to which we refer, a m-uple system of quantities $X_{r\ldots\ldots r_m}$ is transformed to $X'_{r_1\ldots\ldots r_m}$ which is connected to the original system by

$$X'_{r_1\ldots\ldots r_m}=X_{\sigma_1\ldots\ldots\sigma_m}\,q^{r_1}_{\sigma_1}\ldots\ldots q^{r_m}_{\sigma_m}$$

we call it covariant and if a system $X^{r_1\ldots\ldots r_m}$ is transformed to

$$X'^{r_1\ldots\ldots r_m}=X^{\sigma_1\ldots\ldots\sigma_m}\,p^{r_1}_{\sigma_1}\ldots\ldots p^{r_m}_{\sigma_m}$$

we call it contravariant.

The system of covariant partial derivatives of elements of a m-uple covariant system is a $(m+1)$-uple covariant one and that of covariant differentials is a covariant m-uple one. The same thing can be said for contravariant partial derivatives and differentials of elements of contravariant system.

7 Now we shall introduce Riemann's symbol of four indices.

From the equations

$$\overline{d}X_{r_1\ldots\ldots r_m}=dX_{r_1\ldots\ldots r_m}-\left\{\begin{matrix}r_1 & \lambda\\ \tau & \end{matrix}\right\}X_{\tau r_2\ldots\ldots r_m}\,dw_\lambda$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-\left\{\begin{matrix}r_m & \lambda\\ \tau & \end{matrix}\right\}X_{r_1\ldots\ldots r_{m-1}\tau}\,dw_\lambda,$$

$$\overline{\delta}\,\overline{d}X_{r_1\ldots\ldots r_m}=\delta\left(dX_{r_1\ldots\ldots r_m}-\left\{\begin{matrix}r_\sigma & \lambda\\ \tau & \end{matrix}\right\}X_{r_1\ldots\ldots r_{\sigma-1}\tau\, r_{\sigma+1}\ldots\ldots r_m}dw_\lambda\right)$$
$$-\left\{\begin{matrix}r_\sigma & \mu\\ \beta & \end{matrix}\right\}dX_{r_1\ldots\ldots r_{\alpha-1}\beta\, r_{\alpha+1}\ldots\ldots r_m}\,\delta w_\mu$$
$$+\left\{\begin{matrix}r_1 & \mu\\ \beta & \end{matrix}\right\}\delta w_\mu\Bigg(\left\{\begin{matrix}\beta & \lambda\\ \tau & \end{matrix}\right\}X_{\tau r_2\ldots\ldots r_m}\,dw_\lambda$$
$$+\ldots\ldots\ldots\ldots$$
$$+\left\{\begin{matrix}r_m & \lambda\\ \tau & \end{matrix}\right\}X_{\beta r_2\ldots\ldots r_{m-1}\tau}\,dw_\lambda\Bigg)$$
$$+\ldots\ldots\ldots\ldots$$
$$+\left\{\begin{matrix}r_m & \mu\\ \beta & \end{matrix}\right\}\delta w_\mu\Bigg(\left\{\begin{matrix}r_1 & \lambda\\ \tau & \end{matrix}\right\}X_{\tau r_2\ldots\ldots r_{m-1}\beta}\,dw_\lambda$$
$$+\ldots\ldots\ldots\ldots$$
$$+\left\{\begin{matrix}\beta & \lambda\\ \tau & \end{matrix}\right\}X_{r_1\ldots\ldots r_{-1}\tau}\,dw_\lambda\Bigg)$$

we have

$$(33)\quad (\overline{\delta},\ \overline{d})\ X_{r_1\ldots\ldots r_m}=(\delta,\ d)\ X_{r_1\ldots\ldots r_m}$$
$$-\{r_\sigma\ \tau,\lambda\,\mu\}\,X_{r_1\ldots\ldots r_{\sigma-1}\tau\, r_{\sigma+1}\ldots\ldots r_m}\,dw_\lambda\,\delta w_\mu,$$

where

$$(34)\quad \{ik,\, lm\}=\frac{\partial}{\partial w_m}\begin{Bmatrix} i\ l \\ k \end{Bmatrix}-\frac{\partial}{\partial w_l}\begin{Bmatrix} i\ m \\ k \end{Bmatrix}+\begin{Bmatrix} i\ l \\ \rho \end{Bmatrix}\begin{Bmatrix} \rho\ m \\ k \end{Bmatrix}-\begin{Bmatrix} i\ m \\ \rho \end{Bmatrix}\begin{Bmatrix} \rho\ l \\ k \end{Bmatrix}$$
$$+a_{.m\lambda}\begin{Bmatrix} i\ \lambda \\ k \end{Bmatrix}.$$

Similarly we have

$$(35)\quad (\overline{\partial},\ \overline{d}\ X_{\dot{r}_1\ldots\ldots\dot{r}_m}=(\delta,\ d)\ X_{r_2\ldots\ldots r_m}$$
$$+\{\tau\ r_\sigma,\ \lambda\ \mu\}\ X_{r_1\ldots\ldots r_{\sigma-1}\tau\ \ r_{\sigma+1}\ldots\ldots r_m}\ \partial w_\mu.$$

The symbol $\{i\ k,\ l\ m\}$ may be written

$$(36)\quad \{i\ k,\ l\ m\}=\frac{\overline{\partial}}{\partial w_m}\begin{Bmatrix} \textcircled{i}\ l \\ \dot{k} \end{Bmatrix}-\frac{\overline{\partial}}{\partial w_l}\begin{Bmatrix} \textcircled{i}\ m \\ \dot{k} \end{Bmatrix}$$

where $\textcircled{i}$ denotes that the absolute derivation is not effected with respect to the index i.

In virtue of (15), (29) and the third formula for the covariant differential, we have

$$\frac{\overline{\partial}}{\partial w_m}\begin{bmatrix} i\ l \\ j \end{bmatrix}=\frac{\overline{\partial}}{\partial w_m}\left(g_{j\sigma}\begin{Bmatrix} i\ l \\ \sigma \end{Bmatrix}\right)=\frac{\overline{\partial}}{\partial w_m}\begin{Bmatrix} i\ l \\ \dot{\sigma} \end{Bmatrix}g_{j\sigma}.$$

Therefore, if we put

$$(37)\quad (i\ j,\ l\ m)=g_{j\sigma}\{i\ \sigma,\ l\ m\},$$

we have

$$(38)\quad (i j,\, l\ m)=\frac{\overline{\partial}}{\partial w_m}\begin{bmatrix} \textcircled{i}\ \ l \\ j \end{bmatrix}-\frac{\overline{\partial}}{\partial w_l}\begin{bmatrix} \textcircled{i}\ \ m \\ j \end{bmatrix}.$$

From (37) we have

$$\{i\ k,\ l\ m\}=g^{k\sigma}\ (i\ \sigma,\ l\,m)$$

From (38) we have

$$(39)\quad (i j,\, l\,m)=\frac{\partial}{\partial w_m}\begin{bmatrix} i\ l \\ j \end{bmatrix}-\frac{\partial}{\partial w_l}\begin{bmatrix} i\ m \\ j \end{bmatrix}$$
$$-g^{\sigma\tau}\left(\begin{bmatrix} j\ m \\ \tau \end{bmatrix}\begin{bmatrix} i\ l \\ \sigma \end{bmatrix}-\begin{bmatrix} j\ l \\ \tau \end{bmatrix}\begin{bmatrix} i\ m \\ \sigma \end{bmatrix}\right)+a_{lm\sigma}\begin{bmatrix} i\ \sigma \\ j \end{bmatrix}.$$

In virtue of (13) and (39), we have

$$(40)\quad (i\ j,\ l\,m)+(j\ i,\ l\ m)=\frac{\partial}{\partial w_m}\left(\frac{\partial g_{ij}}{\partial w_l}\right)-\frac{\partial}{\partial w_l}\left(\frac{\partial g_{ij}}{\partial w_m}\right)+a_{lm\sigma}\frac{\partial g_{ij}}{\partial w_\sigma}$$
$$=0.$$

