Group-Theory of Semi-Convergent Series

By

Toshizô Matsumoto

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It is desirable to extend the idea of the group of substitutions of a finite number of elements. Consider the sequence of all the natural numbers in the natural order. If we arrange them in a different order and write them successively under the integers of the first sequence, then we may conceive there a substitution of an infinite number of elements. A system of such substitutions may be conditioned to form a group. But such a definition is purely logical and would not be fruitful. To avoid it, I consider a semi-convergent series. If we rearrange the terms of the series in a different order, we arrive at the idea of the substitution of an infinite number of elements. If the series formed by the sum of differences of the corresponding terms of the given series and the terms of the newly rearranged series be absolutely convergent, we say that the given series admits the substitution. In such a case the two series must have equal value. Here we touch the problem of M. Borel. On the other hand the substitution of an infinite number of elements leads to extend the idea of generalised cycles. But we do not give in this paper the full discussions of the cycles and hastily go to define the group. All the substitutions admitted by a series are proved to form a group, the group of the series. The extended symmetric group is a system of all possible substitutions. This group characterizes the absolutely convergent series and the group of the semi-convergent series is a divisor of the symmetric group. From this we may give the series which have same group. But the detailed theorems can not be given here.

The difficulties of our problem lie in the fact that a semi-convergent series contains absolutely convergent series in it. Here we give the definition of the coefficients of substitution which serve to detect theoretically the absolutely convergent series contained in the semi-convergent series.

From the coefficients of substitution, we arrive at the idea of the exponents of substitution which have close relations with the exponent of absolute convergency of the series.

By aid of the notion of the exponents of substitution, the existence-theorem of the divisor of the group of the series is proved.

Next we classify all the semi-convergent series whose terms are the same but in different arrangements. For this we introduce the idea of equivalence of two series and all the series in a class are equivalent to one another and they have the same sum, while the series in different classes are not equivalent. Therefore Borel's problem becomes theoretically to search two classes, such that a series in one class and a series in the other shall have equal sum.

At the end by aid of Threlfall's method of proof to change the sum of semi-convergent series of complex terms, it is proved that under the mere condition that a series and its transformed series shall have the same sum, a group can not in general be defined.

1. Being given a *semi-convergent* series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots\cdots + u_n + \cdots\cdots ,$$

if the series

$$\sum_{n=1}^{\infty} (u_n - u_{s_n}) = (u_1 - u_{s_1}) + (u_2 - u_{s_2}) + \cdots + (u_n - u_{s_n}) + \cdots$$

be *absolutety convergent*, we say that the given series admits the substitution

$$S = \begin{pmatrix} 1 & 2 & \cdots\cdots & n & \cdots & \cdot\cdot \\ s_1 & s_2 & \cdots\cdots & s_n & \cdots\cdots & \end{pmatrix} = \begin{pmatrix} n \\ s_n \end{pmatrix}$$

where $s_1, s_2, \ldots s_n, \ldots$, mean $1, 2, \ldots\ldots, n, \ldots.$, but in a different order. The series $u_1 + u_2 + \cdots\cdots + u_n + \cdots\cdots$ is said to be *transformed* into the series $u_{s_1} + u_{s_2} + \cdots\cdots + u_{s_n} + \cdots\cdots$ by the substitution S, and it is clear that the transformed series $\sum u_{s_n}$ is convergent.

The substitution $S = \begin{pmatrix} n \\ s_n \end{pmatrix}$ is very different from the permutation of a finite number of elements. Take an element, say 1, in the upper row of the substitution S. If $s_1=1$, the substitution S replaces u_1 by u_{s_1}, *i. e.*, S contains a cycle (1) of one element. If $s_1 \neq 1$, write s_1 to the right of 1:

$$1,\ s_1.$$

Let a be the integer standing upon the element 1 of the lower row of S, we write a to the left of 1:

$$a,\ 1,\ s_1.$$

If $a=s_1$, the substitntion S contains the cycle $(1\ s_1)$, if otherwise let $b=s_1$, we write s_b to the right of s_1:

$$a,\ 1,\ s_1,\ s_b.$$

If $a \neq s_b$, let c be the integer standing above the integer a in the lower row of S, and write it to the left of a. Continuing this process, we obtain a system of integers

$$(\cdots\cdots c\ a\ 1\ s_1\ s_b\ s_d\cdots\cdots) \qquad \ldots\ldots(1)$$

where $1=s_a,\quad a=s_c,\cdots\cdots$

$s_1=b,\quad s_b=d,\cdots\cdots$

If this system of integers contains an infinite number of integers, they are different with one another. If we apply S to the series $\sum u_n$, any term whose suffix is an integer of the system, is replaced by the term whose suffix is the next one in the system. Hence this system of integers (1) is the *generalised cycle.* For example

$$\begin{pmatrix} 1 & 2 & 3 & 4\cdots\cdots & 2n & 2n+1\cdots\cdots \\ 3 & 1 & 5 & 2\cdots\cdots & 2n-2 & 2n+3\cdots\cdots \end{pmatrix}$$

$$=(\cdots\cdots 2n+2\ \ 2n\cdots\cdots 6\ 4\ 2\ 1\ 3\ 5\cdots\cdots 2n+1\ \ 2n+3\cdots\cdots).$$

If there remain some integers not contained in the cycle (1), with those integers we may form another cycles. Suppose that 2 is not contained in the cycle (1), and let

$$(\cdots\cdots \gamma\ \alpha\ 2\ s_2\ s_\beta\ s_\delta\cdots\cdots) \qquad \ldots\ldots(2)$$

be the cycle containing 2. The cycle (1) and (2) do not have common integers. For example, if $s_\delta=s_1$, we must have

$$s_\beta=1,\quad s_2=a,\quad 2=c.$$

This is absurd, since by the hypothesis 2 is not contained in (1).

2. *If the series* $\sum u_n$ *admits the substitution* $S=\begin{pmatrix} n \\ s_n \end{pmatrix}$, *we have*

$$\sum_{n=1}^{\infty}(u_n-u_{s_n})=0 \quad or \quad \sum u_n=\sum u_{s_n}$$

Since by the hypothesis the series $\sum(u_n-{}_{s_n}u)$ is absolutely convergent, it may be summed in any order of the terms. Consider a term $(u_1-u_{s_1})$. If $s_1=1$, $(u_1-u_{s_1})=0$; in this case consider $(u_2-u_{s_2})$ instead of $(u_1-u_{s_1})$. If $s_2=2$, we have only to consider $(u_3-u_{s_3})$ and so on. Hence suppose $s_1\neq 1$, then in the series $\sum(u_n-u_{s_n})$, there is only a term

$$(u_a-u_{s_a}) \text{ such that } s_a=1,$$

and we have

$$(u_1-u_{s_1})+(u_a-u_{s_a})=-u_{s_1}+u_a.$$

If $a=s_1$, the above sum is zero and we consider in the series $\sum(u_n-u_{s_n})$ the term next following the term $(u_1-u_{s_1})$ except the term $(u_a-u_{s_a})$. If $a\neq s_1$, there is only a term

$$(u_b-u_{s_b}) \text{ such that } b=s_1,$$

and we have

$$-u_{s_1}+u_a+(u_b-u_{s_b})=u_a-u_{s_b}.$$

Continuing this process, we obtain from the series $\sum(u_n-u_{s_n})$, a partial series

$$(\mathrm{I})\equiv(u_1-u_{s_1})+(u_a-u_{s_a})+(u_b-u_{s_b})+\cdots\cdots\cdots$$

whose sum is zero. For if this partial series has a finite number of terms, it is identically zero by its construction; in the other case it is absolutely convergent. Let (I_ν) be the sum of the fiast ν terms of the partial series and $(u_l-u_{s_l})$ and $(u_m-u_{s_m})$ be the last two consecutive terms of it, then we have

