

On the System of Orbital-planes in the Inner Region of an Atom, and on the O-Triplets in X-Ray Spectra belonging to the L-Series.¹

By

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ABSTRACT

Sommerfeld's theory of the elliptic orbits is modified, by making use of three quantum numbers corresponding to three co-ordinates in space. From the standpoint of this theory a selection principle of the transition of electrons in the inner region of an atom is proposed on the one hand; and on the other hand a discussion is made on the formation of O-triplets in X-ray spectra.

To begin the discussion, let us start from the assumption that the following postulates hold good at least in the inner region of an atom, which is responsible for X-ray radiations.

- I. The stationary orbit of every electron in an atom is a plane orbit, and the stationary orbits taken by the majority of the electrons in an atom are situated in the same plane.
- II. Sommerfeld's quantizing principle [Quantenansatz] and its consequential theory of the elliptic orbits govern the motion of each electron separately.

In the following articles, some slight modifications of Sommerfeld's theory are described, which is founded on the above two postulates, and their consequences are compared with the experimental results.

§ 1. *The Energy of Kepler's Motion of an Electron in the Atom.*

¹ This was read before the Physico-Mathematical Society of Japan, April, 1923, and a brief account of that address is published in the Proceedings of that Society, 3rd Ser., Vol. 5, No. 4, 1923. A slight modification is made in the present paper with regard to the "quintets" noted at that time.

The qualitative and quantitative arguments in connection with Kepler's motion of an electron in an atom of the so-called "Wasserstoffähnlich" elements were put forward by Sommerfeld¹. Now, mainly following him, let us start to get, in reference to any element, the general expression of the energy of Kepler's motion of an electron revolving in the inner region of an atom.

Let f be the degree of freedom of the system under consideration q_i the configurational co-ordinate, T the kinetic energy expressed as a function of q_i and \dot{q}_i , and $p_i = \frac{\partial T}{\partial \dot{q}_i}$ the corresponding impulse co-ordinate. Then, Sommerfeld's quantizing principle, for each one of the f pairs of co-ordinates (p_i, q_i) , can be expressed

$$(1) \quad \int p_i dq_i = nh \quad i = 1, 2, 3, \dots, f.$$

where h : Plack's constant = 6.55×10^{-27} erg. sec.

n : positive integer, independently chosen for each degree of freedom.

As to the integration limits to be adopted in formula (1), Sommerfeld states that "Every co-ordinate q_i must be taken over the whole region [Bereich] which belongs to the unique representation of the phases of the system."

In the present case, the mass of the electron or the planet is regarded as infinitely small in comparison with that of the nucleus or the sun, so that there is no necessity to take the motion of the nucleus into account. Moreover, as the magnitude of the electron is very small compared with that of the atom, it is treated as a material point.

In an atom which has only one electron in it, as in the cases of hydrogen, ionized helium, etc., the state of the electron is determined by two conditions; e.g., the distance from the centre of the nucleus to the electron and the azimuth angle of the electron on the stationary orbit measured from the aphelion or the perihelion of the orbit. On the other hand, in the case of an element of large atomic weight which has many electrons in its atom, the above consideration can no more be held. To determine the state of each electron in its stationary orbit, besides the two co-ordinate above stated, one more co-ordinate is needed, to determine the inclination of the orbital plane. In short, the state of the electron has three degrees of freedom in such a case. Thus, it is necessary in such a case to select one special plane as the equatorial plane from which the inclination of the orbital plane or the ecliptic plane is measured. Fortunately, according to the foregoing postulate I, we can select the plane in which the

¹ Ann. d. Phys. **51**, 44, (1916).
Atombau u. Spektrallinien, 3 Aufl., Kap. 8.

$$p = \frac{\delta T}{\delta \dot{\varphi}}$$

where φ : azimuth angle of the electron on the ecliptic plane.

From these equations, the following expression can easily be obtained by simple calculations concerning the spherical triangle KPO

$$\left. \begin{aligned} \int p_\theta d\theta &= 2\pi p (1 - \cos a_1) = n_2 h \\ \int p_\varphi d\varphi &= 2\pi p \cos a_1 = n_1 h \end{aligned} \right\} (4)$$

$$\text{Thus, } \int p_\theta d\theta + \int p_\varphi d\varphi = (n_1 + n_2) h = 2\pi p \quad (5)$$

As the point P is always situated on the x-y plane, z and \dot{z} should always be $z=0$, and $\dot{z}=0$. Consequently, the equations of motion of the electron should take the same forms as those of Sommerfeld's.

At first, disregarding the effect due to the repulsion of the fellow electrons, let us take only the electrostatic attraction between the nucleus and the electron into account.

Then the total amount of the energy W possessed by the electron should be the same as that of the mono-electronic elements calculated by Sommerfeld.

$$\text{i. e., } W = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) - \frac{eE}{r} \quad (6)$$

where m_0 is constant and equal to the mass of the electron at rest,

$$\beta = \frac{v}{c}$$

v is the instantaneous velocity of the electron,

c is the velocity of light = 3×10^{10} cm. sec⁻¹.,

e is the electronic charge,

E is the charge of the nucleus.

From the solution of the equations of motion given by Sommerfeld, the expression (6) can be transformed into the form.

$$1 + \frac{W}{m_0 c^2} = \left(1 + \frac{(1-\epsilon^2) f_0^2}{p^2 - p_0^2} \right)^{-\frac{1}{2}} = \left(\frac{p^2 - p_0^2}{p^2 - \epsilon^2 p_0^2} \right)^{\frac{1}{2}} \quad (7)$$

$$\text{where } p_0 = \frac{eE}{c}$$

ϵ is the eccentricity of the orbit.

Further, the quantizing condition concerning to the eccentricity of the orbit gives¹

¹ A. Sommerfeld: Ann. d. Phys., **51**, 53 (1916).

$$2\pi pr \left(\frac{1}{\sqrt{1-\epsilon^2}} - 1 \right) = 2\pi \sqrt{p^2 - p_0^2} \left(\frac{1}{\sqrt{1-\epsilon^2}} - 1 \right) = n'h \quad (8)$$

where $\gamma^2 = 1 - \left(\frac{eE}{pc} \right)^2 \quad (9)$

From (5) and from the above expression (8) we get, by

putting $a = \frac{2\pi e^2}{hc}, \quad (10)$

$$\sqrt{p^2 - p_0^2} = \frac{n_1 + n_2}{2\pi} h \sqrt{1 - \frac{p_0^2}{p^2}} = \frac{n_1 + n_2}{2\pi} h \sqrt{1 - \left(\frac{a}{n_1 + n_2} \frac{E}{e} \right)^2},$$

$$\begin{aligned} \frac{1 - \epsilon^2}{p^2 - p_0^2} &= \frac{1}{\left(\frac{n'h}{2\pi} + \sqrt{p^2 - p_0^2} \right)^2} \\ &= \frac{\left(\frac{2\pi}{h} \right)^2}{\left[n' + (n_1 + n_2) \sqrt{1 - \left(\frac{a}{n_1 + n_2} \frac{E}{e} \right)^2} \right]^2}, \quad (11) \end{aligned}$$