From (38) we have

$$(41)\quad (ij,\ l\,m)+(ij,\ m\ l)=0.$$

Supposing that $n>2$, denote by d, δ and Δ the differentiations along three different parameter curves. Then we have

$$(\Delta,\ \delta)\, dw_k = (\delta,\ d)\, \Delta w_k = (d,\ \Delta)\, \delta w_k = 0,$$

and accordingly

$$(\overline{\Delta},\ \overline{\delta})\, dw_k^{\cdot} = \{\lambda\ k,\ \mu\ \nu\}\, dw_\lambda\, \delta w_\mu\, \Delta w_\nu,$$
$$(\overline{\delta},\ \overline{d})\, \Delta w_k^{\cdot} = \{\nu\ k,\ \lambda\ \mu\}\, dw_\lambda\, \delta w_\mu\, \Delta w_\nu,$$
$$(\overline{d},\ \overline{\Delta})\, \delta w_k^{\cdot} = \{\mu\ k,\ \nu\ \lambda\}\, dw_\lambda\, \delta w_\mu\, \Delta w_\nu.$$

But we have

$$(\overline{\Delta},\ \overline{\delta})\, dw_k^{\cdot} + (\overline{\delta},\ \overline{d})\, \Delta w_k^{\cdot} + (\overline{d},\ \overline{\Delta})\, \delta w_k^{\cdot}$$
$$= \overline{\Delta}\,[\overline{\delta}\,(dw_k^{\cdot}) - \overline{d}\,(\delta w_k^{\cdot})] + \overline{\delta}\,(\overline{d}(\Delta w_k^{\cdot}) - \overline{\Delta}(dw_k^{\cdot}))$$
$$+ \overline{d}\,(\overline{\Delta}(\delta w_k^{\cdot}) - \overline{\delta}\,(\Delta w_k^{\cdot})) = 0.$$

Therefore, we have

$$\{j k,\ l m\} + \{m k,\ j l\} + \{l k,\ m j\} = 0$$

and accordingly

(42) $(i j,\ l m) + (i m,\ j l) + (i l,\ m j) = 0.$

From (40), (41) and (42) we have

(43) $(l m,\ i j) = (i j,\ l m).$

CHAPTER III

FUNDAMENTAL EQUATIONS.

DEFINITION OF DARBOUX CURVES.

8 Let A be a point on the hypersurface and $A_1, \ldots\ldots, A_n, A_{n+1}$ be the points

$$\frac{\partial A}{\partial w_1},\ \ldots\ldots,\ \frac{\partial A}{\partial w_n},\quad \frac{1}{n}\, g^{\sigma\tau}\, \frac{\overline{\partial A_\sigma}}{\partial w_\tau}$$

respectively.[1] Then we have

$$(1)\qquad \frac{1}{\sqrt{g}} \left| A\ A_1 \ldots\ldots A_n\ \frac{\overline{\partial A_i}}{\partial w_j} \right| = g_{ij},$$

$$(2)\qquad |\, A\ A_1 \ldots\ldots A_n\ A_{n+1}\, | = \sqrt{g}$$

The equation (2) shows that the points $A, \ldots\ldots, A_{n+1}$ are linearly independent. Therefore, we may write

$$(3)\quad \begin{cases} dA = dw_1\, A_1 + \ldots\ldots\ldots\ldots + dw_n\, A_n, \\ \overline{dA_i} = dw_{i0}\, A + dw_{i1}\, A_1 + \ldots\ldots + dw_{in}\, A_n + dw_{i\,n+1}\, A_{n+1}, \\ \qquad (i = 1, \ldots\ldots, n) \\ dA_{n+1} = dw_{n+1,0} A + dw_{n+1,1}\, A + \ldots\ldots + dw_{n+1,n}\, A_n + dw_{n+1,n+1}\, A_{n+1}, \end{cases}$$

1 We shall denote by capital letters A, B, C,......not only the points but their coordinates.

where dw_{ij} are Pfaffian expressions,

From (1) and (3) we have

(4) $dA_i = dw_{i0} A + (dw_{.\lambda} + \left\{{}^{i}{}_{\lambda}{}^{\sigma}\right\} dw_\sigma) A_\lambda + dw_{in+1} A_{n+1},$

(5) $dw_{in+1} = g_{.\sigma} dw_\sigma .$

From the equation

$$\begin{aligned} d\sqrt{g} &= \sqrt{g}\, d\log \sqrt{g} = \sqrt{g} \left\{{}^{\tau}{}_{\tau}{}^{\sigma}\right\} dw_\sigma \\ &= d \mid A\ A_1 \ldots\ldots\ldots A_n\ A_{n+1} \mid \\ &= \sqrt{g}\,(dw_{\tau\tau} + \left\{{}^{\tau}{}_{\tau}{}^{\sigma}\right\} dw_\sigma + dw_{n+1,n+1}) \end{aligned}$$

we have

$$dw_{11} + \ldots\ldots\ldots + dw_{nn} + dw_{n+1,n+1} = 0$$

On the other hand, we have

$$n A_{n+1} = \frac{\partial}{\partial w_\tau}(g^{\sigma\tau} A_\sigma) + A_\sigma g^{\sigma\lambda} \left\{{}^{\lambda}{}_{\tau}{}^{\tau}\right\}$$

$$nd A_{n+1} = \ldots\ldots\ldots\ldots + A_{n+1} (dw_{\sigma\sigma} + dw_{n+1\ n+1}),$$

where terms not written are linearly dependent on $A, \ldots\ldots, A_n$.

Therefore, we have

$$dw_{n+1,\ n+1} = 0$$

From the equation

$$\frac{\overline{\partial A_i}}{\partial w_j} = \frac{\partial w_{i0}}{\partial w_j} A + \frac{\partial w_{i\lambda}}{\partial w_j} A_\lambda + g_{ij} A_{n+1}$$

we have

$$g^{\sigma\tau} \frac{\partial w_{\sigma k}}{\partial w_\tau} = 0, \quad (k = 0,\ 1, \ldots\ldots, n)$$

9 The necessary and sufficient condition that the system of total differential equations (3) may be completely integrable is that the equations

$$(\delta,\ d)A = 0,$$

$$\overline{(\delta,\ d)}A_i + (i\ \tau,\ \lambda\ \mu)\, dw_\lambda\, \delta w_\mu\, A_\tau = 0,$$

$$(i = 1, \ldots\ldots, n)$$

$$(\delta,\ d)A_{n+1} = 0$$

hold as the consequence of (3).

This condition is equivalent to

(8) $dw_\sigma\, \delta w_{\sigma k} - \delta w_\tau\, dw_\tau = 0, \quad (k = 0, \ldots\ldots, n)$

(9) $dw_{i\sigma} g_{\sigma\mu}\, \delta w_\mu - \delta w_{.\sigma} g_{\sigma\lambda}\, dw_\lambda = 0,$

(10) $\overline{\delta}(dw_{ij}) - \overline{d}(\delta w_{ij}) + dw_{i0}\,\delta w_j - \delta w_{i0}\, dw_j$

$\qquad + dw_{i\lambda}\,\delta w_{\lambda j} - \delta w_{i\lambda}\, dw_{\lambda j}$

$\qquad + dw_{in+1}\,\delta w_{n+1\,j} - \delta w_{in+1}\,\delta w_{n+1\,j} + \{ij,\ \lambda\mu\}\, dw_\lambda\, \delta w_\mu = 0,$

(11) $\overline{\delta}(dw_{i0}) - \overline{d}(\delta w_{i0}) + dw_{i\lambda}\,\delta w_{\lambda 0} - \delta w_{i\lambda}\, dw_{\lambda 0}$

$\qquad + dw_{in+1}\,\delta w_{n+1\,0} - \delta w_{in+1}\, dw_{n+1\,0} = 0,$

$(\ =1,\ldots\ldots,\ )$

(12) $dw_{n+1\sigma}\, g_{\sigma\mu}\, \delta w_\mu - \delta w_{n+1\sigma}\, g_{\sigma\lambda}\, dw_\lambda = 0,$