$$(\mathrm{I}_\nu)=(u_1-u_{s_1})+(u_a-u_{s_a})+(u_b-u_{s_b})+\cdots\cdots+(u_l-u_{s_l})+(u_m-u_{s_m})$$
$$=u_l-u_{s_m} \quad or \quad -u_{s_l}+u_m,$$

according as ν is odd or even. Since the given series and its transformed series are convergent, u_l, u_m, u_{s_l}, u_{s_m} tend to zero for $\nu\to\infty$; we have therefore

$$(\mathrm{I})=\lim_{\nu\to\infty}(\mathrm{I}_\nu)=0.$$

We remark that the construction of the partial series may easily be shown by the cycle. Since $1=s_a$, $s_1=b,\ldots\ldots\ldots$, S contains the cycle

$$(\cdots\cdots\cdots a\ 1\ s_1\ s_b\cdots\cdots\cdots\cdots).$$

From this we construct the partial series

$$(\mathrm{I})=(u_1-u_{s_1})+(u_a-u_{s_a})+(u_b-u_{s_b})+\cdots\cdots\cdots.$$

If all of the terms of $\sum(u_n-u_{s_n})$ are not contained in (I), consider the first remaining term. Suppose $(u_2-u_{s_2})$ be that term. Beginning with $(u_2-u_{s_2})$ we construct a partial series (II) by the same consideration as (I) and by the same reasoning we have

$$(\mathrm{II})=0.$$

The partial series do not have common terms. For let (II) be

$$(\mathrm{II})=(u_2-u_{s_2})+(u_\alpha-u_{s_\alpha})+(u_\beta-u_{s_\beta})+\cdots\cdots\cdots,$$

then as we have remarked all the integers

$$\cdots\cdots\cdots,\ \alpha,\ 2,\ \beta,\cdots\cdots\cdots$$

are different from any one of the integers

$$\cdots\cdots\cdots,\ a,\ 1,\ b,\cdots\cdots\cdots$$

Therefore (I) and (II) have no common terms.

If all the terms of $\sum(u_n-u_{s_n})$ are not contained in either (I) or (II), we continue the construction of the partial series and we have

$$\sum_{n=1}^{\infty}(u_n-u_{s_n})=(\mathrm{I})+(\mathrm{II})+\cdots\cdots\cdots,$$

each series on the right being equal to zero. Hence we have

$$\sum_{n=1}^{\infty}(u_n-u_{s_n})=0 \quad or \quad \sum u_n=\sum u_{s_n}$$

Q. E. D.

This theorem may be stated as follows:

If the series $\sum u_n$ and $\sum u s_n$ are not equal, then the series $\sum(u_n-u_{s_n})$ is not absolutely convergent.

3. The inverse of the previous theorem is not true *i. e.*, when $\sum u_n=\sum u_{s_n}$, the series $\sum(u_n-u_{s_n})$ is not necessarily absolutely convergent. For example let

$$\sum_{n=1}^{\infty}u_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots\cdots+(-1)^{n-1}\frac{1}{n}\pm\cdots\cdots\cdots,$$

$$S=\begin{pmatrix}1 & 2 & 3 & 4\cdots\cdots\cdots 2n-1 & 2n\ \cdots\cdots\cdots\\ 2 & 1 & 4 & 3\cdots\cdots\cdots\cdots 2n & 2n-1\cdots\cdots\cdots\end{pmatrix},$$

then we have

$$\sum_{n=1}^{\infty}(u_n-u_{s_n})=\left(1+\frac{1}{2}\right)-\left(\frac{1}{2}+1\right)+\left(\frac{1}{3}+\frac{1}{4}\right)-\left(\frac{1}{3}+\frac{1}{4}\right)+\cdots=0,$$

$$\sum_{=1}^{\infty}|u_n-u_{s_n}|=\frac{3}{2}+\frac{3}{2}+\frac{7}{12}+\frac{7}{12}+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots=\infty.$$

We may conclude easily that *the series with alternate signs do not admit the substitution.*

$$S=\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots\cdots 2n-1 & 2n\cdots\cdots\cdots\cdots \\ 2 & 1 & 4 & 3\cdots\cdots & 2n & 2n-1 \cdots\cdots\cdots \end{pmatrix}.$$

M. Borel[1] found sufficient conditions that a series and its transformed series shall have the equal sum. Given a semi-convergent series

$$u_1+u_2+\cdots\cdots\cdots+u_n+\cdots\cdots\cdots,$$

let

$$v_1+v_2+\cdots\cdots\cdots+v_n+\cdots\cdots\cdots$$

be its transformed series (without knowing its convergency).
If $u_m=v_n$, he put

$$|\, m-n \,| \equiv a_m$$

which is called the *displacement* of the term of the m[th] order. The maximum of a_1, $a_2,\cdots\cdots\cdots$, a_m is denoted by λ_m and by η_m the maximum of $|\, u_m \,|$, $|\, u_{m+1} \,|,\cdots\ \cdots\cdot$ In either of the following cases $\sum u_n$ does not change its sum:

$$\lim_{m\to\infty} \lambda_m\eta_m=0, \qquad \lim_{m\to\infty} \lambda_m |\, u_m \,| =0 \quad \text{or} \quad \lim_{m\to\infty} a_m\eta_m=0.$$

Either of these conditions is very rough. For even when the series $\sum u_n$ admits the substitution *i. e.*, when the series $\sum(u_n-v_n)$ is absolutely convergent, the condition of Borel may not be satisfied. For example let

$$\sum u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\cdots$$

$$\sum u_{s_n} = \frac{1}{3}-\frac{1}{2}+1-\frac{1}{4}+\frac{1}{9}-\frac{1}{6}+\frac{1}{5}-\frac{1}{8}+\frac{1}{7}-\frac{1}{10}+\cdots$$

where

$$S=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\cdots\cdots\cdots \\ 3 & 2 & 1 & 4 & 9 & 6 & 5 & 8 & 7 & 10\cdots\cdots\cdots \end{pmatrix}$$

After the element 4, the corresponding elements of S are given by

$$\begin{matrix} 3^\nu+1 & 3^\nu+2 & 3^\nu+3 & 3^\nu+4\cdots\cdots 3^{\nu+1} & 3^{\nu+1}+1 \\ 3^\nu+1 & 3^{\nu+1} & 3^\nu+3 & 3^\nu+2\cdots\cdots 3^{\nu+1}-2 & 3^{\nu+1}+1 \end{matrix}$$

$$\nu=1,\ 2,\ 3,\cdots\cdots\cdots.$$

The series $\sum(u_n-u_{s_n})$ is absolutely convergent, for

$$\sum_{n=1}^{\infty}|u_n-u_{s_n}| = \left|1-\frac{1}{3}\right|+0+\left|\frac{1}{3}-1\right|+\cdots\cdots+0+\left|\frac{1}{3^\nu+2}-\frac{1}{3^{\nu+1}}\right|+0$$

1. Méthodes et problèmes de thèorie des fonctions, 68—73. (1922).

$$+\left|\frac{1}{3^\nu+4}-\frac{1}{3^\nu+2}\right|+0+\cdots\cdots+\left|\frac{1}{3^\nu+2i+2}-\frac{1}{3^\nu+2i}\right|+0+\cdots$$

$$+\left|\frac{1}{3^{\nu+1}}-\frac{1}{3^{\nu+1}-2}\right|+0+\cdots\cdots\cdots,$$

$$\sum_{\nu,i}\left|\frac{1}{3^\nu+2i+2}-\frac{1}{3^\nu+2i}\right|=\sum_{\nu,i}\frac{2}{(3^\nu+2i+2)(3^\nu+2i)}$$

where $\sum_{\nu,i}$ means the sum of all the terms excepting the terms such as $\left|\frac{1}{3^\nu+2}-\frac{1}{3^{\nu+1}}\right|$. This series is clearly convergent. For the remaining terms

$$\sum_{\nu=1}^{\infty}\left|\frac{1}{3^\nu+2}-\frac{1}{3^{\nu+1}}\right|=\sum_{\nu=1}^{\infty}\frac{2(3^\nu-1)}{(3^\nu+2)3^{\nu+1}}$$

$$<\sum_{\nu=1}^{\infty}\frac{1}{3^\nu}$$

which is also convergent. Therefore the series $\sum(u_n-u_{s_n})$ is absolutely convergent. Now consider that

$$a_{3^{\nu+1}}=3^{\nu+1}-3^\nu-2=2(3^\nu-1)=\lambda_{3^{\nu+1}},$$

$$u_{3^{\nu+1}}=\frac{1}{3^{\nu+1}}=\eta_{3^{\nu+1}},$$

$$\lambda_{3^{\nu+1}}\eta_{3^{\nu+1}}=\frac{2}{3}\left(1-\frac{1}{3^\nu}\right)\to\frac{2}{3}\neq 0.$$

Thus Borel's condition is not satisfied although we have by the theorem of N° 2, $\sum u_n=\sum u_{s_n}$.