Substituting the above values of (11) into (7), we get

$$\begin{aligned} 1 + \frac{W}{m_0 c^2} &= \left\{ 1 + \frac{\left(a \frac{E}{e} \right)^2}{\left[n' + (n_1 + n_2) \sqrt{1 - \left(\frac{a}{n_1 + n_2} \frac{E}{e} \right)^2} \right]^2} \right\}^{-\frac{1}{2}} \\ &= 1 - \frac{1}{2} \frac{a^2}{A^2} \left(\frac{E}{e} \right)^2 + \frac{3}{8} \frac{a^4}{A^4} \left(\frac{E}{e} \right)^4 - \frac{5}{16} \frac{a^6}{A^6} \left(\frac{E}{e} \right)^6 \\ &+ \frac{35}{128} \frac{a^8}{A^8} \left(\frac{E}{e} \right)^8 - \dots \dots \dots (12) \end{aligned}$$

where $A = n' + (n_1 + n_2) \sqrt{1 - \left(\frac{a}{n_1 + n_2} \frac{E}{e} \right)^2}$

Let N be Rydberg's frequency

$$N = \frac{2\pi^2 m_0 e^4}{h^3} = 1097 \ 37 \text{cm.}^{-1}$$

Then, the total energy of the electron should be

$$\begin{aligned} W &= -Nh \left(\frac{E}{e} \right) \left\{ \frac{1}{A^2} - \frac{3}{4} \frac{a^2}{A^4} \left(\frac{E}{e} \right)^2 + \frac{5}{8} \frac{a^4}{A^6} \left(\frac{E}{e} \right)^4 \right. \\ &\quad \left. - \frac{35}{64} \frac{a^6}{A^8} \left(\frac{E}{e} \right)^6 + \dots \dots \dots \right\} \quad (13) \end{aligned}$$

By expanding every term of the right side of the above expression (13)

in an ascending power-series of α^2 and adding together, we get the following energy expression

$$\begin{aligned}
 W = & - \frac{Nh}{(n_1+n_2+n')^2} \left(\frac{E}{e} \right)^2 \left[1 + \frac{1}{4} \frac{\alpha^2}{(n_1+n_2+n')^2} \left(\frac{E}{e} \right)^2 \right. \\
 & \left\{ 1 + 4 \frac{n'}{n_1+n_2} \right\} + \frac{1}{8} \frac{\alpha^4}{(n_1+n_2+n')^4} \left(\frac{E}{e} \right)^4 \left\{ 1 + 6 \frac{n'}{n_1+n_2} \right. \\
 & \left. + 12 \left(\frac{n'}{n_1+n_2} \right)^2 + 2 \left(\frac{n'}{n_1+n_2} \right)^3 \right\} + \frac{1}{64} \frac{\alpha^6}{(n_1+n_2+n')^6} \\
 & \left(\frac{E}{e} \right)^6 \left\{ 5 + 40 \frac{n'}{n_1+n_2} + 128 \left(\frac{n'}{n_1+n_2} \right)^2 + 184 \left(\frac{n'}{n_1+n_2} \right)^3 \right. \\
 & \left. + 64 \left(\frac{n'}{n_1+n_2} \right)^4 + 8 \left(\frac{n'}{n_1+n_2} \right)^5 \right\} + \dots \dots \dots \left. \right] \quad (14)
 \end{aligned}$$

In the above expression (14), the total energy W can simply be expressed in the terms of $n_1+n_2=n$ and n' ; it is not necessary to express it by using the three quantum numbers n_1 , n_2 , and n' separately. Thus, at first sight, it seems quite meaningless to consider such an orientation of the orbital plane in space. But in reality, the above expression (14) has its foundation on the assumption that there is no effect due to the repulsion of the fellow electrons on the revolving electron and consequently E is a constant for any orbit in the atom.

If we take into account such effects, some corrections should be needed. It will appear later that E in (14) should express the so-called "effective nuclear charge" and should be a function of n_1 , n_2 , and n' except in the case of the mono-electronic elements.

Now if we express the effective nuclear charge of the electronic orbit of quantum numbers n_1 , n_2 , n' by $E(n_1, n_2, n')$, then the energy expression $W(n_1, n_2, n')$ corresponding to the "Term" of Sommerfeld's fine structure theory becomes.

$$\begin{aligned}
 W(n_1, n_2, n') = & - \frac{Nh}{(n_1+n_2+n')^2} \left(\frac{E(n_1, n_2, n')}{e} \right)^2 \left[1 + \frac{1}{4} \right. \\
 & \frac{\alpha^2}{(n_1+n_2+n')^2} \left(\frac{E(n_1, n_2, n')}{e} \right)^2 \left\{ 1 + 4 \frac{n'}{n_1+n_2} \right\} \\
 & + \frac{1}{8} \frac{\alpha^4}{(n_1+n_2+n')^4} \left(\frac{E(n_1, n_2, n')}{e} \right)^4 \left\{ 1 + 6 \frac{n'}{n_1+n_2} \right. \\
 & \left. + 12 \left(\frac{n'}{n_1+n_2} \right)^2 + 2 \left(\frac{n'}{n_1+n_2} \right)^3 \right\} + \dots \dots \dots \left. \right] \quad (14b)
 \end{aligned}$$

Table I, The numerical Values of $\frac{\nu}{N}$ in the L Series.