(13) $\overline{\delta}(dw_{n+1\,k}) - \overline{d}(\delta w_{n+1\,k}) + dw_{n+1\,0}\,\delta w_k - \delta w_{n+1\,0}\, dw_k$

$\qquad + dw_{n+1\,\lambda}\,\delta w_{\lambda k} - \delta w_{n+1\,\lambda}\, dw_{\lambda k} = 0,$

(14) $\delta(dw_{n+1\,0}) - d(\delta w_{n+1\,0}) + dw_{n+1\,\lambda}\,\delta w_{\lambda 0} - \delta w_{n+1\,\lambda}\, dw_{\lambda 0} = 0.$

From (8) we have

(15) $\dfrac{\partial w_{ik}}{\partial w_j} - \dfrac{\partial w_{jk}}{\partial w_i} = 0.$ $(k=0,\ 1,\ldots\ldots,\ )$

From (9) we have

(16) $\dfrac{\partial w_{i\sigma}}{\partial w_j} g_{\sigma l} - \dfrac{\partial w_{\ \sigma}}{\partial w_l} g_{\sigma j} = 0.$ $(i,\ j,\ =1,\ldots\ldots,\ n)$

From (12) we have

(17) $\dfrac{\partial w_{n+1\,\sigma}}{\partial w_j} g_{\sigma l} - \dfrac{\partial w_{n+1\,\sigma}}{\partial w_l} g_{\sigma j} = 0.$

From (3) we have

$$d^2A = d(dw_\lambda\, A_\lambda)$$
$$= \overline{d}(dw_\lambda)\, A_\lambda + dw_\lambda\, \overline{d}A_\lambda$$
$$= dw_\lambda\, dw_{\lambda 0}\, A + (\overline{d}(dw_\mu) + dw_\lambda\, dw_{\lambda\mu})\, A_n + \varphi\, A_{n+1},$$

(*a*) $d^3A = \ldots\ldots\ldots\ldots + \left(\dfrac{3}{2} d\varphi + dw_\lambda\, dw_{\lambda\mu}\, g_{\mu\nu}\, dw_\nu\right) A_{n+1},$

where terms not written are linearly dependent on $A,\ldots\ldots,\ A_n$.

From (*a*) we have

(*b*) $\varphi = -dw_\lambda\, dw_{\lambda\sigma}\, g_{\sigma\nu}\, dw_\nu$

$\qquad = k_{\lambda\mu\nu}\, dw_\lambda\, dw_\mu\, dw_\nu.$

From (15), (16) and (*b*) we have

(18) $k_{ijl} = -\dfrac{\partial w_{\ \sigma}}{\partial w_j} g_{\sigma l},$

(19) $\dfrac{\partial w_{im}}{\partial w_j} = -k_{ij\sigma}\, g^{\sigma m}$

From (7) and (19) we have

(20) $g^{\sigma\tau} k_{\sigma\tau i}=0, \quad (\ =1,\ldots\ldots, r)$

and accordingly

$$(21) \quad g^{\sigma\tau} \frac{\overline{\partial k}_{\sigma\tau i}}{\partial w_j}=0. \quad (i, j=1,\ldots\ldots, r)$$

10 From (10) and (19) we have

$$\frac{\overline{\partial k}_{ijm}}{\partial w_l}-\frac{\overline{\partial k}_{ijl}}{\partial w_m}+g^{\sigma\tau}(k_{i\sigma l} k_{j\tau m}-k_{i\sigma m} k_{j\tau l})$$

$$+\frac{\partial w_{i0}}{\partial w_l} g_{jm}-\frac{\partial w_{i0}}{\partial w_m} g_{jl}$$

$$+\frac{\partial w_{n+1\,\sigma}}{\partial w_m} g_{\sigma j} g_{il}-\frac{\partial w_{n+1\,\sigma}}{\partial w_l} g_{\sigma j} g_{im}+(i\,j,\ l\,m)=0,$$

and accordingly

$$(c) \quad 2\left(\frac{\overline{\partial k}_{ijm}}{\partial w_l}-\frac{\overline{\partial k}_{ijl}}{\partial w_m}\right)+\frac{\partial w_{i0}}{\partial w_e} g_{jm}+\frac{\partial w_{j0}}{\partial w_l} g_{im}-\frac{\partial w_{i0}}{\partial w_m} g_{jl}-\frac{\partial w_{j0}}{\partial w_m} g_{il}$$

$$+\frac{\partial w_{n+1}}{\partial w_m} g_{\sigma j} g_{il}+\frac{\partial w_{n+1\,\sigma}}{\partial w_m} g_{\sigma i} g_{jl}$$

$$-\frac{\partial w_{n+1\,\sigma}}{\partial w_l} g_{\sigma j} g_{im}-\frac{\partial w_{n+1\,\sigma}}{\partial w_l} g_{\sigma}\, g_{jm}=0,$$

$$(d) \quad 2\, g^{\sigma\tau}(k_{i\sigma l} k_{j\tau m}-k_{i\sigma m} k_{j\tau l})+2(ij,\ l\,m)$$

$$+\frac{\partial w_{i0}}{\partial w_e} g_{jm}-\frac{\partial w_{j0}}{\partial w_l} g_{im}-\frac{\partial w_{i0}}{\partial w_m} g_{jl}-\frac{\partial w_{j0}}{\partial w_m} g_{il}$$

$$+\frac{\partial w_{n+1\,\sigma}}{\partial w_m} g_{\sigma j} g_i-\frac{\partial w_{n+1\,\sigma}}{\partial w_m} g_{\sigma i} g_{ji}$$

$$-\frac{\partial w_{n+1\,\sigma}}{\partial w_l} g_{\sigma j} g_{im}+\frac{\partial w_{n+1\,\sigma}}{\partial w_l} g_{\sigma}\, g_{jm}=0.$$

From (c) we have

$$(22) \quad g^{\lambda r} g^{\mu s}\left(\frac{\overline{\partial k}_{\lambda\mu m}}{\partial w_l}-\frac{\overline{\partial k}_{\lambda\mu l}}{\partial w_m}\right)=0, \quad (r \neq s \neq l \neq m)$$

$$(23) \quad \frac{\partial w_{\sigma 0}}{\partial w_m} g^{\sigma s}-\frac{\partial w_{n+1\,s}}{\partial w_m}=2 g^{\lambda l} g^{\mu s}\left(\frac{\overline{\partial k}_{\lambda\mu m}}{\partial w_l}-\frac{\overline{\partial k}_{\lambda\mu l}}{\partial w_m}\right), \quad (s \neq l \neq m)$$

$$(24)\quad \frac{\partial w_{\sigma 0}}{\partial w_m} g^{\sigma m} - \frac{\partial w_{\sigma 0}}{\partial w_l} g^{\sigma l} - \frac{\partial w_{n+1\,m}}{\partial w_m} + \frac{\partial w_{n+1\,l}}{\partial w_l}$$
$$= 2 g^{\lambda l} g^{\mu m} \left(\frac{\overline{\partial} k_{\lambda\mu m}}{\partial w_l} - \frac{\overline{\partial} k_{\lambda\mu l}}{\partial w_m} \right). \quad (l \neq m)$$

From (d) we have

$$(25)\quad g^{\lambda r} g^{\mu s} g^{\sigma\tau} (k_{\lambda\sigma l} k_{\mu\tau m} - k_{\lambda\sigma m} k_{\mu\tau l}) + 2 g^{\lambda r} g^{\mu s} (\lambda\,\mu,\ l\,m) = 0,$$
$$(r \neq s \neq l \neq m)$$

$$(26)\quad \frac{\partial w_{\sigma 0}}{\partial w_m} g^{\sigma s} + \frac{\partial w_{n+1\,s}}{\partial w_m} = 2 g^{\lambda l} g^{\mu s} g^{\sigma\tau} (k_{\lambda\sigma m} k_{\mu\tau} - k_{\lambda\sigma l} k_{\mu\tau m})$$
$$+ 2 g^{\lambda l} g^{\mu s} (\lambda\,\mu,\ m\,l),$$
$$(s \neq l \neq m)$$

$$(27)\quad \frac{\partial w_{\sigma 0}}{\partial w_l} g^{\sigma l} + \frac{\partial w_{\sigma 0}}{\partial w_m} g^{\sigma m} + \frac{\partial w_{n+1\,m}}{\partial w_m} + \frac{\partial w_{n+1\,l}}{\partial w_l}$$
$$= 2 g^{\lambda l} g^{\mu m} g^{\sigma\tau} (k_{\lambda\sigma m} k_{\mu\tau} - k_{\lambda\sigma} k_{\mu\tau m})$$
$$+ 2 g^{\lambda l} g^{\mu m} (\lambda\,\mu,\ m\,l). \quad (l \neq m).$$