4. *If the series $\sum u_n$ admits two substitutions, it admits their product.*

Let two substitutions be

$$S=\begin{pmatrix}1 & 2 & \cdots\cdots & n & \cdots\cdots\\ s_1 & s_2 & \cdots\cdots & s_n & \cdots\cdots\end{pmatrix},$$

$$T=\begin{pmatrix}1 & 2 & \cdots\cdots & n & \cdots\cdots\\ t_1 & t_2 & \cdots\cdots & t_n & \cdots\cdots\end{pmatrix}.$$

That in general the series $\sum u_n$ admits a substitution, say S, is nothing but the series $\sum(u_n-u_{s_n})$ is absolutely convergent. Therefore the effect of the substitution is indefferent of the order of the integers of the upper row in S, provided the integers standing under them are the same as the integers of the upper row were in the natural order.

Therefore the effect of the substitution T is not affected when the integers of the upper row in T are in the order of $s_1, s_2, \cdots\cdots, s_n, \cdots\cdots$. Hence we may write

$$T = \begin{pmatrix} s_1 & s_2 & \cdots\cdots & s_n & \cdots\cdots \\ r_1 & r_2 & \cdots\cdots & r_n & \cdots\cdots \end{pmatrix}.$$

To determine the integers of the lower row, suppose

$$s_n = m,$$

then in T, under m, there is the integer t_m; hence we have

$$r_n = t_m,$$

and consequently all the integers $r_1, r_2, \cdots\cdots\cdots, r_n, \cdots\cdots\cdots$ are determined uniquely.

By the *product* ST, we understand the substitution

$$ST \equiv R \equiv \begin{pmatrix} 1 & 2 & \cdots\cdots\cdots & n & \cdots\cdots\cdots \\ r_1 & r_2 & \cdots\cdots\cdots & r_n & \cdots\cdots\cdots \end{pmatrix},$$

Now the theorem stated above can easily be proved. Since the series $\sum u_n$ admits the substitution T, the series

$$\sum(u_n - u_{t_n}) = (u_1 - u_{t_1}) + (u_2 - u_{t_2}) + \cdots\cdots + (u_n - u_{t_n}) + \cdots\cdots$$

is absolutely convergent; consequently its terms may be rearranged in any order. We arrange its terms such as the first numbers in the brackets shall be $u_{s_1}, u_{s_2}, \cdots\cdots\cdots, u_{s_n}, \cdots\cdots\cdots$, then the second numbers in the brackets become $u_{r_1}, u_{r_2}, \cdots\cdots, u_{r_n}, \cdots\cdots$. Therefore we have

$$\sum(u_n - u_{t_n}) = (u_{s_1} - u_{r_1}) + (u_{s_2} - u_{r_2}) + \cdots\cdots + (u_{s_n} - u_{r_n}) + \cdots$$

which is absolutely convergent.
On the other hand we have

$$\sum(u_n - u_{r_n}) = \sum(u_n - u_{s_n}) + \sum(u_{s_n} - u_{r_n}).$$

Since the series $\sum u_n$ admits the substitution S, the series $\sum(u_n - u_{r_n})$ is absolutely convergent, or the series $\sum u_n$ admits the substitution ST.

Q. E. D.

By this proof, to determine whether the series $\sum u_n$ admits the substitution ST or not, we may proceed as follows: First apply S on $\sum u_n$ and obtain the series which we designate by $\sum u_n S$, *i. e.*,

$$\sum u_n S \equiv u_{s_1} + u_{s_2} + \cdots\cdots + u_{s_n} + \cdots\cdots\cdots$$

and suppose the series $\sum (u_n - u_{s_n})$ be absolutely convergent ($\sum u_n$ admits S). Secondly apply T on the new series $\sum u_n S$ and obtain the series

$$\sum u_n ST = u_{r_1} + u_{r_2} + \cdots\cdots + u_{r_n} + \cdots\cdots\cdots$$

and suppose the series $\sum (u_{s_n} - u_{r_n})$ be absolutely convergent ($\sum u_n S$ admits T). Then the series $\sum u_n$ admits the product ST.

5. *If the series $\sum u_n$ admits a substitution, then it admits the inverse of the substitution.*

Let
$$S = \begin{pmatrix} 1 & 2 & \cdots\cdots & n & \cdots\cdots \\ s_1 & s_2 & \cdots\cdots & s_n & \cdots\cdots \end{pmatrix}$$

be the substitution. We denote by S^{-1} its inverse, then

$$S^{-1} = \begin{pmatrix} s_1 & s_2 & \cdots\cdots & s_n & \cdots\cdots \\ 1 & 2 & \cdots\cdots & n & \cdots\cdots \end{pmatrix}$$

such that

$$SS^{-1} = S^{-1}S = 1\ ;$$

1 signifies the identical substitution.

We may rearrange the integers of the upper row in S^{-1} in their natural order without affecting the effect of the substitution. We write therefore

$$S^{-1} = \begin{pmatrix} 1 & 2 & \cdots\cdots & n & \cdots\cdots \\ p_1 & p_2 & \cdots\cdots & p_n & \cdots\cdots \end{pmatrix}$$

such that if $s_n \equiv m$, we have $n = p_m$ and conversely.

Since the series $\sum u_n$ admits S, the series

$$\sum (u_n - u_{s_n}) = (u_1 - u_{s_1}) + (u_2 - u_{s_2}) + \cdots\cdots + (u_n - u_{s_n}) + \cdots\cdots$$

is absolutely convergent, hence the series

$$(u_{s_1} - u_1) + (u_{s_2} - u_2) + \cdots\cdots + (u_{s_n} - u_n) + \cdots\cdots$$

is also absolutely convergent. Therefore we may rearrange its terms such as the first numbers in the brackets shall be $u_1, u_2, \cdots\cdots, u_n, \cdots\cdots$;

then the second numbers in the brackets will become $u_{p_1}, u_{p_2}, \cdots\cdots, u_{p_n}, \cdots$ Thus the series

$$(u_1 - u_{p_1}) + (u_2 - u_{p_2}) + \cdots\cdots + (u_n - u_{p_n}) + \cdots\cdots$$

is absolutely convergent, *i. e.*, the series $\sum u_n$ admits the inverse substitution S^{-1}. Q. E. D.

A system of all substitutions which are admitted by the series $\sum u_n$ forms a group.

For the system contains the identical substitution and if S be any one of the substitutions of the system, then as we have proved, its inverse S^{-1} is admitted by the series; hence it is contained in the system. Moreover if S and T be any two substitutions of the system, then as we have proved in the preceding paragraph, their product ST is admitted by the series; hence the product is contained in the system. It is clear that the product of three substitutions obeys the law of association. Therefore the system of all substitutions admitted by the series forms a group.

This group is called the *group of the series* $\sum u_n$. If all the substitutions of a group be admitted by a series, we say that the *series admits the group.*

We remark that the series $\sum u_n$ and all its transformed series by the substitutions of the group have the same value. (N° 2)

6. We call the system of all possible substitutions the *symmetric group. An absolutely convergent series admits the symmetric group*, for we may rearrange its terms in any order without affecting its property. Conversely *if a series $\sum a_n$ admits the symmetric group, then it must be absolutely convergent.* (It is clear if all the terms have the same sign.)