Atomic Number	ϵ	$\alpha_2 (\alpha')$	$\alpha_1 (\alpha)$	$\alpha_3 (\alpha'')$	$\eta (\eta)$	$\beta_4 (\beta')$	$\beta_6 (\beta)$	$\beta_1 (\beta)$	$\beta_5 (\beta)$	$\beta_2 (\beta)$	β_8	β_7	$\beta_5 (\zeta)$	β_9	$\beta_5 (\theta, \delta)$	$\gamma_5 (\kappa)$	$\gamma_1 (\delta)$	γ_7	$\gamma_6 (\theta)$	$\gamma_2 (\zeta')$	$\gamma_3 (\chi)$	$\gamma_4 (\Psi)$
29 Cu	—	—	68.47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
30 Zn	—	—	74.55	—	—	—	—	76.25	—	—	—	—	—	—	—	—	—	—	—	—	—	—
32 Ge	—	—	87.50	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
33 As	—	—	94.42	—	—	—	—	97.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—
34 Se	—	—	101.58	—	—	—	—	104.53	—	—	—	—	—	—	—	—	—	—	—	—	—	—
35 Br	—	—	109.04	[109.41] _s	—	—	—	112.39	—	—	—	—	—	—	—	—	—	—	—	—	—	—
37 Rb	—	—	124.78	[125.28] _s	—	—	—	129.06	—	—	—	—	—	—	—	—	—	—	—	—	—	—
38 Sr	—	—	133.07	—	—	—	—	137.87	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39 Y	—	—	141.61	[142.12] _s	—	—	—	147.01	—	—	—	—	—	—	—	—	—	—	—	—	—	—
40 Zr	—	—	150.47	[151.08] _s	—	—	—	156.50	—	—	—	—	—	—	—	—	—	—	—	—	—	—
41 Nb	—	159.4	159.55	[159.96] _s	—	—	—	166.29	—	172.09	—	—	—	—	—	—	—	—	—	—	—	—
42 Mo	—	168.7	168.93	[169.54] _s	—	—	—	176.40	—	—	—	—	—	—	—	—	193.42	—	—	—	—	—
44 Ru	—	188.13	188.44	[189.28] _s	—	—	—	197.62	—	—	—	—	—	—	—	—	218.38	—	—	—	—	—
45 Rh	—	198.29	198.62	[199.43] _s	—	—	—	208.77	—	—	—	—	—	—	—	—	231.53	—	—	—	—	—
46 Pd	—	208.69	209.07	[209.70] _s	—	224.1	—	220.25	226.4	233.7	—	—	—	—	—	—	245.19	—	—	253.6*	253.6*	—
47 Ag	—	219.37	219.80	[220.50] _s	—	236.01	—	232.06	238.27	246.69	—	—	—	—	—	—	259.15	—	—	—	—	—
48 Cd	—	230.32	230.82	—	—	248.00	—	244.29	250.59	259.8	—	—	—	—	—	—	273.81	—	—	—	—	—
49 In	—	241.55	242.12	—	—	—	—	256.84	—	273.5	—	—	—	—	—	—	288.79	—	—	—	—	—
50 Sn	—	253.06	253.69	—	—	273.4	—	269.79	276.35	287.59	—	—	—	—	—	—	304.29	—	—	314.6	315.8	322.3
51 Sb	—	264.84	265.53	—	—	286.5	—	283.13	289.73	302.0	—	—	—	—	—	—	320.29	—	—	328.0*	328.0*	—
52 Te	—	276.91	277.67	—	—	299.77	—	296.85	303.65	316.7	—	—	—	—	—	—	336.73	—	—	—	—	—
53 J	—	289.23	290.07	—	—	313.6	—	310.94	317.8	331.87	—	—	—	—	—	—	353.59	—	—	—	—	—
55 Cs	—	314.73	315.76	—	—	342.55	—	340.33	347.46	363.5	—	—	—	—	—	—	389.05	—	—	406.1*	406.1*	—
56 Ba	—	327.94	329.05	—	—	357.38	—	355.64	362.89	379.80	—	—	—	—	—	—	407.48	—	—	—	—	—
57 La	—	341.47	342.66	—	—	372.86	—	371.48	378.90	396.5	—	—	—	—	—	—	426.45	—	—	—	—	—
58 Ce	—	355.28	356.57	—	—	388.57	—	387.66	395.40	413.6	—	—	—	—	—	—	445.8	—	—	456.8*	456.8*	—
59 Pr	—	369.33	370.82	—	—	404.3	—	404.36	412.14	431.00	—	—	—	—	—	—	465.77	—	—	471.80	473.29	—
60 Nd	—	383.64	385.31	—	—	421.5	—	421.50	429.44	448.8	—	—	—	—	—	—	486.40	—	—	506.2*	506.2*	514.0
62 Sm	—	413.22	415.21	—	—	—	—	457.17	465.4	485.5	—	—	—	—	—	—	528.96	—	—	550.1*	550.1*	—
63 Eu	—	428.42	430.65	—	—	474.4	—	475.61	483.2	504.3	—	—	—	—	—	—	549.2	—	—	570.1	573.8	—
64 Gd	—	444.02	446.34	—	—	493.0	—	494.68	503.8	523.3	—	—	—	—	—	—	573.75	—	—	586.5	588.3	—
65 Tb	—	459.78	462.30	—	471.4	511.6	—	514.16	523.05	542.78	—	—	550.2	—	—	—	596.1	—	—	618.2	620.9	635.2
66 Dy	—	475.79	478.54	—	—	530.5	—	534.09	542.6	562.71	—	—	—	—	—	—	621.2	—	—	642.2	644.0	—
67 Ho	—	492.12	495.09	—	—	551.3	—	554.58	563.75	582.91	—	—	—	—	—	—	645.4	—	—	667.3	669.3	—
68 Er	—	508.78	511.93	—	529.2	571.1	—	575.63	585.5	603.16	—	—	—	—	—	—	668.4	—	—	690.5	694.0	—
70 Yb	482.2	542.90	546.51	—	563.9	612.5	—	619.0	629.1	645.4	—	—	—	—	—	—	720.4	—	—	743.6	746.2	—
71 Lu	498.0	560.43	564.20	—	—	635.6	—	642.9	653.4	667.2	—	—	—	—	—	—	746.9	—	—	769.6	772.5	—
73 Ta	528.50	595.83	600.20	—	621.83	678.93	686.87	688.51	699.23	711.42	715.60	723.30	728.65	—	—	778.86	803.04	—	820.98	826.90	831.28	857.74
74 W	544.02	613.85	618.45	—	642.78	701.66	708.03	712.39	723.23	733.76	737.60	746.45	751.56	(753.74) _o	(785.87) _s	807.03	831.81	(845.18) _o	854.08	854.98	859.97	887.77
76 Os	—	651.75	656.45	—	—	748.25	756.33	762.83	774.08	780.58	—	—	—	—	—	—	891.25	—	—	—	—	—
77 Ir	—	670.35	675.84	—	—	774.62	777.75	788.99	800.82	804.39	808.82	—	—	—	—	—	921.96	—	945.64*	945.64*	952.64	—
78 Pt	608.64	689.73	695.58	—	734.82	799.52*	799.52*	815.65	828.80*	828.80*	833.90	844.93	851.57	—	[850.1] _s	924.71	953.77	—	978.07*	978.07*	984.52	1018.17
79 Au	625.63	709.22	715.53	—	759.97	825.15	820.51	843.02	—	853.46	858.94	870.77	877.70	—	[880.5] _s	955.01	985.83	—	1011.12*	1011.12*	1017.41	1051.86
80 Hg	—	—	—	—	—	—	—	[868.7] _s	—	[872.66] _s	—	—	—	—	—	—	1011.46	—	—	—	—	—
81 Tl	—	749.39	756.42	—	810.30	878.64	869.49	899.88	913.23	904.16	—	922.64	931.47	—	[932.7] _s	1019.14	1053.12	—	1082.69	1078.83	1087.54	1125.00
82 Pb	676.70	769.96	777.51	—	835.90	907.01	894.50	929.98*	943.30	929.98*	936.03	950.22	959.72	—	—	1054.83	1088.37	—	1119.97*	1113.87	1119.97*	1162.89
83 Bi	694.07	790.20	798.54	—	862.32	934.22	918.97	959.93	973.85	956.28	—	—	987.98	—	[987.3] _s	1087.75	1124.10	—	1157.55*	1149.20	1157.55*	1196.89
84 Po	—	—	[832.5] _s	—	—	—	—	[990.5] _s	—	—	—	—	—	—	—	—	—	—	—	—	—	—
88 Ra	—	—	[906.5] _s	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
90 Th	819.19	944.08	955.78	—	—	1155.00	1102.78	1194.94*	1211.67	1151.93	—	—	1194.94*	—	—	—	1399.74	—	1146.20	—	—	—
92 U	855.84	990.37	1003.23	—	1134.95	1222.53	1158.70	1269.08	1286.29	1210.70	—	1238.14	1258.43	—	—	—	1486.98	—	1537.76	1507.82	1526.41	—

§ 2. *The Lines belonging to the L Series and their Wavelengths*

The X-ray spectra, as are well known, consist of three groups of lines. i. e., the K, L, and M Series.