11 For a point M on the hypersurface in the vicinity of A, we have

$$M = A + dA + \frac{1}{2} d^2 A + \frac{1}{6} d^3 A + \ldots\ldots$$
$$= A\left[1 + \frac{1}{2} dw_\lambda\, dw_{\lambda 0} + \ldots\ldots\right]$$
$$+ A_\mu \left[dw_\mu + \frac{1}{2} \overline{d}(dw_\mu) + dw_\lambda\, dw_{\lambda\mu} + \ldots\ldots\right]$$
$$+ A_{n+1}\left[\frac{1}{2}\varphi + \frac{1}{4} d\varphi - \psi + \ldots\ldots\right].$$

Let $\xi_0, \ldots\ldots, \xi_{n+1}$ be projective coordinates referred to the coordinate frame of reference whose vertices and unit point are

$$A,\ A_1, \ldots\ldots,\ A_{n+1},\ A + A_1 + \ldots\ldots + A_{n+1}$$

respectively and put

$$z_i = \frac{\xi_1}{\xi_0}. \quad (= 1, \ldots\ldots, n+1).$$

Then, we have for then point M

$$(28)\quad \begin{cases} z_i = dw_i + \frac{1}{2}\overline{d}(dw_i) + \frac{1}{2} dw_\lambda\, dw_{\lambda i} + \ldots\ldots \\ \qquad (i = 1, \ldots\ldots, n) \\ z_{n+1} = \frac{1}{2}\varphi + \frac{1}{4} d\varphi - \frac{1}{6}\psi + \ldots\ldots \end{cases}$$

In virtue of (28), z_{n+1} can be expanded in a power series of $z_1, \ldots\ldots, z_n$ which is convergent as long as the absolute values of $z_1, \ldots\ldots, z_n$ are sufficiently small and of which the terms to the third degree inclusive are

$$\frac{1}{2} g_{\sigma\tau} \, z_\sigma \, z_\tau + \frac{1}{3} k_{\sigma\tau\rho} \, z_\sigma \, z_\tau \, z_\rho .$$

Therefore, the hypersurface of the second degree Q which has the contact of the second order with the given hypersurface at A is of the form

$$z_{n+1} = \frac{1}{2} g_{\sigma\tau} \, z_\sigma \, z_\tau + c_\sigma \, z_\sigma \, z_{n+1} + c \, z_{n+1}^2 .$$

For a point on Q in the vicinity of A, we have

$$z_{n+1} = \frac{1}{2} g_{\sigma\tau} \, z_\sigma \, z_\tau + \frac{1}{2} c_\rho \, g_{\sigma\tau} \, z_\sigma \, z_\tau \, z_\rho + \ldots\ldots$$

The projection from A_{n+1} on the tangent hyperplane at A of the variety at which Q intersects the given hypersurface is of the form

$$(k_{\sigma\tau\rho} - c_\rho \, g_{\sigma\tau}) \, z_\sigma \, z_\tau \, z_\rho + \ldots\ldots = 0.$$

Therefore, the hypersurface of the second degree which has the contact of the second order with the given hypersurface at A and intersects the given hypersurface at the variety such that the cone of its tangents at A is apolar[1] to the cone of the asymptotic tangents at A is of the form

$$z_{n+1} = \frac{1}{2} g_{\sigma\tau} \, z_\sigma \, z_\tau + c \, z_{n+1}^2 .$$

We shall call this hypersurface of the second degree the *semi-canonical osculating quadric* at A of the given hypersurface.

The cone of the tangents of the variety at which the given hypersurface intersects the semi-canonical osculating quadric is

$$k_{\sigma\tau\rho} \, z_\sigma \, z_\tau \, z_\rho = 0$$

We shall call these tangents *Darboux tangents* and the curve whose tangent at any point on it is *a* Darboux tangent, a *Darboux curve*

12 Up to here, we have considered only point coordinates in this chapter.

But the same can be said for hyperplane coordinates.

Denote by a, not only the tangent hyperplane at A, but its coordinates, and by $a_1, \ldots\ldots, a_n, a_{n+1}$ the hyperplanes

$$\frac{\partial a}{\partial w_1}, \ldots\ldots, \frac{\partial a}{\partial w_n}, \quad \frac{(-1)_{n+1}}{n} g^{\sigma\tau} \frac{\overline{\partial a_\sigma}}{\partial w_\tau}$$

respectively.

1 Grace and Young, Algebra of invariants. p. 303.

Then, if we write

$$(30)\quad \begin{cases} da = dw_1\ a_1 + \ldots\ldots\ldots\ldots + dw_n\ a_n, \\ \overline{da_i} = d\Omega_{i0} a + d\Omega_{i1}\ a_1 + \ldots\ldots + d\Omega_{in}\ a_n + d\Omega_{in+1}\ a_{n+1}, \\ \qquad (\ =1,\ldots\ldots, n) \\ da_{n+1} = d\Omega_{n+1\,0}\ a + \ldots\ldots + d\Omega_{n+1\,n}\ a_n + d\Omega_{n+1,\,n+1}\ a_{n+1}, \end{cases}$$

we have, in virtue of (5) and (8) in 1, chap. I,

$$(31)\quad \begin{cases} |a\ a_1\ldots\ldots a_n\ a_{n+1}| = \sqrt{g}, \\ d\Omega_{n+1,\,n+1} = 0, \\ d\Omega_{i\,n+1} = (-1)^{n+1} g_{i\sigma}\, dw_\sigma, \\ \dfrac{\partial\Omega_{im}}{\partial w_j} = k_{ij\sigma}\ g^{\sigma m} = -\dfrac{\partial w_{im}}{\partial w_j}. \\ \qquad (i, j, m = 1,\ldots\ldots, n) \end{cases}$$

From (3) and (30) we have

$$(dA_i\ a) = g_{i\sigma}\, dw_\sigma (A_{n+1}\ a),$$

$$(A\ da_i) = (-1)^{n+1}\ g_{i\sigma}\ dw_\sigma (A\ a_{n+1}).$$

On the other hand, we have from (4) in 1, chap. I

$$(dA_i\ a) = -(A_i\ a_j) = -(A_j\ a_i) = (A\ da_i) = g_{\ \sigma}\ dw_\sigma.$$

Thenefore, we have

$$(32)\quad (A_{n+1}\ a) = (-1)^{n+1} (A\ a_{n+1}) = 1.$$

From (6), (31) and (32) we have

$$(33)\quad (dA\ a_{n+1}) = (da\ A_{n+1}) = 0.$$

13 Referring to the coordinate frame of reference whose vertices and unit point are

$$A,\ A_1,\ldots\ldots,\ A_{n+1},\ A + A_1 + \ldots\ldots + A_{n+1}$$

respectively, the equation of the semi-canonical osculating quadric Q is

$$\xi_0\ \xi_{n+1} = \frac{1}{2} g_{\sigma\tau}\ \xi_\sigma\ \xi_\tau + c\,\xi_{n+1}^2$$

and accordingly, the polar of the point $(\lambda_0,\ \lambda_1,\ldots\ldots,\ \lambda_n,\ 0)$ with respect to Q is

$$\lambda_0\ \xi_{n+1} = g_{\sigma\tau}\ \lambda_\sigma\ \xi_\tau$$

This hyperplane may be expressed

$$\lambda_0\ a + \lambda_1\ a_1 + \ldots\ldots + \lambda_n\ a_n,$$

for the necessary and sufficient condition that the point

$$\xi_0\ A + \xi_1\ A_1 + \ldots\ldots + \xi_{n+1}\ A_{n+1}$$

may lie on the hyperplane

$$\lambda_0\ a + \lambda_1\ a_1 + \ldots\ldots + \lambda_n\ a_n$$

is that

$$\lambda_0\ \xi_{n+1} = \lambda_\sigma\ \xi_\tau\ g_{\sigma\tau}.$$

We shall speak of the point $\lambda_\sigma\ A_\sigma$ and the hyperplane $\lambda_\sigma\ a_\sigma$ as the reciprocals of each other.