For let $b_1, b_2, \cdots\cdots, b_n, \cdots\cdots$ be the positive terms of the series $\sum a_n$ and $c_1, c_2, \cdots\cdots, c_n, \cdots\cdots$ its negative terms. To construct a new series, we arrange the terms of the series $\sum a_n$ as follows:

If a_1 be positive, take c_1 as the first term of the new series; if a_1 be negative, instead of c_1, take b_1 as the first term of the new series. Suppose for simplicity $a_1 > 0$ and we write $c_1 = a_1'$. If a_2 be positive, take c_2 as the second term of the new series; if otherwise b_1 the second term. Suppose for simplicity $a_2 < 0$ and we write $b_1 = a_2'$. We proceed in this way and obtain a new series

$$a_1' + a_2' + \cdots\cdots + a_n' + \cdots\cdots$$

by the rearrangement of the terms of the series $\sum a_n$, where a_n and a_n'

have the different signs. The rearrangements is a substitution and hence the series

$$(a_1-a_1')+(a_2-a_2')+\cdots\cdots+(a_n-a_n')+\cdots\cdots$$

should be by the hypothesis absolutely convergent. That is impossible in so far as the given series $\sum a_n$ is not absolutely convergent, for by the construction

$$|a_1-a_1'|+|a_2-a_2'|+\cdots\cdots+|a_n-a_n'|+\cdots\cdots=2\sum|a_n|.$$

To conclude these, *the necessary and sufficient condition that a series is absolutely convergent is that the series admits the symmetric group*, or *the group of a semi-convergent series is a divisor of the symmetric group.*

Certain series admit the same group. For example the series $\sum u_n$ and $\sum(\lambda u_n+a_n)$ admit the same group, where λ is an arbitrary constant and $\sum a_n$ is an absolutely convergent series. From this we have the following theorem: *Given two series $\sum u_n$ and $\sum v_n$, if we can find a constant μ such that the series $\sum(u_n+\mu v_n)$ be absolutely convergent, then the series $\sum u_n$ and $\sum v_n$ admit the same group.* For in this case the series $\sum u_n$ and $\sum\{-u_n+(u_n+\mu v_n)\}=\sum\mu v_n$ admit the same group, *i. e.*, $\sum u_n$ and $\sum v_n$ admit the same group.

Under the same condition, consider the limit

$$\lim_{n\to\infty}\frac{u_n}{v_n}.$$

If there be a partial sequence $\frac{u_m'}{v_m'}$, $(m=1,\ 2,\cdots)$ *of the sequence* $\frac{u_n}{v_n}$, $(n=1,\ 2,\cdots\cdots)$ *such that for any given positive number however small, we have*

$$\left|\frac{u_m'}{v_m'}+\mu\right|\geqq\varepsilon$$

then the partial series $\sum u'_m$ and $\sum v_m'$ of the series $\sum u_n$ and $\sum v_n$ respectively are absolutely convergent. For put

$$u_n+\mu v_n\equiv a_n,\quad n=1,\ 2,\cdots\cdots,$$

then by the hypothesis the series $\sum a_n$ is absolutely convergent. Hence writing

$$u_m' + \mu v_m' \equiv a_m',$$

we have

$$\frac{u_m'}{v_m'} + \mu = \frac{a_m'}{v_m'},$$

and by the assumption, we have

$$\left| \frac{a_m'}{v_m'} \right| \geqq \varepsilon.$$

Since $\sum a_m'$ is a partial series of $\sum a_n$ and any partial series of it must be absolutely convergent, the partial series $\sum v_m'$ is absolutely convergent. Consequently the partial series $\sum u_m'$ is also absolutely convergent.

From this it follows that one of the limiting points of the set of numbers $\frac{u_n}{v_n}$, $(n=1, 2, \cdots\cdots)$ must be $-\mu$.

We remark that even when the terms of the series $\sum u_n$ and $\sum v_n$ satisfy the condition

$$\lim_{n \to \infty} \frac{u_n}{v_n} = -\mu \neq 0,$$

$\sum u_n + \mu v_n$ may not be absolutly convergent. For example take

$$u_n = (-1)^n \frac{1}{n}, \quad v_n = (-1)^n \frac{1}{n}\left(1 + \frac{1}{\log n}\right), \quad n = 2, \ 3, \cdots\cdots,$$

then

$$\lim_{n \to \infty} \frac{u_n}{v_n} = 1, \quad \mu = -1.$$

But the series

$$\sum (u_n + \mu v_n) = \sum (-1)^{n-1} \frac{1}{n \log n}$$

is not absolutely convergent.

On the contrary let

$$\sum u_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots,$$

$$\sum v_n = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots\cdots$$

then $$\sum(u_n-v_n)=\frac{1}{2}-\frac{1}{3\cdot4}+\frac{1}{5\cdot6}-\frac{1}{7\cdot8}+\cdots\cdots$$

is absolutely convergent, hence the series $\sum u_n$ and $\sum v_n$ admit the same group and we notice that

$$\lim_{n\to\infty}\frac{u_n}{v_n}=1.$$

7. We assume as usual the semi-convergent series

$$\sum_{n=1}^{\infty}u_n=u_1+u_2+\cdots\cdots+u_n+\cdots\cdots$$

admits the substitution

$$S=\begin{pmatrix}1 & 2 & \cdots\cdots & n & \cdots\cdots\\ s_1 & s_2 & \cdots\cdots & s_n & \cdots\cdots\end{pmatrix}$$

then the series

$$\sum_{n=1}^{\infty}(u_n-u_{s_n})=(u_1-u_{s_1})+(u_2-u_{s_2})+\cdots\cdots+(u_n-u_{s_n})+\cdots\cdots$$

is absolutely convergent. Now put

$$\theta_n\equiv\left(1-\frac{u_{s_n}}{u_n}\right),\quad n=1,\ 2,\cdots\cdots$$

which we call the *coefficients of substitution* S (multiplied into u_n), then we have

$$\sum(u_n-u_{s_n})=\sum\theta_n u_n$$

where the series $\sum|\theta_n||u_n|$ is convergent. Therefore *the inferior limit of the coefficients of substitution θ_n for $n\to\infty$ must be zero.* For if there be a positive number ε, such as

$$\lim_{n\to\infty}|\theta_n|\geqq\varepsilon,$$

we should have

$$\sum|u_n-u_{s_n}|\geqq\varepsilon\sum|u_n|$$

which is impossible, for the series $\sum|u_n|$ is divergent.

From this it follows that *there is an infinite number of pairs* (u_n, u_{s_n}) *where u_n and u_{s_n} have the same sign.* Therefore if there be only a finite number of pairs (u_n, u_{s_n}) where u_n and u_{s_n} have the same sign, the series $\sum u_n$ can not admit the substitution S. Moreover *the partial series formed by some of u_n whose coefficients of substitution θ_n is greater than in absolute value than any positive number however small is absolutely convergent.* For let θ_m', $(m=1, 2, \cdots\cdots)$ be a partial sequence of the sequence θ_n, $(n=1, 2, \cdots\cdots)$ such that

$$|\theta_m'| \geqq \varepsilon,$$

then the partial series $\sum \theta_m' u_m'$ of the series $\sum \theta_n u_n$ is absolutely convergent where u_n' mean the terms u_n which correspond to θ_m'. But since

$$\sum |\theta_m' u_m'| \geqq \varepsilon \sum |u_m'|,$$

the partial series $\sum u_m'$ of the series $\sum u_n$ is absolutely convergent.

From this it follws that if there be the terms u_n such that u_n and u_{s_n} have the different signs, the partial series formed by such terms u_n is absolutely convergent. For in sush a case the corresponding coefficients are not less than unity, in absolute value.