If we mention the lines in order of their wavelengths (Roughly because some lines interchange their positions owing to the different elements), we get for

the K Series: $K\alpha_2, K\alpha_1, K\beta_3, K\beta_1, K\beta_2$;

the L Series: $Ll, L\alpha_2, L\alpha_1, L\alpha_3, L\eta, L\beta_1, L\beta_6, L\beta_1, L\beta_3, L\beta_2, L\beta_8, L\beta_7, L\beta_5, L\beta_9, L\beta_{5(o,s)}, L\gamma_5, L\gamma_1, L\gamma_7, L\gamma_6, L\gamma_2, L\gamma_3, L\gamma_4$;

the M Series: $M\alpha_2, M\alpha_1, M\beta, M\gamma, M\delta, M\epsilon$.

With regard to the wavelengths of the L series many experimental data have been obtained by numerous investigators. Among these data let us select the up-to-date values given in A. Sommerfeld's paper¹ concerning the elements lighter than tantalium, while concerning the heavier elements let us select the values given in D. Coster's paper², as the latter seems to me quite complete for all the elements between tantalium and uranium; and let us rely on them in our further arguments.

From these numerical values of the wavelengths, we can get the values of $\frac{\nu}{N}$ as shown in Table I, where ν is the wave number in cm^{-1} and N the Rydberg's constant, which is equal to 109737 cm^{-1} .

In this table, the values in ()₀, ()_s, are interpolated from the measurements of Overn³ and Siegbahn⁴, and those in [] are taken from the old data of Siegbahn and Friman⁵.

To get rid of the effect of the general shift of all the experimental values of the wavelengths of the lines, due to the difference of the numerical values given to the grating constants of the crystals used, all these interpolations (except the spectra of P_o and R_a) & exterpolarations are so made that the ratio of the distances between an interpolated line and its neighbouring lines on both sides is equal to the corresponding ratio in the data used, i. e., equal to the ratio of the distances between the original line to be interpolated and its neighbouring lines on both sides. To interpolate the lines $L\alpha_1$, and $L\beta_1$, of P_o and R_a , which have no neighbouring lines, we determine their wavelengths in such way that: with regard to every line to be interpolated (e. g. $L\alpha_1$), the ratio of the differences

1 Atombau u. Spektrallinien, 3 Aufl., 3 Kap., s. 200.

2 Z. S. f. Phys., 6, 188, (1921).

3 Phys. Rev., 16, 137, (1919).

4 Phil. Mag., 38, 639, (1919).

5 Phys. Z. S., 17, 17, 61, 176, (1916).

Ann. d. Phys., 49, 623, (1916).

between the wavelength of that line (L_{α_1}) of the element P_0 or R_{α} , and those of the same line (L_{α_1}) of the elements on both sides i. e., Bi and Th, remain the same as that of the original data (Siegbahn and Friman's data.).

The notations in () on the right side of the names of lines are their other names given by Sommerfeld¹. Among those lines, there is a line $L_{\beta_{3(e,s)}}$ which is the line denominated L_{β_3} by Overn and Siegbahn.

§ 3. On the Regular Doublets of X-Ray Spectra.

The further arguments we now attempt to make, aim at the explanation of some experimental fact, by the adoption of our foregoing theory. Before entering into these arguments, however, it will be better to give the explanations of the so-called "regular doublets" in the X-ray spectra which have been excellently discussed by Sommerfeld in this theory of elliptic orbits.

Generally, let us consider the case when an electron falls down from the same initial stationary orbit to two adjacent final stationary orbits of the quantum numbers n_1, n_2, n' , and m_1, m_2, m' , respectively suddenly radiating out the differences of energy in the form of monochromatic radiations. In such a case, we can express, by (14), the wave number differences $\Delta \nu$ of these monochromatic radiation as follows :

$$\Delta \nu = \frac{W(m_1, m_2, m') - W(n_1, n_2, n')}{h} = (n_1, n_2, n') - (m_1, m_2, m') \quad (15)$$

$$\text{where } (n_1, n_2, n') = N \left(\frac{E(n_1, n_2, n')}{e} \right)^2 \left\{ \frac{1}{(n_1 + n_2 + n')^2} + \frac{\alpha^2}{(n_1 + n_2 + n')^4} \left(\frac{E(n_1, n_2, n')}{e} \right)^2 \left(\frac{1}{4} + \frac{n'}{n_1 + n_2} \right) \right\} \quad (16)$$

Two terms on the right side of the formula (15) should represent the magnitude of the energy levels measured from the same standard state.

According to Broglie², Hertz³, Duane and Patterson⁴, there exist one, three, and five absorption edges in the regions of the K, L, and M series respectively, each one of these absorption edges, as is noticed by Duane⁵, Smekal⁶, and others, corresponds to one of the energy levels of the stationary orbits.

1 Atombau u. Spektrallinien, 3 Aufl., 3 Kapital, s. 200.

2 C. R., **158**, 1493, (1914).

3 Z. S. f. Phys., **3**, 19, (1920).

4 Proc. Nat. Ac. Sc. Washington, Sept., (1920).

5 Proc. Nat. Ac. Sc. Washington, **6**, 477, (1920).

6 Z. S. f. Phys., **5**, 121, (1921).

Now, following the general designation, let K, L₁, L₂, L₃, M₁, M₂, M₃, &c., be respectively the stationary orbits counted from the inside of the atom, corresponding to the absorption edges in the regions of the K, L, M, Series. These are considered to the same as those denoted by the same symbols in the case of emission.

It is well known that there exist several pairs of lines in X-Ray spectra which are called "regular doublets". If we mention these regular doublets in order of increasing wavelengths, we get for

the L₁ L₂ doublets: $K\alpha_1-K\alpha_2, L_1-L_2, L\alpha_3-L\beta_1, L\beta_6-L\gamma_5, L\beta_7-L\gamma_7, L\beta_5-L\gamma_6,$

the M₁ M₂ doublets: $L\alpha_1-L\alpha_2, M\beta-M\alpha_2,$

the M₃ M₄ doublets: $K\beta_1-K\beta_3, L\beta_3-L\beta_1,$

the N₁ N₂ doublets: $M\alpha-M\alpha_2,$

the N₅ N₆ doublets: $K\gamma-K\beta_2, L\gamma_3-L\gamma_2.$

Among these regular doublets, let us restrict ourselves, for the present to the consideration of the L₁ L₂ doublets only.

With regard to these L₁ L₂ doublets, it has already been noticed by Sommerfeld that the following data should hold:

- (a) The wavelength-differences between the components of the doublets remain always constant throughout all the atomic numbers of the elements.
- (b) For each element the wave number-difference between the components of the L₁ L₂ doublets are not only equal to each other, but also equal to the wave number-differences between the absorption edges, L₁ and L₂.
- (c) The 4th roots of the wave number-differences between the components of the regular doublets have always a linear relation to the atomic numbers of the elements.

Not only is the appearance of these doublets very similar to that of the hydrogen doublets, but there exists the following simple relation between the wave number-differences of these two kinds of doublets:

$$\Delta\nu_L (Z - 3.5)^4 \doteq \Delta\nu_H \quad (17)$$

where $\Delta\nu_L$ and $\Delta\nu_H$ are the wave number differences between the components of the L₁ L₂ doublets and the hydrogen doublets respectively. What we have briefly stated above in the case of the L₁ L₂ doublets, generally holds with regard to the other regular doublets, viz.:

- (i) The wavelength-differences between the components of a regular doublet remain nearly constant throughout all the elements.
- (ii) For each element, the wave numbers-differences between the components of the regular doublets belonging to the same kind (e.g., the regular doublets $K\alpha_1-K\alpha_2, L_1-L_2, L\alpha_3-L\beta_1,$ &c., belonging to the L₁ L₂ doublets) are not only equal to each other, but also equal to the wave number-differences between the absorption edges corresponding to this

kind of regular doublets (e. g., the wave number-difference between the absorption edges L_1 and L_2 corresponding to the $L_1 L_2$ doublets are equal to those of the regular doublets belonging to the $L_1 L_2$ doublets for each element)

(iii) The 4th roots of the wave number-differences between the components of the regular doublets have always a linear relation to the atomic numbers of the elements.