Two tangents at a point A on the hypersurface are said to be conjugate to each other, if they are conjugate with respect to the cone of the asymptotic tangents at A. Let t and t' be two tangents at A and denote by

$$dw_1, \ldots\ldots\ldots\ldots, dw_n$$

and

$$\delta w_1, \ldots\ldots\ldots\ldots, \delta w_n$$

the values of the referred Pfaffian expressions along t and t' respectively. Then the necessary and sufficient condition that t and t' may be conjugate to each other is that

$$g_{\sigma\tau}\ dw_\sigma\ \delta w_\tau = 0.$$

Therefore, we know that the locus of the tangents conjugate to t is the $(n-1)$-flat at which the hyperplanes a and da intersect. We shall say this $(n-1)$-flat is conjugate to the tangent t.

CHAPTER IV.

CLASSIFICATION OF HYPERSURFACES.

14 Non-developable surfaces in the three dimensional space are classified projectively in two classes: 1° ruled surfaces, 2° curved surfaces. It is well known that for surfaces of the second degree which are ruled surfaces having two families of generating lines, Darboux curves are indeterminate, i. e., all $k_{ije} = 0$; in ruled surfaces having only one family of generating lines, Darboux curves always coincide with the generating lines and upon a curved surface there are three distinct families of Darboux curves.

If all $k_{ije} = 0$ for a hypersurface in a space of any dimension, it is a quadric.[1] Excluding this case, the following nine cases can occur for hypersurfaces in the four dimensional space.

I. The case where a cone of Darboux tangents (D) at any point on the hypersurface degenerates into three coincident planes.

II. The case where (D) degenerates into two coincident planes and another plane.

III. The case where (D) degenerates into three distinct coaxial planes.

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1 J. Kanitani. These memoirs, **8**, 378. (1925).

IV. The case where (D) degenerates into three distinct non-coaxial planes.

V. The case where (D) degenerates into a proper cone of the second degree K and a plane L tangent to K.

VI. The case where (D) degenerates into a proper cone of the second degree K and a plane L which intersects K at two distinct lines.

VII. The case where (D) is cuspidal.

VIII. The case where (D) is nodal.

IX. The case where (D) is anautotomic.

The cone (D) and that of the asymptotic tangents (H) at any point on the hypersurface intersects at six tangents. Choose the fundamental Pfaffian expressions so that the plane $dw_1=0$ is a tangent plane of (H) along one of the said six tangents, the plane $dw_2=0$ is another tangent planeof (H) and the plane $dw_3=0$ passes through the lines of contact of these tangent planes. Then φ may be reduced to the form

$$2\,dw_1\,dw_2+dw_3^2$$

and we have

$$(1)\quad \begin{cases} k_{222}=0, \\ 2k_{i12}+k_{i33}=0. \quad (i=1,\ 2,\ 3). \end{cases}$$

On (H) the ratios of the fundamental Pfaffian expressions may be expressed by a parameter λ in the form

$$dw_1 : dw_2 : dw_3 = -2\lambda^2 : 1 : -2\lambda.$$

In virtue of (1), the values of λ for the said six tangents are the roots of the equation

$$8k_{111}\,\lambda^6+24k_{113}\,\lambda^5+30k_{331}\,\lambda^4+20k_{333}\,\lambda^3-15k_{233}\,\lambda^2+6k_{233}\lambda=0,$$

the root $\lambda=0$ corresponding to the tangent l.

By examining the order of the multiplicity of the root $\lambda=0$, we can easily prove the following theorems.

Theorem I. *If* (D) *has the contact of the fifth order with* (H), (D) *degenerates into three coincident planes.*

Theorem II. *If* (D) *has the contact of the fourth order with* (H), (D) *degenerates into two coincident planes and a plane which passes through the line of contact.*

Theorem III. *If* (D) *has the contact of the third order with* (H), (D) *degenerates into a proper cone of the second degree* K *and a plane which touches* K *along the line of contact.*

Theorem IV. *If* (D) *has the contact of the second order with* (H), (D) *is nodal and along the nodal generating line one of the tangent planes of* (D) *coincides with that of* (H).

We shall call the case where (D) is nodal and the nodal generating line is an asymptotic tangent, case VIII_1 and the case where the nodal generating line is not asymptotic, case VIII_2

Theorem V. *If* (D) *has the contact of the second order with* (H) *along two tangents*, (D) *degenerates into a plane through the lines of contact and a proper cone of the second degree which touches* (H) *along the lines of contact.*

The theorems reciprocal to these also hold.

Theorem VI. *In case I, the three coincident planes touch* (H).

In fact, in this case, ψ may be reduced to the form

$$k_{111}\ dw_1^3$$

so that we have the equation

$$g^{11}=0,$$

which shows that the plane $dw_1=0$ is tangent to (H).

In a similar way, we can prove the following theorems:—

Theorem VII. *In case* II, (H) *touches the two coincident planes along the line of intersection of the planes to which* (D) *degenerates.*

Theorem VIII. *In case* V, *if* t *be the tangent along which the plane* L *touches the cone* K, *then* (H) *touches* K *and* L *along* t.

Theorem IX. *In case* VIII_1, *along the double generating line, one of the tangent planes of* (D) *coincides with that of* (H).

Theorem X. *In case* VI, *let* t_1 *and* t_2 *be two tangents along which* L *intersects* K. *Then* (H) *touches* K *along* t_1 *and* t_2.

15 Now consider case I.

Refer to the non-homogeneous coordinates (y) defined by

$$y_i=\frac{x_i}{x_0}. \quad (i=1, 2, 3, 4).$$

Then, since (1, 0, 0, 0, 0) is a system of the solutions of the system of the total differential equations (3) in 8 chap. III, we have

$$(2)\quad dw_{i0}=0, \quad (i=1, 2, 3, 4).$$

Next choose the fundamental Pfaffian expressions so that $dw_1^3=0$ represents the three coincident planes to which (D) degenerates, the plane $dw_2=0$ a tangent plane of (H) and the plane $dw_3=0$ passes through the lines of contact of $dw_1=0$ and $dw_2=0$ with (H). Then φ and ψ may be reduced to the forms

$$\psi=dw_1^3,$$
$$\varphi=2\ dw_1\ dw_2+dw_3^2$$

and we have

(3) $\dfrac{\overline{\partial k_{111}}}{\partial w_l} = -3 \begin{Bmatrix} 1 & l \\ & 1 \end{Bmatrix},$

(4) $\dfrac{\overline{\partial k_{11m}}}{\partial w_l} = -\begin{Bmatrix} m & l \\ & 1 \end{Bmatrix},$ $(m \neq 1)$

(5) $\dfrac{\overline{\partial k_{ijm}}}{\partial w_l} = 0.$ (two of $i, j, m \neq 1$)

From (2), (3), (4), (5) and the equation

$$\frac{\partial w_{\sigma 0}}{\partial w_m} g^{\sigma s} - \frac{\partial w_{n+1\ s}}{\partial w_m} = 2g^{\lambda l} g^{\mu s} \left(\frac{\overline{\partial k_{\lambda\mu m}}}{\partial w_l} - \frac{\overline{\partial k_{\lambda\mu l}}}{\partial w_m} \right) \quad (s \neq l \neq m)$$

[(23) in 10, chap. III]

we have

$$\frac{\partial w_{42}}{\partial w_3} = -2 \begin{Bmatrix} 2 & 3 \\ & 1 \end{Bmatrix} = 2\begin{bmatrix} 2 & 3 \\ & 2 \end{bmatrix} = 0,$$

$$\frac{\partial w_{43}}{\partial w_1} = 2 \begin{Bmatrix} 3 & 2 \\ & 1 \end{Bmatrix} = 2\begin{bmatrix} 3 & 2 \\ & 2 \end{bmatrix} = 2a_{231}.$$