8. *Given a semi-convergent series $\sum u_n$ we can determine a number ρ such that for any given positive number ε however small, the series $\sum |u_n|^{1+\rho+\varepsilon}$ is convergent while the series $\sum |u_n|^{1+\rho-\varepsilon}$ is divergent. ρ is a number positive or zero, but sometimes ∞.*

To prove this, rearrange the series $\sum |u_n|$ in the order of the magnitude of its terms. Let us denote it by $\sum a_n$. For a number x, the series $\sum a_n^{1+x}$ is convergent or divergent. Suppose it be convergent, then for any number y greater than x, the series $\sum a_n^{1+y}$ is convergent. The lower limit ρ of such numbers is the required. Since the series $\sum a_n$ is divergent, ρ must be positive or zero. If for any number x however great the series $\sum a_n^{1+x}$ be divergent, then ρ is infinite. In any case ρ is called the *exponent of (absolute) convergency.*

For the series

$$\sum (-1)^{n-1} \frac{1}{n}, \qquad \rho = 0,$$

$$\sum (-1)^{n-1} \frac{1}{\sqrt{n}}, \quad \rho = 1,$$

$$\sum (-1)^{n} \frac{1}{\log n}, \quad \rho = \infty.$$

For the last series, since $\lim\limits_{n\to\infty} \dfrac{(\log n)^{1+x}}{n} = 0$ for any number x, we have for n sufficiently great,

$$\frac{1}{n} < \frac{1}{(\log n)^{1+x}}$$

which shows $\rho = \infty$.

The exponent of convergency is sometimes useful, For if we choose the pairs of terms (u_n, u_{s_n}) such that the coefficients of substitution satisfy the relation, ρ being finite,

$$|\theta_n| = o(|u_n|^{\rho+\varepsilon}),$$

the series $\sum(u_n - u_{s_n})$ is absolutely convergent. Therefore the series $\sum u_n$ will admit the substitution $S = \begin{pmatrix} n \\ s_n \end{pmatrix}$.

We remark that when $x < \rho$, the series $\sum u_n |u_n|$ is not necessarily convergent. For example consider the convergent series

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \cdots\cdots.$$

From it we construct the series, p and q being different integers,

$$\underbrace{\frac{1}{p\sqrt{1}} + \cdots + \frac{1}{p\sqrt{1}}}_{p \text{ terms}} - \underbrace{\frac{1}{q\sqrt{1}} - \cdots - \frac{1}{q\sqrt{1}}}_{q \text{ terms}} + \underbrace{\frac{1}{p\sqrt{2}} + \cdots + \frac{1}{p\sqrt{2}}}_{p \text{ terms}}$$

$$\underbrace{- \frac{1}{q\sqrt{2}} - \cdots - \frac{1}{q\sqrt{2}}}_{q \text{ terms}} + \cdots\cdots$$

which is clearly convergent. But the series

$$\frac{1}{(p\sqrt{1})^{1+x}} + \cdots + \frac{1}{(p\sqrt{1})^{1+x}} - \frac{1}{(q\sqrt{1})^{1+x}} - \cdots - \frac{1}{(q\sqrt{1})^{1+x}} + \frac{1}{(p\sqrt{2})^{1+x}}$$

$$+ \cdots + \frac{1}{(p\sqrt{2})^{1+x}} - \frac{1}{(q\sqrt{2})^{1+x}} - \cdots - \frac{1}{(q\sqrt{2})^{1+x}} + \cdots\cdots$$

is not convergent in so far as $x \leqslant 1$. For its general terms are

$$\frac{1}{(p\sqrt{n})^{1+x}} + \cdots + \frac{1}{(p\sqrt{n})^{1+x}} - \frac{1}{(q\sqrt{n})^{1+x}} - \cdots - \frac{1}{(q\sqrt{n})^{1+x}}$$

$$= \left(\frac{1}{p^x} - \frac{1}{q^x}\right) \frac{1}{(\sqrt{n})^{1+x}}.$$

Hence the series is equal to the series

$$\left(\frac{1}{p^x}-\frac{1}{q^x}\right)\sum\frac{1}{(\sqrt{u})^{1+x}}$$

which is divergent in so far as $x\leqq 1$.

9. In the following for simplicity instead of $u_n|u_n|^x$, we write u_n^{1+x}, *i. e.*, u_n^{1+x} has the same sign with u_n and its absolute value is $|u_n|^{1+x}$.

If the exponent of convergency ρ of the series $\sum u_n$ be a positive number and the series $\sum u_n^{1+x}$ be convergent, where $0<x<\rho$, then the series $\sum u_n^{1+x}$ admits the group of the series $\sum u_n$.

Let $S=\binom{n}{s_n}$ be any substitution of the group of the series $\sum u_n$, then the series $\sum(u_n-u_{s_n})$ is absolutely convergent. Consider the ratio

$$f\equiv\frac{u_n^{1+x}-u_{s_n}^{1+x}}{u_n-u_{s_n}}>0,$$

(1) If u_n and u_{s_n} have different signs, excepting a finite number of terms, we have

$$|u_n^{1+x}|<|u_n|,\quad |u_{s_n}^{1+x}|<|u_{s_n}|$$

therefore f is less than unity.

(2) If u_n and u_{s_n} have the same sign, at first consider such coefficients of substitution θ_n which satisfy the inequalities

$$0<\varepsilon\leqq|\theta_n|\leqq g,$$

where ε is less than unity and g greater than unity. By the relation

$$1-\frac{u_{s_n}}{u_n}=\theta_n,\ or\ \frac{u_{s_n}}{u_n}=1-\theta_n>0,$$

we have

$$f=|u_n|^x\,\frac{1-(1-\theta_n)^{1+x}}{\theta_n}.$$

Since $u_n\rightarrow 0$, we have

$$f<\frac{1+(1+g)^{1+x}}{\varepsilon},$$

i. e., f is less than a number.

Secondly consider the coefficients such as $|\theta_n| < \varepsilon$. Put $1 - \theta_n \equiv \eta_n$, then η_n is positive and we have

$$f = |u_n|^x \frac{1 - \eta_n^{1+x}}{1 - \eta_n}$$

Let m be the positive integer next greater than the integral part of x, we have

$$f < 1 + \eta_n + \cdots\cdots \eta_n^m .$$

Since $\eta_n < 1 + \varepsilon$, we have

$$f < 1 + (1 + \varepsilon)^m,$$

i. e., f is less than a number.

Thirdly consider the case where $|\theta_n| > g$. In this case since g is greater than unity and $1 - \theta_n$ is positive, the coefficients θ_n must be negative; hence

$$-\theta_n > g, \qquad \frac{u_{s_n}}{u_n} = 1 - \theta_n > 1 + g, \qquad \frac{u_n}{u_{s_n}} < \frac{1}{1+g} .$$

Now

$$f = |u_{s_n}|^x \frac{\left(\dfrac{u_n}{u_{s_n}}\right)^{1+x} - 1}{\dfrac{u_n}{u_{s_n}} - 1} < \frac{1}{1 - \dfrac{1}{1+g}}$$

or

$$f < \frac{1+g}{g} ,$$

i. e., f is less than a number. (If $u_n = u_{s_n}$, then $u_n^{1+x} - u_{s_n}^{1+x} = 0$ which is trivial.)

Therefore we have in all the cases

$$|u_n^{1+x} - u_{s_n}^{1+x}| = O(|u_n - u_{s_n}|)$$

which shows that the series $\sum u_n^{1+x}$ admits the substitution S and hence the group of the series $\sum u_n$.

We assumed in the above theorem that the positive number x is less than the exponent of the series. But the theorem is also true for

$x \leqq \rho$. For $x > \rho$, the hypothesis upon the series $\sum u_n^{1+x}$ is unnecessary. In this case since the series $\sum u_n^{1+x}$ is absolutely convergent, it admits the symmetric group, (N° 6), *a fortiori* the group of the series $\sum u_n$. For $x = \rho$, the condition of convergency of the series $\sum u_n^{1+\rho}$ is necessary.