Now modeling ourselves on Sommerfeld's idea, let us assume that the regular doublets in X-ray spectra are generated, similarly to the hydrogen doublet, by the splitting of the lines due to the difference in the radial quantum numbers n' of the initial or final stationary orbits, corresponding to the same values of $n_1 + n_2 + n'$ and n_2 . That is, we suppose that the regular doublets are formed either by the falling of electrons from the same initial stationary orbit down to two different final stationary orbits specified above or by the falling from the two different initial stationary orbits specified above down to the same final stationary orbit.

With regard to the $L_1 L_2$ doublets, it is generally accepted that they are formed by the falling of the electrons from the initial stationary orbits L_1 and L_2 down to the same final stationary orbit (as in the case of the $K\alpha_1-K\alpha_2$ doublet), or from the same initial stationary orbit to the final stationary orbits L_1 and L_2 (as in the cases of the L_i-L_μ , $L\alpha_2-L\beta_1$, $L\beta_6-L\gamma_5$, $L\beta_7-L\gamma_7$, $L\beta_5-L\gamma_6$, doublets) respectively.

According to the view generally accepted, let us suppose that the sum of the three quantum numbers corresponding to the K, L, M, N, &c., orbits be respectively 1, 2, 3, 4, &c. Then it is clear from the above conclusion, that the numerical values of $n_1 + n_2 + n'$ corresponding to the L_1 and L_2 stationary orbits remain always equal to 2, while those of the radial quantum numbers are 1 and 0 respectively.

As to the choice of the quantum values among n_1 , n_2 , and n' , we have omitted the case when $n_1=0$ by the intensity law which will be stated in the next chapter. Now, it is presumable that the effective nuclear charges of the stationary orbits, each one of which is situated near by the others, are equal. As the stationary orbits corresponding to the quantum numbers $[1, 0, 1]$ and $[2, 0, 0]$ satisfy the above condition we may put $E_{(1,0,1)} = E_{(2,0,0)} = (Z-\epsilon)e$ where ϵ is a certain correction term. Then the wave number-difference $\Delta\nu_L$ corresponding to this pair of quantum numbers will be obtained from (15) as written in (18)

$$\Delta\nu_L = (1, 0, 1) - (2, 0, 0)$$

$$\begin{aligned} & \doteq N \left[\frac{1}{2^2} \left\{ \left(\frac{E_{(1,0,1)}}{e} \right)^2 - \left(\frac{E_{(2,0,0)}}{e} \right)^2 \right\} + \frac{\alpha^2}{2^4} \left\{ \left(\frac{E_{(1,0,0)}}{e} \right)^4 \right. \right. \\ & \left. \left. - \left(\frac{E_{(2,0,0)}}{e} \right)^4 \right\} \frac{1}{4} + \frac{\alpha^2}{2^4} \left(\frac{E_{(1,0,1)}}{e} \right)^4 \right] = \frac{N\alpha^2}{2^4} (Z - \iota)^4 \\ & = \Delta\nu_H (Z - \iota)^4 = L_2 - L_1 \quad (18) \end{aligned}$$

As this equation represents the relation which must be satisfied by a regular doublet, we have equated the value, thus calculated, to $L_2 - L_1$. Consequently we may assign the quantum numbers $[1,0,1]$ and $[2,0,0]$ to L_1 and L_2 respectively.

The above expression (18) which represents the wave number-difference between the components of the $L_1 L_2$ doublets, is the same as the corresponding expression in Sommerfeld's paper¹. Accordingly, if we substitute $\iota = 3.5$ as does Sommerfeld, the expression (17) previously given is reduced from (18)

What we have illustrated above concerning the $L_1 L_2$ doublets, can also be explained to hold, with regard to the other regular doublets, from our preceding idea of the 3 quantum numbers. Thus, the part concerning the elliptic orbits of Sommerfeld's theory is included as the special case in our foregoing considerations of the 3 quantum numbers. Moreover, the difficulty of his theory with regard to the number of the absorption edges is removed in the present consideration by the abolition of limitation of all the electronic orbits to the coplaner orbits.

§ 4. *On the Selection Principle of X-Ray Spectra.*

As has already been stated, there are one, three, and five absorption edges in the regions of the K, L, M series respectively, and these absorption edges correspond to the energy levels of the stationary orbits represented by the same alphabetical symbols. It may, therefore, be expected that the wave number of each line in X-ray spectra should be equal to the wave number-difference between two absorption edges corresponding to the initial and final stationary orbits.

In point of fact, by adequately combining the wave numbers of the lines and those of the absorption edges, Coster², Sommerfeld³ and others determined the initial and final stationary orbits of electrons to produce the X-ray spectra as shown in Table 2.

From the following Table 2, it is clear that the wave number of every line can be equated to the wave number-differences between two absorp-

1. *Atombau u. Spektrallinien*, 3 aufl., s. 608.
2. *Z. f. Phys.*, **6**, s. 188, (1921).
3. *Atombau u. Spektrallinien*, 3 Aufl., s. 635.

Table 2.

The initial and final stationary Orbits of Electrons in X-Ray Radiations

$K\alpha_3; I_3 \rightarrow K$	$L\gamma_2; N_6 \rightarrow L_3$	$L\beta_2; N_3 \rightarrow L_1$
$K\alpha_2; L_2 \rightarrow K$	$L\gamma_3; N_5 \rightarrow L_3$	$L\gamma_1; N_1 \rightarrow L_2$
$K\alpha_1; L_1 \rightarrow K$	$L\delta; M_5 \rightarrow L_1$	$M\alpha_1; N_1 \rightarrow M_1$
$K\beta_3; M_1 \rightarrow K$	$L\eta; M_5 \rightarrow L_2$	$M\alpha_2; N_2 \rightarrow M_2$
$K\beta_1; M_3 \rightarrow K$	$L\alpha_2; M_2 \rightarrow L_1$	$M\beta; N_2 \rightarrow M_2$
$K\beta_2; N_6 \rightarrow K$	$L\alpha_1; M_1 \rightarrow L_1$	$M\gamma; N_3 \rightarrow M_3$
$K\gamma; N_5 \rightarrow K$	$L\beta_1; M_2 \rightarrow L_2$	$M\delta; N_4 \rightarrow M_3$
$L\beta_4; M_4 \rightarrow L_3$	$L\beta_6; N_7 \rightarrow L_1$	
$L\beta_3; M_3 \rightarrow L_3$	$L\gamma_5; N_1 \rightarrow L_2$	

tion edges; but, the inverse statement does not hold and there are many missing lines corresponding to some pairs of two absorption edges.

In this chapter we shall, therefore, limit ourselves to the explanation of such a selective transition of electrons in an atom, from our consideration of taking the three different kinds of quantum numbers.