But, in virtue of the equation

$$\frac{\partial w_{n+1\ \sigma}}{\partial w_j} g_{\sigma l} - \frac{\partial w_{n+1\ \sigma}}{\partial w_l} g_{\sigma j} = 0, \quad \text{[(17) in 9, chap. III]}$$

we have

$$\frac{\partial w_{42}}{\partial w_3} = \frac{\partial w_{43}}{\partial w_1}.$$

Therefore, we have

$$a_{231} = b\left[a_1^1 \left(\frac{\partial a_2^1}{\partial u_3} - \frac{\partial a_3^1}{\partial u_2} \right) + a_2^1 \left(\frac{\partial a_3^1}{\partial u_1} - \frac{\partial a_1^1}{\partial u_3} \right) + a_3^1 \left(\frac{\partial a_1^1}{\partial u_2} - \frac{\partial a_2^1}{\partial u_1} \right) \right] = 0$$

i.e. $dw_1 = 0$ is completely integrable. In other words, the locus of Darboux curves is a surface contained in the hypersurface.

Upon this surface, there is only one family of asymptotic curves defined by

$$\begin{cases} dw_1 = 0, \\ dw_3 = 0, \end{cases}$$

or

$$\frac{du_1}{b_1^2} = \frac{du_2}{b_2^2} = \frac{du_3}{b_3^2} = dt,$$

where t is an auxiliary variable.

Along the curves of this family, u_1, u_2, u_3, are the functions of t and we have

$$\frac{dA}{dt}=\frac{\partial A}{\partial w_2}=A_2,$$

$$\frac{d^2A}{dt^2}=\frac{\partial A_2}{\partial w_2}.$$

Therefore, we have

$$\frac{d^2A}{dt^2}=\begin{Bmatrix}22\\2\end{Bmatrix}\frac{dA}{dt}$$

and accordingly, the said family of asymptotic curves is that of straight lines.

Any point P on a line of this family may be expressed in the form

$$P=\lambda A+\mu A_2$$

and we can easily see that the differential dP along any curve on the surface of Darboux curves through this line is linearly dependent only on A, A_2 and A_3.

Therefore, we have the theorem.

Theorem XI. *In case* I, *the locus of Darboux curves is a family of developable surfaces contained in the hypersurface.*

16 From the theorems in **14** we can conclude that, if the order of contact of (D) with (H) is less than the second, at most the five cases III, IV, VII, VIII_2 and IX can occur.

First, consider case III. In this case, (H) *can not pass through the axis of the coaxial planes to which* (D) *degenerates.*

In fact, if we choose the fundamental Pfaffian expressions so that $dw_1=0$, $dw_2=0$ represent two of the coaxial planes, ψ is of the form

$$\psi=3dw_1\,dw_2\,(k_{112}\,dw_1+k_{122}\,dw_2)$$

and we have

$$(6)\qquad \begin{cases} g^{11}\,k_{121}+2g^{12}\,k_{122}=0\\ 2g^{12}\,k_{121}+g^{22}\,k_{122}=0\end{cases}$$

Since k_{112}, k_{122} and g are different from zero, we can see from (6) that g^{11}, g^{12} and g^{22} are different from zero and

$$(7)\qquad g^{11}g^{22}-4\,(g^{12})^2=0$$

If (H) passes through the axis

$$\begin{cases} dw_1=0,\\ dw_2=0\end{cases}$$

we must have the equation

$$g_{33}=g\{g^{11}\,g^{22}-(g^{12})^2\}=0$$

which is inconsistent with (7).

Now choose the fundamental Pfaffian expressions so that $dw_1=0$, $dw_2=0$ are the tangent planes of (H) passing through the said axis and $dw_3=0$ passes through the lines of contact of these planes.

Then φ and ψ can be reduced to the forms

$$(8)\quad \begin{cases} \varphi=2\,dw_1\,dw_2+dw_3^2, \\ \psi=k_{111}\,dw_1^3+k_{222}\,dw_2^3. \end{cases}$$

Therefore, (D) intersects (H) along six distinct tangents in this case.

Next, consider case IV.

Choose the fundamental Pfaffian expressions so that $dw_1=0$, $dw_2=0$, $dw_3=0$ represent the three planes to which (D) degenerates. Then φ and ψ can be reduced to the forms

$$(9)\quad \begin{cases} \varphi=dw_1^2+dw_2^2+dw_3^2 \\ \psi=6\,k_{123}\,dw_1\,dw_2\,dw_3 \end{cases}$$

Therefore, we know that in this case (D) *intersects* (H) *along six distinct tangents and the triangular pyramid to which* (D) *degenerates is self conjugate with respect to* (H).

Next, consider case VII. In this case (D) has only one inflectional generating line. Choose the fundamental Pfaffian expressions so that the plane $dw_1=0$ is the cuspidal tangent plane of (D), $dw_2=0$ is the inflectional tangent plane of (D) and $dw_3=0$ passes through the cuspidal and the inflectional generating lines of (D). Then ψ is of the form

$$3\,k_{223}\,dw_2^2\,dw_3+k_{111}\,dw_1^3$$

and we have

$$g_{11}=0,$$

$$g_{13}=0,$$

$$g_{22}\,g_{33}-(g_{23})^2=0$$

$$g=-(g_{12})^2\,g_{33}\neq 0$$

Accordingly, by the transformation of the form

$$dw_1'=\sqrt{g_{12}}\;dw_1,$$

$$dw_2'=\sqrt{g_{12}}\;dw_2,$$

$$dw_3'=\sqrt{g_{33}}\;dw_3+\frac{g_{23}}{\sqrt{g_{33}}}dw_2,$$

φ and ψ are reduced to the form

$$(10)\quad \begin{cases} \varphi=2\,dw_1\,dw_2+dw_3^2, \\ \psi=k_{111}\,dw_1^3+k_{222}\,dw_2^3+3\,k_{223}\,dw_2^2\,dw_3. \end{cases}$$

In (10) k_{111} and k_{223} are different from zero and if $k_{222}=0$, $dw_3=0$ is the inflectional tangent plane.

Therefore, we know that in this case *the tangent planes of* (H) *passing through the cuspidal generating line of* (D) *are the cuspidal tangent planes of* (D) *and the plane passing through the cuspidal and the inflectional generating lines and the line of contact of the former lies on the inflectional tangent plane of* (D). The plane conjugate to the cuspidal generating line intersects (D) at three distinct tangents, unless the inflectional generating line of (D) is an asymptotic tangent.

In case $VIII_2$, choose the fundamental Pfaffian expressions so that the planes $dw_1=0$ and $dw_2=0$ are the tangent planes of (H) passing through the nodal generating line of (D) and $dw_3=0$ is conjugate to it.

Then φ and ψ may be reduced to the forms

$$(11)\quad \begin{cases}\varphi=2dw_1\,dw_2+dw_3^2,\\ \psi=k_{111}\,dw_1^3+k_{222}\,dw_2^3+3k_{113}\,dw_1^2\,dw_3+3k_{223}\,dw_2^2\,dw_3\end{cases}$$

In (11) not both of k_{111} and k_{222} are zero and if one of them, e.g., k_{111} is zero, $dw_3=0$ is an inflectional tangent plane and the tangent

$$\begin{cases}dw_2=0\\ dw_3=0\end{cases}$$

is an inflectional generating line.

17 If t be a non-asymptotic tangent such that its polar planes[1] with respect to (D) and (H) coincide, by choosing the fundamental Pfaffian expressions so that the planes $dw_1=0$ and $dw_2=0$ are the tangent planes of (H) passing through t and $dw_3=0$ is conjugate to t, φ and ψ can be reduced to the forms

$$\varphi=2dw_1\,dw_2+dw_3^2,$$
$$\psi=k_{111}\,dw_1^3+k_{222}\,dw_2^3+3(k_{113}\,dw_1^2+2k_{123}\,dw_1\,dw_2+k_{223}\,dw_2^2)\,dw_3+k_{333}\,dw^3.$$

Now we shall examine if such a tangent exists.