Consider for example the series $\sum u_n$ whose terms are monotone decreasing in absolute value. Then by Abel's theorem the series $\sum u_n |u_n|^x$ where x is positive, is also convergent, for the factors $|u_n|^x$ are positive and decrease monotonely to zero. Therefore the series $\sum u_n^{1+x}$ admits the group of the series $\sum u_n$. This result will be applied for the criterion of convergency. For the seriss $\sum u_{s_n}{}^{1+x}$, transformed of the series $\sum u_n^{1+x}$ by a substitution $S = \binom{n}{s_n}$ must be convergent if the series $\sum u_n$ admits the substitution S.

10. Consider as usual the semi-convergent series $\sum u_n$ and the substitution $S = \binom{n}{s_n}$. We write the coefficients of substitution θ_n as follows:

$$|\theta_n| \equiv |u_n|^{\omega_n}, \quad n = 1, \ 2, \cdots\cdots\cdots$$

and we call ω_n the *exponents of substitution*. Accordingly we have

$$\sum (u_n - u_{s_n}) = \sum \theta_n u_n = \sum \pm u_n^{1+\omega_n}.$$

Since u_n tends to zero for $n \to \infty$, *if the series* $\sum u_{s_n}$ *be convergent, the number of the exponents of substitution* ω_n *which are not greater than* -1 *must be finite, i. e., the inferior limit of* ω_n *must be greater or equal to* $-1+0$. For under the condition $u_n^{1+\omega_n}$ must tend to zero for $n \to \infty$.

If the series $\sum u_n$ *admits the substitution* S, *the superior limit of the exponents of substitution must not be less than the exponent of convergency of the series* $\sum u_n$. For by the definition of the exponent of convergency ρ, the series $\sum u_n^{1+\rho-\varepsilon}$ is not absolutely convergent where ε is any positive number however small. (N° 8) Hence if the superior limit of the exponents of substitution ω_n be less than ρ, then except some finite number of terms, we have

$$\sum |u_n - u_{s_n}| = \sum |u_n|^{1+\omega_n} > \sum |u_n|^{1+\rho-\varepsilon}$$

which is contrary to the assumption that the series $\sum u_n$ admits the substitution S.

For example consider the series $\sum(-1)^{n-1}\frac{1}{n}$ whose exponents of convergency $\rho=0$ and the substitution

$$S=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7\cdots\cdots \\ 3 & 2 & 1 & 4 & 7 & 6 & 5\cdots\cdots \end{pmatrix}$$

which is admitted by the series. Here

$$\frac{1}{4n+1}-\frac{1}{4n+3}=\left(\frac{1}{4n+1}\right)^{1+\omega_{4n+1}}$$

or

$$\omega_{4n+1}=\frac{\log(4n+3)-\log 2}{\log(4n+1)}.$$

In the same way

$$\omega_{4n+3}=\frac{\log(4n+1)-\log 2}{\log(4n+3)};$$

but since the terms of the even order are not replaced, it may be designated by

$$\omega_{2n}=\infty.$$

Thus the inferior limit of the exponents of substitution is unity while the superior limit is ∞; both limits being greater than the exponent of convergency.

11. *Given a number of substitutions, if the inferior limit of the exponents of each substitution be greater than zero, then those substitutions form a group. This group is a divisor of the group of the series.*

(I) Let $S=\begin{pmatrix} n \\ s_n \end{pmatrix}$ be any one of the given substitution. We put

$$u_n-u_{s_n}=\pm u_n^{1+\lambda_n}, \quad n=1,\ 2,\ \ldots\ldots\ldots(1)$$

Let $T=\begin{pmatrix} n \\ t_n \end{pmatrix}$ be another one of the given substitutions, then we may write (N° 4)

$$T=\begin{pmatrix} s_n \\ r_n \end{pmatrix}.$$

Put $$u_{s_n}-u_{r_n}=\pm u_{s_n}^{1+\mu_n}, \quad n=1,\ 2,\ \ldots\ldots(2)$$

then the sequence μ_n ($n=1, 2, \cdots\cdots$) is nothing but the sequence of the exponents of the substitution T. We have by the hypothesis

$$\varliminf_{n\to\infty} \lambda_n > 0, \quad \varliminf_{n\to\infty} \mu_n > 0,$$

i. e., except some finite number, λ_n, μ_n are greater than a positive number, say $\varkappa$.

Let ω_n ($n=1, 2, \cdots\cdots$) be the exponents of the product ST, then we have

$$u_n - u_{r_n} = \pm u_n^{1+\omega_n}, \quad n=1, 2, \cdots\cdots.$$

Now we want to prove that the inferior limit of ω_n is also positive By (1) we have

$$u_{s_n} = u_n \mp u_n^{1+\lambda_n}.$$

Hence by (2) we have

$$u_n - u_{r_n} = \pm u_n^{1+\lambda_n} \pm u_{s_n}^{1+\mu_n}$$
$$= \pm u_n^{1+\lambda_n} \pm (u_n \mp u_n^{1+\lambda_n})^{1+\mu_n} \; \ldots\ldots (3)$$

At first consider the exponents such as

$$\varkappa < \lambda_n \leqq g, \quad \varkappa < \mu_n \leqq h,$$

where g and h are any positive number however great. By (3) we have

$$|u_n - u_{r_n}| < |u_n|^{1+\lambda_n} + |u_n|^{1+\mu_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n}.$$

Hence if $\lambda_n \leqq \mu_n$,

$$|u_n - u_{r_n}| < |u_n|^{1+\lambda_n}\{1 + |u_n|^{\mu_n - \lambda_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n}\}.$$

Since λ_n and μ_n are greater than $\varkappa > 0$ and u_n tends to zero for $n \to \infty$, the second factor on the right lies between 1 and 3. Hence if we put

$$|u_n|^{\delta_n} \equiv 1 + |u_n|^{\mu_n - \lambda_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n},$$

we have $\qquad \omega_n > \lambda_n + \delta_n,$

where δ_n is negative but tends to zero. Therefore $\varepsilon > 0$ being given ($\varkappa > \varepsilon$), for sufficiently great n, say $n > N_1$,

$$\omega_n > \varkappa - \varepsilon > 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

When $\lambda_n > \mu_n$, we obtain the same result interchanging simply λ_n and μ_n in the above discussion.

Secondly consider the exponents such as

$$\lambda_n > g, \quad \varkappa < \mu_n \leqslant h_1,$$

where we assume $g > h_1$, $h_1 \geqslant h$.

By (3) we have

$$|u_n - u_{r_n}| < |u_n|^{1+\lambda_n} + |u_n|^{1+\mu_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n}$$

$$= |u_n|^{1+\mu_n}\{|u_n|^{\lambda_n-\mu_n} + (1 + |u_n|^{\lambda_n})^{1+\mu_n}\} \cdots (5)$$

Since $\lambda_n > g > h_1 \geqslant \mu_n$ and u_n tends to zero with n, $|u_n|^{\lambda_n-\mu_n}$ tends to zero with n. On the other hand since $|u_n|^{\lambda_n} < |u_n|^{\mu_n}$, we have

$$(1 + |u_n|^{\lambda_n})^{1+\mu_n} = \left(\frac{1 + |u_n|^{\lambda_n}}{1 + |u_n|^{\mu_n}}\right)^{1+\mu_n}(1 + |u_n|^{\mu_n})^{1+\mu_n}$$

$$< (1 + |u_n|^{\mu_n})^{1+\mu_n}$$

which tends to unity for $n \to \infty$, for μ_n are finite and positive, *i. e.*, the second factor on the right of the inequality (5) is greater than unity and tends to it for $n \to \infty$. Hence we conclude quite in the same way that for sufficiently great n, say $n > N_2$,

$$\omega_n > \varkappa - \varepsilon. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Thirdly consider the exponents such as

$$\varkappa < \lambda_n \leqslant g_1, \quad \mu_n > h,$$

where we assume $g_1 < h$, $g \leqslant g_1$. The inequality

$$|u_n - u_{r_n}| < |u_n|^{1+\lambda_n}\{1 + |u_n|^{\mu_n-\lambda_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n}\}$$

is also valid. Under the condition of λ_n, since $|u_n|^{\lambda_n}$ tends to zero for $n \to \infty$,

$$(1 + \mu_n)\log(1 + |u_n|^{\lambda_n}) < (1 + \mu_n)\ |u_n|^{\lambda_n}.$$

Consequently we have

$$|u_n|^{\mu_n-\lambda_n}(1 + |u_n|)^{1+\mu_n} < |u_n|^{\mu_n-\lambda_n} e^{(1+\mu_n)|u_n|^{\lambda_n}}$$

$$= (|u_n| e)^{\mu_n-\lambda_n} e^{(1+\mu_n)|u_n|^{\lambda_n} - \mu_n + \lambda_n}.$$