Now, let the quantum numbers of the initial and the final stationary orbits be $[m_1, m_2, m']$ and $[n_1, n_2, n']$ respectively. Then, from the energy law, we have obviously, for every actual passage of the electron, associated with radiation.

$$m_1 + m_2 + m' > n_1 + n_2 + n' \quad (19)$$

If we assume that Sommerfeld's first law of quantum inequalities holds also good in the present case, we can suppose that under the normal state only such passages as are restricted by the following relations are possible.

$$m_1 \geq n_1 \quad m_2 \geq n_2 \quad m' \geq n' \quad (20)$$

Moreover, we shall assume that even if radiations which do not follow the above restriction (20) occur, these exceptions will be of small intensity.

As the measure of the possibility P of the occurrence of an electron on the stationary orbits corresponding to $[n_1, n_2, n']$, let us consider, following Sommerfeld's proposition,

$$P = \frac{n_1}{n_1 + n'} \quad (21)$$

Then from the above expression (21), it can easily be seen that the numerical value of P corresponding to a stationary orbit on the longitudinal plane of the atom and that corresponding to a rectilinear orbit passing through the nucleus of the atom are equal to zero; while that corresponding to a circular orbit remains always 1. In the other cases,

P always takes numerical values between 0 and 1.

It is generally known that the sum of the three quantum numbers $n_1 + n_2 + n'$ of the K, L, M, N, and O orbits are 1, 2, 3, 4 and 5, respectively. By keeping constant the total sum of n_1, n_2, n' , and giving various values to every one of these quantum numbers we get the combinations given in Table 3.

From these quantum numbers given in Table 3, the numerical values of P can be derived as the consequence of the foregoing expression (21).

As has already been stated, every one of these numerical values of P is the measure of the possibility of occurrence of an electron in the corresponding stationary orbit. Thus, to a larger numerical value of P of a stationary orbit, there corresponds an absorption edge of the larger intensity. In the above Table 3, the cases when $n_1 = 0$ are excluded, because the numerical values of P remains always 0 in such cases.

Now, let us consider what distribution of the spectrum lines should naturally be expected, as the necessary consequence of our foregoing arguments which are based on the assumption of three kinds of quantum numbers. To begin with, however, it would be advisable to limit ourselves to the discussion of the stationary orbits up to the N region. As regards the stationary orbits in the O region there are some interesting points, and to these points we shall come back later in Chapter 5.

Now, suppose that out of all possible stationary orbits only those which correspond to the numerical values of P larger than $\frac{1}{2}$, are responsible for X-ray phenomena observed up to the present time.

The quantum numbers corresponding to such stationary orbits are underlined in Table 3. (Notwithstanding, both quantum numbers [2, 0, 2] and [1, 2, 1] of the stationary orbits in the N series correspond to the same $P = \frac{1}{2}$, but the former only is excluded from underlining.

This can be explained by the fact that the stationary orbit of the quantum numbers [2, 0, 2] only is responsible — as will be stated later — for the radiations of the longer wavelengths as in the M and N series.)

Then, the stationary orbits of the following quantum numbers which are underlined, should be detectable in the region of each series: i. e., we have

in the K series: [1, 0, 0]
 in the L series: [1, 0, 1] ↔ [2, 0, 0] [1, 1, 0]
 in the M series; [2, 0, 1] ↔ [3, 0, 0] [1, 1, 1] ↔ [2, 1, 0] [1, 2, 0]
 in the N series; [3, 0, 1] ↔ [4, 0, 0] [2, 1, 1] ↔ [3, 1, 0]
 [1, 2, 1] ↔ [2, 2, 0] [1, 3, 0]

Consequently, there are 1, 3, 5, and 7 stationary orbits in the regions of the K, L, M, and N series respectively.

The stationary orbits corresponding to the equal radial quantum numbers n_2 and the sum of the three quantum numbers $n_1 n_2 n'$, are connected by the symbol \longleftrightarrow in the above expression. Such orbits should, according to our foregoing consideration, be responsible for the formation of regular doublets. Thus we arrive, theoretically, at the conclusion — a conclusion which agrees well with the experiments — that there should exist 1, 3, 5, and 7 energy levels in the regions of the K, L, M, and N series respectively: moreover, there should exist 1, 2, and 3 regular doublets in the spectra belonging to the L, M, and N series respectively.

Thus having confirmed the above, it may be well to discuss the selection principle in X-ray spectra, as the furtherance of our foregoing argument. Let us first consider the effect due to the repulsion of the inner electrons in the same atom, on a revolving electron. Now, suppose that this effect is larger, when the angle between the equatorial plane and the ecliptic plane of this electron is smaller: i. e., to the larger spatial quantum number n_2 , there should correspond the larger effective nuclear charge.

It is quite obvious that, as the consequence of the foregoing expression (15), the wave number-difference between the components of a regular doublet is proportional to the 4th power of the numerical value of the effective nuclear charge. Thus, among a number of the regular doublets belonging to the same series, the one corresponding to the larger spatial quantum number should be of the larger wave number difference.

Consequently, it would be legitimate to consider

$$\left. \begin{aligned} \Delta \nu_{M_{(1,1,1)} - M_{(2,1,0)}} &> \Delta \nu_{M_{(2,0,1)} - M_{(3,0,0)}} \\ \Delta \nu_{N_{(1,2,1)} - N_{(2,2,0)}} &> \Delta \nu_{N_{(2,1,1)} - N_{(3,1,0)}} \Delta > \nu_{N_{(3,0,1)} - N_{(4,0,0)}} \end{aligned} \right\} (22)$$

where $\Delta \nu_{X_{(k_1, k_2, k')}} - X_{(l_1, k_2, l')}$ represents the wave number-difference of the regular doublet corresponding to the two energy levels characterized by $X_{(k_1, k_2, k')}$ and $X_{(l_1, k_2, l')}$ respectively.

With regard to the regular doublets, the following conclusion are well known as the outcome of the experiments:—

(1) The wave number-difference between the energy levels L_1 and L_2 in the L series, M_1 and M_2 , M_3 and M_4 in the M series, N_1 and N_2 , N_3 and N_4 , N_5 and N_6 in the N series are equal to the wave number-differences of the corresponding regular doublets in the same series.

(2) These regular doublets in each series can be arranged in order

of decreasing wave number-differences as follows:

for the L series; $L_1 L_2$ doublet,

for the M series; $M_3 M_4$ doublet, $M_1 M_2$ doublet,

for the N series; $N_4 N_6$ doublet, $N_3 N_4$ doublet, $N_1 N_2$ doublet.

Now, from our foregoing expression (22), the quantum numbers corresponding to each one of the energy levels can be assigned. This assignment is so made that it agrees with the experimental results above mentioned, referring to the order of the wave number-differences of the regular doublets. According to this assignment, there should exist a correspondence between the quantum numbers and the energy levels as shown in Table 3:

Thus, referring to the planes of the orbits, we can affirm the following propositions:—

(1) The electron can move from a certain outer stationary orbit to the next one as from L to K and M to L, without changing its spatial quantum number, or by losing it in even number.

(2) In every radiation caused by the transition of electrons from a certain outer orbit to the inner one passing through some intermediate orbits as from M to K, from N to L, and from O to K, the spatial quantum numbers of the initial and final orbits are not equal; the electron loses its spatial quantum number by 1 or by an odd number every time when it passes through one of the intermediate orbits.