Suppose that φ is reduced to the form

$$2dw_1\,dw_2+dw_3^2.$$

The tangents whose polar planes with respect to (D) and (H) coincide, are defined by the equations

$$(12)\quad \frac{k_{1\sigma\tau}\,dw_\sigma\,dw_\tau}{dw_2}=\frac{k_{2\sigma\tau}\,dw_\sigma\,dw_\tau}{dw_1}=\frac{k_{3\sigma\tau}\,dw_\sigma\,dw_\tau}{dw_3}.$$

Evidently, singular generating lines of (D) satisfy (12). Any tangent

1 All points on a tangent at a point A on the hypersurface, except A itself, have the same polar plane with respect to an algebraic cone in the tangent hyperplane at A whose vertex is A. We shall call this plane the plor plane of the tangent with respect to the cone.

which satisfies (12) and is not a singular generating line of (D) has the said property. If it is an asymptotic tangent, (D) and (H) touch along it. Now we shall prove that if any two of the said tangents coincide, they can not be asymptotic in the case where the order of the contact of (D) with (H) is not higher than the first.

Let t_1 be an asymptotic tangent which has the said property and choose the fundamental Pfaffian expressions so that

$$\begin{cases} dw_1=0, \\ dw_3=0 \end{cases}$$

represent the tangent t_1. Then, since $dw_1=0$ is the common tangent plane of (H) and (D), we have

$$k_{222}=0, \quad k_{223}=0$$

and by assumption

$$2k_{122}=-k_{233}\neq 0$$

The said tangents are the lines of the intersection of the cubic cones

$$(k_{111}\,dw_1^2+2k_{113}\,dw_1\,dw_3+k_{133}\,dw_3^2)\,dw_1$$
$$+(k_{112}\,dw_1^2-k_{233}\,dw_3^2)\,dw_2-k_{122}\,dw_1\,dw_2^2=0$$

and

$$(k_{111}\,dw_1^2+2k_{113}\,dw_1\,dw_3+k_{133}\,dw_3^2)\,dw_3$$
$$-(k_{113}\,dw_1^2+3k_{133}\,dw_1\,dw_3+2k_{333}\,dw_3^2)\,dw_2$$
$$+(5k_{212}\,dw_3-2k_{312}\,dw_1)\,dw_2^2=0$$

The tangent t_1 is one of the intersections and the tangent planes along t_1 of these cones are

$$k_{212}\,dw_1-0,$$
$$5k_{212}\,dw_3-2k_{312}\,dw_1=0$$

which do not coincide, for $k_{212}\neq 0$. Therefore, two of the intersections can not coincide with t_1 and accordingly, at least six of the intersections can not be asymptotic, for (H) and (D) can not touch along more than three tangents. In case IX, all the tangents which satisfy (12) have the said property and accordingly, there are nine such tangents among which at least six are not asymptotic.

CHAPTER V.

DARBOUX LINES AND SEGRE CONICS.

18 Consider the family of curves on the hypersurface defined by

$$\frac{dw_1}{\lambda_1}=\frac{dw_2}{\lambda_2}=\frac{dw_3}{\lambda_3}$$

and let C be the curve of this family passing through A.

Any point P on the tangent of C at A, except the point A, may be expressed in the form

$$P=\mu A+\lambda_1 A_1+\lambda_2 A_2+\lambda_3 A_3$$

When the point A moves along a curve on the hypersurface, AP generates a ruled surface. In particular, if A moves along the curve C, AP generates a developable surface. Now we shall find another such curve.

Let C' be such a curve and P be the point of contact of AP with the edge of the regression of the developable surface corresponding to C'. Then we have

$$(1)\quad \begin{cases} \dfrac{\lambda_\sigma\left(\delta w_{\sigma 1}+\left\{{\sigma\tau \atop 1}\right\}\delta w_\tau\right)+\mu\,\delta w_1+\delta\lambda_1}{\lambda_1} \\ =\dfrac{\lambda_\sigma\left(\delta w_{\sigma 2}+\left\{{\sigma\tau \atop 2}\right\}\delta w_\tau\right)+\mu\,\delta w_2+\delta\lambda_2}{\lambda_2} \\ =\dfrac{\lambda_\sigma\left(\delta w_{\sigma 3}+\left\{{\sigma\tau \atop 3}\right\}\delta w_\tau\right)+\mu\,\delta w_3+\delta\lambda_3}{\lambda_3} \\ g_{\sigma\tau}\,\lambda_\sigma\,\delta w_\tau=0 \end{cases}$$

δ denoting the differentiation along C'.
Eliminating δw_1, δw_2 and δw_3 from (1), we have

$$\begin{vmatrix} 0 & \lambda_\sigma g_{\sigma 1} & \lambda_\sigma g_{\sigma 2} & \lambda_\sigma g_{\sigma 3} \\ \lambda_1 & m_{11}+\mu & m_{12} & m_{13} \\ \lambda_2 & m_{21} & m_{22}+\mu & m_{23} \\ \lambda_3 & m_{31} & m_{32} & m_{33}+\mu \end{vmatrix}=0,$$

where

$$m_{ij}=\frac{\partial\lambda_i}{\partial w_j}+\lambda_\sigma\left(\frac{\partial w_{\sigma i}}{\partial w_j}+\left\{{\sigma j \atop i}\right\}\right)$$

Since the equation (2) is of the second degree with respect to μ, there are two curves of the said property besides C and accordingly, two points of contact of AP with the edges of the regression corresponding to them.

The harmonic conjugate of A with respect to these two points is

(3) $$-\frac{1}{2}\left[\frac{\lambda_\sigma\lambda_\tau\lambda_\rho k_{\sigma\tau\rho}}{g_{\sigma\tau}\lambda_\sigma\lambda_\tau}-\lambda_\rho\left(\left\{{\rho\ \sigma \atop \sigma}\right\}-\frac{1}{2}\frac{\partial}{\partial w_\rho}\log\frac{\lambda_\rho^2}{g_{\sigma\tau}\lambda_\sigma\lambda_\tau}\right]A+\lambda_\sigma A_\sigma.$$

We shall call this point the ray point of A with respect to the given family of curves.

19 When the point A moves along a curve on the hypersurface, the plane L conjugate to AP generates a hypersurface, In particular, if A moves along the curve C, L generates a developable hypersurface which is the envelope of the tangent hyperplanes of the given hypersurface along C. Now we shall find another such curve.

Any hyperplane passing through L, except the hyperplane a, may be expressed in the form

$$p = \nu a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3.$$

For the envelope of p along a curve C′ to be the locus of L along C′, it is necessary and sufficient that the intersection of p and dp is L. From this reason, by a similar method to that employed in the preceding article, we can prove that there are two curves having the said property besides C and accordingly, two tangent hyperplanes through L of the developable surfaces corresponding to them. The harmonic conjugate of the hyperplane a with respect to these hyperplanes is

$$\frac{1}{2}\left[\frac{\lambda_\sigma\lambda_\tau\lambda_\rho k_{\sigma\tau\rho}}{g_{\sigma\tau}\lambda_\sigma\lambda_\tau}+\lambda_\rho\left(\left\{{\rho\ \sigma \atop \sigma}\right\}-\frac{1}{2}\frac{\partial}{\partial w_\rho}\log\frac{\lambda_\rho^2}{g_{\sigma\tau}\lambda_\sigma\lambda_\tau}\right)\right]a+\lambda_\sigma a_\sigma.$$

and the point reciprocal to it is

(4) $$\frac{1}{2}\left[\frac{\lambda_\sigma\lambda_\tau\lambda_\rho k_{\sigma\tau\rho}}{g_{\sigma\tau}\lambda_\sigma\lambda_\tau}+\lambda_\rho\left(\left\{{\rho\ \sigma \atop \sigma}\right\}-\frac{\partial}{\partial w_\rho}\log\frac{\lambda_\rho^2}{g_{\sigma\tau}\lambda_\sigma\lambda_\tau}\right)\right]A+\lambda_\sigma A_\sigma$$

we shall call them respectively the axis hyperplane and the axis point of A with respect to the given family of curves.