For n sufficiently great

$$|u_n| e < 1,$$

while the exponent of the second factor is

$$\lambda_n + |u_n|^{\lambda_n} - (1 - |u_n|^{\lambda_n})\mu_n$$

which becomes negative when n increases, for $|u_n|^{\lambda_n}$ tends to zero for $n \to \infty$, and $\mu_n > h > g_1 \geqq \lambda_n$. Hence we have

$$\lim_{n\to\infty} |u_n|^{\mu_n - \lambda_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n} = 0.$$

Using the same notation as the first case, we have

$$\omega_n > \lambda_n + \delta_n,$$

where δ_n is negative but tends to zero, or for sufficiently great n, say $n > N_3$,

$$\omega_n > \varkappa - \varepsilon \quad \text{..............................} \quad (7)$$

Fourthly consider the exponents such as

$$\lambda_n > g, \qquad \mu_n > h.$$

By (3) we have

$$|u_n - u_{r_n}| < |u_n|^{1+\lambda_n} + |u_n|^{1+\mu_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n}.$$

Here we have

$$(1 + |u_n|^{\lambda_n})^{1+\mu_n} < e^{|u_n|^{\lambda_n}(1+\mu_n)}.$$

Therefore

$$|u_n|^{1+\mu_n}(1 + |u_n{}^{\lambda_n}|)^{1+\mu_n} < e^{(1+\mu_n)\log|u_n| + |u_n|^{\lambda_n}(1+\mu_n)}.$$

The exponents on the right is equal to

$$(1+\mu_n)\log|u_n| \cdot \left\{1 + \frac{|u_n{}^{\lambda_n}}{\log|u_n|}\right\}.$$

If we put

$$-\eta_n \equiv \frac{|u_n|^{\lambda_n}}{\log|u_n|},$$

since $\log|u_n|$ tends to $-\infty$ and $|u_n|^{\lambda_n}$ to zero with n, η_n is positive and tends to zero with n. Now the exponent is equal to

$$(1+\mu_n)(1-\eta_n)\log|u_n|,$$

therefore we have

$$|u_n|^{1+\mu_n}(1 + |u_n|^{\lambda_n})^{1+\mu_n} < |u_n|^{(1+\mu_n)(1-\eta_n)}.$$

Now the exponent on the right is

$$(1+\mu_n)(1-\eta_n)=1+\mu_n\left(1-\eta_n-\frac{\eta_n}{\mu_n}\right).$$

Since $\mu_n>h$ where h is positive and great, and η_n tends to zero for $n\to\infty$, for n sufficiently great, say $n>N_4$,

$$1-\eta_n-\frac{\eta_n}{\mu_n}>\frac{1}{2}.$$

Consequently for $n>N_4$,

$$|u_n-u_{r_n}|<|u_n|^{1+\lambda_n}+|u_n|^{1+\frac{\mu_n}{2}}.$$

Therefore we have

$$|u_n-u_{r_n}|<2|u_n|^{1+\lambda_n},$$

or

$$|u_n-u_{r_n}|<2|u_n|^{1+\frac{\mu_n}{2}};$$

hence we have according to the case, for $n>N_5$, $(N_5\geq N_4)$,

$$\omega_n>g-\varepsilon>\varkappa-\varepsilon,$$

or

$$\omega_n>\frac{h}{2}-\varepsilon>\varkappa-\varepsilon. \qquad \text{...............(8)}$$

(We may assume from the beginning $h>2\varkappa$.)

Concluding these four cases, let N be the greatest of N_1, N_2, N_3, N_5, then by (4), (6), (7), (8) for any given positive number ε however small, we have

$$\omega_n>\varkappa-\varepsilon, \textit{ for all } n>N;$$

hence the inferior limit of the exponents of the substitution ST is greater than zero *i. e.*, the product of the substitutions S and T belongs to the same category as S and T.

(II) Again let $S=\begin{pmatrix} n \\ s_n \end{pmatrix}$ be any one of the given substitutions and put

$$u_n-u_{s_n}=\pm u_n^{1+\lambda_n}, \qquad n=1,\ 2,\text{.........}(9)$$

Put for simplicity

$$u_{s_n} - u_n = \pm u_{s_n}^{1+\omega_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where ω_n are the exponents of the inverse substitution S^{-1}. Now we want to prove that the inferior limit of ω_n is also positive.

By (9) we have

$$|u_{s_n}| > |u_n|(1 - |u_n|^{\lambda_n}) \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Since the inferior limit of λ_n is positive, we may take

$$\lambda_n > \varkappa > 0$$

excepting some finite number of the exponents. Moreover u_n tends to zero for $n \to \infty$. Therefore we have for sufficiently great n, say $n > N$,

$$1 - |u_n|^{\lambda_n} > 0,$$

$$-(1+\lambda_n)\log(1 - |u_n|^{\lambda_n}) < \frac{|u_n|^{\lambda_n}}{1 - |u_n|^{\lambda_n}}(1+\lambda_n).$$

The term on the right of the second inequality is clearly finite (even when $\lambda_n \to \infty$). Therefore we can find a positive number M such that for $n > N$,

$$\frac{|u_n|^{\lambda_n}}{1 - |u_n|^{\lambda_n}}(1+\lambda_n) < M.$$

Consequently for $n > N$,

$$(1 - |u_n|^{\lambda_n})^{1+\lambda_n} > e^{-M}.$$

Therefore, since by (9) and (10) $|u_n|^{1+\lambda_n} = |u_{s_n}|^{1+\omega_n}$, we have by (11)

$$|u_{s_n}|^{1+\lambda_n} > |u_{s_n}|^{1+\omega_n}(1 - |u_n|^{\lambda_n})^{1+\lambda_n}$$
$$> |u_{s_n}|^{1+\omega_n} e^{-M}$$

Hence
$$\lambda_n + \frac{M}{\log|u_{s_n}|} < \omega_n.$$

Therefore given a positive number ε however small, we can find $N_1 (N_1 \geqslant N)$ such that

$$\omega_n > \varkappa - \varepsilon, \text{ for all } n > N_1.$$

Hence the inferior limit of the exponents of the inverse substitution S^{-1} is greater than zero *i. e.*, the inverse substitution of the substitution S belongs to the same category as S.

Concluding (I) and (II), given a number of substitutions, if the inferior limit of the exponents of each substitution be greater than zero, then those substitutions form a group. Since all these substitutions are contained in the group of the series $\sum u_n$, the group formed by these substitutions is a divisor of the group of the series.