These propositions give physical basis to the "transition law" previously deduced by the present writer¹, as well as to Coster's a-b law² in certain respects. The only difficulty in our conclusions is that there is an exception with regard to a line $K\alpha_3$ observed by Duane and Stenstrom³. But this line is very feeble and has only been mentioned in Duane's paper. Thus the exceptions to our propositions, if any, should occur very infrequently and be of very small intensity.

Owing to our propositions mentioned above, the ecliptic plane of revolving electron tends to the equatorial plane, on every occasion when the transition takes place by passing through some intermediate orbits. This may probably be due to the magnetic field caused by the majority of electrons revolving in the equatorial plane.

Previously, we omitted the energy level $[2, 0, 2]$ in our foregoing discussion, while the energy level $[1, 2, 1]$ was taken into account in spite of the fact that the relative intensities in both cases are the same and equal

¹ Proc. Roy. Soc., **105**, 59, (1924).

² Z. f. Phys., **6**, 202, (1921).

³ Phys. Rev., **15**, 328, (1920).

to $\frac{1}{2}$.

The reason for this may be explained as follows:—

As the energy level $[2, 0, 2]$ corresponds to the numerical value 0 of the spatial quantum number n_2 , it follows from the preceding propositions that the radiations concerned in this energy level should neither belong to the K nor to the L series. They should belong to the M or the N series. In such regions of X-ray spectra, the lines are hardly detectable. The lines observed in these regions are all of a stronger intensity. Consequently, it is inconceivable that the lines which are concerned in the stationary orbits of such a weak intensity as $P = \frac{1}{2}$, can have been observed up to the present time.

On the other hand, the energy level $[1, 2, 1]$ corresponds to the numerical value 2 of the spatial quantum number n_2 . There should exist some lines in the K and the L series, which are formed by the electrons having $[1, 2, 1]$ at their initial orbit. It should, therefore, have some effect upon the experimental results, even tho the relative intensity P be $\frac{1}{2}$.

This is the reason, why the energy level $[1, 2, 1]$ was only mentioned in our previous discussion.

In a similar way, we can explain the fact that there do not exist any detectable energy levels in the O series, when $n_2=0$, and $n_2=1$.

Not only those stated above, with regard to the O orbit, the following can be presupposed from our second proposition:—

The lines produced by the electrons falling from the O orbit down to the L orbit, the numerical value of n_2 of the initial orbit should be 2 (in the case when the final orbit is L_1 or L_2), or 3 (in the case when the final orbit is L_3), while if the lines belong to the K series, the spatial quantum number n_2 of the initial O orbit should be 3.

§ 5. On the O-Triplets in X-Ray Spectra belonging to the L-Series.

Coster noticed that besides the regular doublets, there exist some lines in the L series between which the wavelength differences remain the same throughout all the elements. Table 4 shows such differences of wavelengths.

From Table 4, it is easy to see that the differences of the wavelength between $L_{\beta_7} - L_{\beta_5}$ and $L_{\beta_8} - L_{\beta_7}$ etc., remain constant throughout the whole element. Table 5 shows the wavelength differences between $L_{\beta_7} - L_{\beta_5}$, $L_{\beta_8} - L_{\beta_7}$, and $L_{\gamma_6} - L_{\gamma_7}$.

Generally, let us consider the case when an electron has fallen down from one of the initial stationary orbits of the quantum number $M = m_1 +$

$m_2 + m'$, to one of the final stationary orbits of the quantum number $N = n_1 + n_2 + n'$, and it suddenly radiates out that difference of energy in the form of a mono-chromatic wave. In such a case, if we neglect the effect due to the repulsion of the fellow electrons, the wave number ν of the mono-chromatic wave should be expressed by

$$\nu = (n_1, n_2, n') - (m_1, m_2, m') \quad (23)$$

where n_1, n_2, n' , are the azimuthal, spatial, and radial quantum numbers of the final orbits and m_1, m_2, m' are those of the initial as stated in § 1.

The stationary orbits in the equatorial plane of the quantum number

Table 4.
The Constant Differences of the Wavelengths (λ in 10^{-11} cm)

Atomic Number	$\beta_2 - \beta_5$	$\gamma_1 - \gamma_6$	$\gamma_2 - \gamma_3$	$\beta_4 - \beta_3$	$\beta_2 - \beta_3$	$\beta_2 - \beta_7$
73 Ta	30.4	24.71	5.8	38.9	7.5	21.0
74 W	29.41	23.53	6.19	38.74	6.5	21.11
76 Os	28.38	—	—	37.8	—	—
77 Ir	29.87	24.8	7.0	38.5	6.2	—
78 Pt	29.40	23.75	6.1	40.3	6.7	21.10
79 Au	29.55	23.1	5.57	—	6.8	21.25
81 Tl	29.56	23.59	6.8	39.3	—	19.9
82 Pb	30.38	23.38	4.5	38.67	6.4	20.9
83 Bi	30.63	23.25	5.5	39.7	—	—
90 Th	28.49	20.93	—	36.9	—	—
92 U	28.55	20.23	7.4	37.0	—	19.68

Table 5.

Atomic Number	Differences of the wavelengths		Differences of the wavenumber		Ratio $\gamma_{\beta_5} - \gamma_{\beta_7} / \gamma_{\beta_7} - \gamma_{\beta_3}$	Differences of wavelengths. $\gamma_7 - \gamma_6$	Differences of wavenumber $\gamma_6 - \gamma_7$
	$\beta_7 - \beta_5$	$\beta_3 - \beta_7$	$\nu_{\beta_5} - \nu_{\beta_7}$	$\nu_{\beta_7} - \nu_{\beta_3}$			
73 Ta	9.4	13.8	5.35	7.70	3 : 4.3	6.8	4.90
74 W	8.3	14.6	5.11	9.15	3 : 5.3		
76 Os	—	—	—	—	—		
77 Ir	—	—	—	—	—		
78 Pt	8.4	14.3	6.64	11.03	3 : 5.1		
79 Au	8.3	14.4	6.93	11.83	3 : 5.1		
81 Tl	9.7	—	8.83	(13.92)	3 : 4.9		
82 Pb	9.48	14.5	9.51	14.19	3 : 4.5		
83 Bi	—	—	—	—	—		
90 Th	—	—	—	—	—		
92 U	11.87	—	20.29	—	—		

N consist of N elliptic orbits of different ellipticities. With respect to the stationary orbits in the planes characterized by $n_2=2$ of the quantum number M the foregoing consideration should, also hold and they consist of M-2 different elliptic orbits.

Now, suppose that the quantum numbers of the initial orbits M be 5 and that of the final orbits N be 2, and the final orbits are situated in the equatorial plane. Then the initial orbits should, according to our foregoing proposition in § 4, be situated in the planes characterized by $n_2=2$.

For the sake of convenience we shall call these orbits in the order of their eccentricities as 1st. final orbit, 2nd. final orbit and 1st. initial orbit, 2nd initial orbit, 3rd. initial orbit, &c. respectively.