From (3) and (4) we know that for Darboux curves the axis point and the ray point coincide.

20 Suppose that φ and ψ are reduced to the forms

$$\varphi = 2\,dw_1dw_2 + dw_3^2,$$
$$\psi = k_{111}\,dw_1^3 + k_{222}dw_2^3 + 3\,(k_{113}dw_1^2 + 2k_{123}dw_1dw_2 + k_{223}dw_2^2)\,dw_3 + k_{333}dw_3^2.$$

In this case, Darboux curves whose tangents are in the plane $dw_3 = 0$ are defihed by the equation

$$\begin{cases} k_{111}\,dw_1^3 + k_{222}\,dw_2^3 = 0, \\ dw_3 = 0. \end{cases}$$

If both k_{111} and k_{222} are different from zero, there are three distinct Darboux tangents on the plane $dw_3=0$. If one of k_{111} and k_{222} is equal to zero, $dw_3=0$ is a cuspidal tangent plane of (D).

Assume that both k_{111} and k_{222} are different from zero. Then, in virtue of (3) the three ray points of A with respect to Darboux curves whose tangents are in the plane $dw_3=0$ are

$$\sqrt[3]{k_{222}}\ B_1-\theta^i\sqrt[3]{k_{111}}\ B_2,\quad (i=1,\ 1,\ 3),$$

where

$$B_1=\frac{1}{2}\left(\left\{\begin{matrix}1\ \sigma\\ \sigma\end{matrix}\right\}-\frac{1}{6}\frac{\partial}{\partial w_1}\log\frac{k_{222}}{k_{111}}\right)A+A_1,$$

$$B_2=\frac{1}{2}\left(\left\{\begin{matrix}2\ \sigma\\ \sigma\end{matrix}\right\}-\frac{1}{6}\frac{\partial}{\partial w_2}\log\frac{k_{111}}{k_{222}}\right)A+A_2$$

and θ is an imaginary cubic root of 1.

Therfore, we have the theorem.

Theorem XII *Let P be a plane which is a polar plane of a non.-asymptotic tangent at a point A on the hypersurface with respect to* (D) *as well as with respect to* (H). *If P is not tangent to* (D), *there are three distinct Darboux curves whose tangents are in P and the three ray points of A with respect to them lie on a straight line,*

We seall call this line *the Darboux line of A.*

In cases III, VII *and* VIII_2, *the same thing holds with regard to the plane conjugate to the singular generating line of* (D).

We shall call the curve which is conjugate to a Darboux curve a *Segre curve.* Segre curves whose tangents are in the plane $dw_3=0$ and conjugate to one of the Darboux tangents in it are defined by the equation

$$k_{111}\ dw_1^3-k_{222}\ dw_2^3=0.$$

The three ray points and the three axis points of A with respect to these curves are

$$-\frac{1}{2}\left(\theta^i\sqrt[3]{k_{111}\,k_{222}}\right)^2A+\sqrt[3]{k_{222}}\ B_1+\theta^i\sqrt[3]{k_{111}}\ B_2$$

and

$$\frac{1}{2}\left(\theta^i\sqrt[3]{k_{111}\,k_{222}}\right)^2A+\sqrt[3]{k_{222}}\ B_1+\theta^i\sqrt[3]{k_{111}}\ B_2\quad (i=1,2,3)$$

respectively.

These six points lie on a conic whose equation referred to the coordinate frame of reference in the plane $dw_3=0$ whose vertices and unit point are respectively

$$A,\ B_1,\ B_2,\ A+B_1+B_2$$

is

$$4\,\xi_0^2 = k_{111}\,k_{222}\,\xi_1\,\xi_2.$$

Therefore, we have the theorem,

Theorm XIII *Let* P *be a plane which is a polar plane of a non-asymptotic tangent at a point* A *on the hypersurface with respect to* (D) *as well as with respect to* (H). *If* P *is not tangent to* (D), *there are three distinct Segre curves whose tangents are on* P *and conjugate to one of the Darboux tangents on* P *and the three ray points and the three axis points of* A *with respect to them lie on a conic which touches the asymptotic tangents at* A *at the points where the* Darboux *line oj* A *intersects the asymptotic tangents.*

We shall call this conic the *Segre conic* of A.

In cases III, VII, VIII$_2$. *the same thing holds with regard to the plane conjugate to the singular generating line of* (D).

21 Consider case IV. In this case φ and ψ can be reduced to

$$\varphi = dw_1^2 + dw_2^2 + dw_3^2$$
$$\psi = 6\,k_{123}\,dw_1\,dw_2\,dw_3$$

The ray points of A with respect to three Darboux curves

$$\begin{cases} dw_2 = 0, \\ dw_3 = 0, \end{cases} \qquad \begin{cases} dw_3 = 0, \\ dw_1 = 0, \end{cases} \qquad \begin{cases} dw_1 = 0, \\ dw_2 = 0, \end{cases}$$

are

$$B_1 = \frac{1}{2}\left\{ {1\ \sigma \atop \sigma} \right\} A + A_1,$$

$$B_2 = \frac{1}{2}\left\{ {2\ \sigma \atop \sigma} \right\} A + A_2,$$

$$B_3 = \frac{1}{2}\left\{ {3\ \sigma \atop \sigma} \right\} A + A_3$$

respectively.

The tangents whose polar planes with respect to (D) and (H) coincide are

$$dw_1 = \sigma\,dw_2 = \sigma_1\,dw_3 \quad (\sigma,\ \sigma_1 = +1 \text{ or } -1)$$

and their polar planes are

$$(5)\quad dw_1 + \sigma\,dw_2 + \sigma_1\,dw_3 = 0.$$

The Darboux curves whose tangents lie on one of the planes (5) are

$$\begin{cases} dw_1 = 0, \\ \sigma dw_2 = -\sigma_1\,dw_3, \end{cases} \qquad \begin{cases} dw_2 = 0, \\ \sigma_1\,dw_3 = -dw_1, \end{cases} \qquad \begin{cases} dw_3 = 0, \\ dw_1 = -\sigma\,dw_2. \end{cases}$$

Corresponding to each of the planes (5), there are three Segre curves whose tangents are on it and conjugate to one of the Darboux tangents on it. They are defined by the equations

$$\frac{dw_1}{-2}=\sigma\, dw_2=\sigma_1\, dw_3,$$

$$dw_1=\frac{\sigma\, dw_2}{-2}=\sigma_1\, dw_3,$$

$$dw_1=\sigma\, dw_2=\frac{\sigma_1\, dw_3}{-2}.$$

The ray points of A with respect to these Darboux curves are

$$\sigma B_2-\sigma_1 B_3,\quad \sigma_1 B_3-B_1,\quad B_1-\sigma B_2.$$

The ray points and the axis points of A with respect to these Segre curves are

$$\pm 2k_{123} A-2B_1+\sigma B_2+\sigma_1 B_3,$$
$$\pm 2k_{123} A+B_1-2\sigma B_2+\sigma_1 B_3,$$
$$\pm 2k_{123} A+B_1+\sigma B_2-2\sigma_1 B_3,$$

where in each expression, the plus sign in the coefficient of A corresponds to the ray point and the minus sign corresponds to the axis point. These points lie on the conicoid whose equation, referred to the coordinate frame of reference whose vertices and unit point are respectively

$$A,\ B_1,\ B_2,\ B_3,\ A+B_1+B_2+B_3$$

is

$$2(k_{123})^2(\xi_1^2+\xi_2^2+\xi_3^2)=3\xi_0^2.$$

Therefore, we have the theorem.

Theorem XIII. *In case* IV, *all the Darboux lines of a point A on the hypersurface lie on the plane which is determined by the ray points of A with respect to Darboux curves whose tangents are the edges of the triangular pyramid to which* (D) *degenerates, and all the Segre conics lie on the conicoid which touches the cone of the asymptotic tangents at A along the curve at which the cone of the asymptotic tangents is cut by the plane on which the Darboux lines of A lie.*

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