In this case quantum numbers of the initial and final orbits are expressed by $N = n_1 + n_2 + n' = 2$ } (24)
 $M = m_1 + m_2 + m' = 5$ }

The radiations are emitted when an electron passes from one of the initial orbits to one of the final orbits. Therefore we should have two sets of triplets; one caused by the falling down of an electron from 3 different initial orbits to one of the final orbits, and the other is formed by combining the 3 initial orbits with the other of the final orbits. Thus the wave numbers of the components of one of the triplets are,

$$\left. \begin{aligned} \nu_{1-1} &= (2, 0, 0) - (3, 2, 0) \\ \nu_{1-2} &= (2, 0, 0) - (2, 2, 1) \\ \nu_{1-3} &= (2, 0, 0) - (1, 2, 2) \end{aligned} \right\} \dots\dots\dots (25)$$

and those of the other set of triplets are,

$$\left. \begin{aligned} \nu_{2-1} &= (1, 0, 1) - (3, 2, 0) \\ \nu_{2-2} &= (1, 0, 1) - (2, 2, 1) \\ \nu_{2-3} &= (1, 0, 1) - (1, 2, 2) \end{aligned} \right\} \dots\dots\dots (26)$$

where ν_{s-t} means the wave number of the radiation emitted by the transition of an electron from the t^{th} initial orbit to the s^{th} final orbit.

It will easily be seen from (25) and (26) that the wave number-differences between any two corresponding numbers of the two sets of triplets must be the same, as shown by the following relations.

$$\nu_{2-1} - \nu_{1-1} \equiv \nu_{2-2} - \nu_{1-2} \equiv \nu_{2-3} - \nu_{1-3} \equiv (1, 0, 1) - (2, 0, 0) \dots\dots (27)$$

By (25) and (26) the wave number differences between two successive components of a triplet should be expressed by

$$\left. \begin{aligned} \Delta \nu_1 &= \nu_{1-2} - \nu_{1-1} = \nu_{2-2} - \nu_{2-1} = (2, 2, 1) - (3, 2, 0) \\ \Delta \nu_2 &= \nu_{1-3} - \nu_{1-2} = \nu_{2-3} - \nu_{2-2} = (1, 2, 2) - (2, 2, 1) \end{aligned} \right\} \dots\dots\dots (28)$$

The above propositions can be applied to the case of x-ray radia-

tion by replacing the nuclear charge $\frac{E}{e}$ in the formulae (27) and (28) with the proper effective nuclear charges.

If we consider the K. L. M. N. O. P. orbits as the stable orbits in an atom, then the K. L. M. N. O. P. terms should have the quantum numbers 1, 2, 3, 4, 5, 6..... respectively. Now let us assume $L\beta_5$ to be generated by the falling down of an electron from one of the O orbits to one of the L orbits, then this line should be a component of one of the triplets. It can naturally be supposed that $L\gamma_6$ line is also a component corresponding to $L\beta_5$ of the other triplet, because $L\gamma_6$ forms an L doublet with $L\beta_5$.

Let o be the correction due to the effect of the other fellow electrons in the atom, to be applied to the nuclear charge for O term; then $(Z-o)e$ should be the effective nuclear charge for this term. Thus for the triplet due to the O term, the wave number differences between them can be obtained by putting the total effective nuclear charge E equal to $(Z-o)e$ in (28)

$$\left. \begin{aligned} \Delta \nu_1 &\doteq \frac{Na^2}{5^4} - \frac{1}{4} (Z-o)^4 \\ \Delta \nu_2 &\doteq \frac{Na^2}{5^4} - \frac{5}{12} (Z-o)^4 \end{aligned} \right\} \dots\dots\dots (29)$$

Therefore

$$\Delta \nu_1 : \Delta \nu_2 \doteq 3 : 5 \dots\dots\dots (30)$$

The differences of the wave numbers between the lines and the ratio between them are shown in Table 5. Actually it is very remarkable that for all the elements examined the ratio of the wave number differences between $L\beta_5-L\beta_7$, $L\beta_7-L\beta_3$ remain always the same and equal to $\Delta \nu_1 : \Delta \nu_2 = 3 : 5$ as shown in (30) within the limit of experimental errors.

In the case of tungsten, the line $L\gamma_6$ which forms a doublet with $L\beta_5$ accompanies a feeble line $L\gamma_7$, and the difference of the wave numbers between them is equal to that between $L\beta_5$ and $L\beta_7$, within the limit of experimental errors.

Thus, it seems to be legitimate to consider that the lines $L\beta_5$, $L\beta_7$, $L\beta_3$ belong to one of the two sets of triplets, and that $L\gamma_6$ and $L\gamma_7$ belong to the other set of triplets, and the wave numbers of these lines will be represented by.

$$\left. \begin{aligned} L\beta_5 & \quad \nu = L_1 - O_1 \\ L\beta_7 & \quad \nu = L_1 - O_2 \end{aligned} \right\}$$

$$\left. \begin{aligned} L\beta_3 \quad \nu = L_1 - O_3 \\ L\gamma_6 \quad \nu = L_2 - O_1 \\ L\gamma_7 \quad \nu = L_2 - O_2 \end{aligned} \right\} \dots\dots\dots (31)$$

where $L_1 - O_1$, $L_1 - O_2$ and $L_1 - O_3$ are the wave numbers of the radiation emitted by an electron which comes down to the L_1 orbit from the O_1 , O_2 , and O_3 orbits respectively, and $L_2 - O_1$, $L_2 - O_2$ the wave numbers corresponding between L_2 orbits and the O_1 , O_2 orbits respectively, O_1 , O_2 , O_3 , are the wave numbers corresponding to the 1st, 2nd, 3rd, initial orbits of the triplet.

Now adopting the theory of triplet, and replacing the nuclear charge with the effective nuclear charge, we get the following terms corresponding to the O 's.

$$\left. \begin{aligned} O_1, (3, 2, 0) &= N(Z-o)^2 \left\{ \frac{1}{5^2} + \frac{\alpha^2}{5^4} (Z-o)^2 \frac{1}{4} \right\} \\ O_2, (2, 2, 1) &= N(Z-o)^2 \left\{ \frac{1}{5^2} + \frac{\alpha^2}{5^4} (Z-o)^2 \frac{1}{2} \right\} \\ O_3, (1, 2, 2) &= N(Z-o)^2 \left\{ \frac{1}{5^2} + \frac{\alpha^2}{5^4} (Z-o)^2 \frac{11}{12} \right\} \end{aligned} \right\} \dots\dots\dots (32)$$

We can calculate from (29) the numerical values of α using the numerical values of the wave number differences of the lines given in Table 5. The numerical values of α thus obtained are shown in Table 6.

The above consideration with regard to the O triplets in the L series is the modification of the theory of quintets in the L series proposed by the writer¹.

Table 6. Triplets.

Atomic Numbers	$\frac{\nu L\beta_5}{N} - \frac{\nu L\beta_7}{N}$	Z-o	o	$\frac{\nu L\beta_7}{N} - \frac{\nu L\beta_8}{N}$	Z-o	o
73 Ta	5.35	125	-52	7.70	119	-46
74 W	5.11	123	-49	9.15	124	-50
78 Pt	6.64	132	-54	11.03	130	-52
79 Au	6.93	133	-54	11.83	132	-53
81 Tl	8.83	141	-60	13.92	137	-56
82 Pb	9.51	144	-62	14.19	137	-55
92 U	20.29	172	-80	—	—	—

From Coster's conclusion represented in Table 3, and our foregoing consideration in § 5, we get the initial and final stationary orbits, as shown in Fig. 2, from X-ray radiation.

¹ H. Hirata: Proc. Roy. Soc., 105, 48, (1924).

The numerals on the left side of Fig. 2, represent the numerical values of n_2 of the energy levels in this figure.

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Fig. 2.