12. *Let* $P=\begin{pmatrix} n \\ p_n \end{pmatrix}$ *be any substitution,* $S=\begin{pmatrix} n \\ s_n \end{pmatrix}$ *be any one of the substitutions of the group* Γ *of the series* $\sum u_n$, *the series* $\sum u_{p_n}$ *has the same group* Γ, *provided the series* $\sum u_{p_n}$ *be convergent.*

As we have said above, without changing the effect, we may write the substitution S as follows:

$$S=\begin{pmatrix} p_1 & p_2 \cdots\cdots p_n \cdots\cdots \\ r_1 & r_2 \cdots\cdots r_n \cdots\cdots \end{pmatrix}$$

where if $p_n=m$, then $r_n=s_m$ and conversely. On the other hand, since by the hypothesis, the series $\sum(u_n-u_{s_n})$ is absolutely convergent, we may rearrange its terms as we please without changing its property. Now we arrange the terms of the series $\sum(u_n-u_{s_n})$ so as the first numbers in the brackets shall be $u_{p_1}, u_{p_2}, \cdots\cdots, u_{p_n}, \cdots\cdots$ then we have

$$\sum(u_n-u_{s_n})=(u_{p_1}-u_{r_1})+(u_{p_2}-u_{r_2})+\cdots\cdots+(u_{p_n}-u_{r_n})+\cdots\cdots$$

which is absolutely convergent; moreover we have using the notation of N° 4

$$\sum u_{p_n} S=\sum u_{r_n},$$

provided the series $\sum u_{p_n}$ be convergent.

Therefore the series $\sum u_{p_n}$ admits the substitution S.

Conversely let S be any substitution admitted by the series $\sum u_{p_n}$ then clearly the series $\sum u_n$ admits the substitution S; hence S is contained in the group Γ. Hence the group of the series $\sum u_{p_n}$ is also Γ.

If in the preceding theorem we write

$$u_{p_n}=v_n, \quad n=1, 2, \cdots\cdots\cdots,$$

the group of the series $\sum v_n$ is $P\Gamma P^{-1}$. For if $S=\begin{pmatrix} n \\ s_n \end{pmatrix}$ be a substitution of the group Γ, as before it may be written as follows

$$S=\begin{pmatrix} p_n \\ r_n \end{pmatrix}.$$

Now if $p_n=m$, then $r_n=s_m$. Find the integer such as $s_m=p_l$. Now $u_{p_n}=v_n$, and by S, u_{p_n} is replaced by u_{r_n}, where $u_{r_n}\equiv u_{s_m}\equiv u_{p_l}$. Hence u_{p_n} is replaced by u_{p_l}. But $u_{p_l}=v_l$, hence by S, v_n is replaced by v_l. On the other hand by P, n is replaced by p_n. By S, p_n is replaced by r_n, where $r_n\equiv s_m\equiv p_l$. Now by P^{-1}, p_l is replaced by l. Hence by PSP^{-1}, n is replaced by l. Therefore the series $\sum v_n$ admits the substitution PSP^{-1}, *i. e.*, the group of the series $\sum v_n$ is $P\Gamma P^{-1}$.

If a substituion P and any substitution S of Γ are such that

$$PSP^{-1}=T, \textit{ or } PS=TP$$

where T is a substitution of Γ, then we say the substitution P and the group are *permutable* which is written as follows:

$$P\Gamma P^{-1}=\Gamma \textit{ or } P\Gamma=\Gamma P.$$

If we write as before

$$u_{p_n}\equiv v_n, \quad n=1, 2,\ldots\ldots\ldots,$$

and if P and Γ be permutable, then the series $\sum u_n$ and $\sum v_n$ have the same group Γ. This case occurs especially when the given substitution is a substitution of the group Γ itself. Now let $S=\begin{pmatrix} n \\ s_n \end{pmatrix}$ be any one of the substitution of the group Γ of the series $\sum u_n$, then writing

$$u_{s_n}\equiv v_n, \quad n=1, 2,\ldots\ldots\ldots,$$

the series $\sum v_n$ has the same group Γ. If we put

$$u_n=v_n+a_n, \quad n=1, 2,\ldots\ldots\ldots,$$

then by the definition of the substitution S, the series $\sum a_n$ is absolutely convergent. This is coincident with a theorem of N° 6.

13. If two series $\sum u_n$ and $\sum v_n$ are such that the series $\sum(u_n-v_n)$ is absolutely convergent, then we say the series $\sum u_n$ and $\sum v_n$ are *equivalent*. We denote it by $\sum u_n\sim\sum v_n$.

Two series which are equivalent to a series are equivalent to each other. For if $\sum u_n \sim \sum v_n$, $\sum u_n \sim \sum w_n$, then $\sum(u_n - v_n)$ and $\sum(u_n - w_n)$ are absolutely convergent; consequently $\sum(v_n - w_n)$ is absolutely convergent *i. e.*, $\sum v_n \sim \sum w_n$.

Let S be any substitution of the group Γ of the series $\sum u_n$, then any series $\sum u_n S$ is equivalent to the series $\sum u_n$. Such series form a class. Let P be any substitution which is not contained in the group Γ, then the series $\sum u_n P$ and $\sum u_n PS$ are equivalent, provided the series are convergent. Such series form another class. The series $\sum u_n$ and $\sum u_n P$ are not equivalent by the hypothesis. Therefore all the series of the former class are not equivalent to any series of the latter class.

Therefore *all the convergent series having the same terms as those of the series $\sum u_n$, excepting their order, can be devided into classes, such that all the series of each class are equivalent to one another, while the series of one class is equivalent to none of the other classes.*

We remark that all the series of one class have the same value, (N° 5) Borel's problem now becomes to find the classes such that the sum of a series of one of the classes is equal to the sum of a series of another class.

14. It is natural to ask whether all the substitutions which do not change the value of a semi-convergent series form a group. But in general this question is answered *negatively*. To prove this we have only to give a series $\sum u_n$ and two substitution P and Q such that

$$\sum u_n = \sum u_n P = \sum u_n Q,$$

but

$$\sum u_n Q \neq \sum u_n PQ.$$

Herr Threlfall[1] gave a simple proof for the possibility of changing the sum of the semi-convergent series of complex numbers. To find the substitutions P, Q we follow his arguments.

Consider a series

$$f = a_1^2 + a_1 + a_2 + a_3^2 + a_3 + a_4 + \cdots\cdots + a_{2n-1}^2 + a_{2n-1} + a_{2n} + \cdots\cdots,$$

where $a_{2n-1} > 0$ $(n = 1, 2, \cdots\cdots)$ and $a_{2n-1} \to 0$ such that

$$\sum_{n=1}^{\infty} a_{2n-1} = \infty.$$

We may take the numbers $a_{2n}(n = 1, 2, \cdots\cdots)$ so as the series f is semi-convergent and its sum is equal to zero. Let P be the substitution which

1. Bedingt convergente Reihen; Math. Zeit., **24**, 212 (1925).

transforms f into

$$fP = a_1 + a_1^2 + a_2 + a_3 + a_3^2 + a_4 + \cdots\cdots + a_{2n-1} + a_{2n-1}^2 + a_{2n} + \cdots\cdots.$$

It is clear that the series f does not chang its value by the substitution P. Hence we have

$$f = fP = 0.$$

Now consider the series of complex numbers

$$F \equiv fP + if = (a_1 + ia_1^2) + (a_1^2 + ia_1) + (a_2 + ia_2) + \cdots\cdots$$
$$+ (a_{2n-1} + ia_{2n-1}^2) + (a_{2n-1}^2 + ia_{2n-1}) + (a_{2n} + ia_{2n}) + \cdots\cdots,$$

then the series is semi-convergent and

$$F = 0.$$

Consider the term $(a_{2n-1} + ia_{2n-1}^2)$, then since $a_{2n-1} > 0$, and $a_{2n-1} \to 0$,

$$\lim_{n\to\infty} \frac{a_{2n-1}^2}{a_{2n-1}} = +0, \quad \text{or } arc\ (a_{2n-1} + ia_{2n-1}^2) \to +0,$$

i. e., there is an infinite number of points in the first quadrant approaching, at the limit, the real axis and the sum of the real part of such terms is ∞ by the condition given above. Now from the first term of the sequence $a_{2n-1} + ia_{2n-1}^2$ $(n = 1, 2, \cdots\cdots)$, we take h_1 terms whose arguments are less than $\dfrac{\pi}{4}$ so that the sum of their real parts becomes greater than unity. We denote those terms by $\alpha_1, \alpha_2, \cdots\cdots, \alpha_{h_1}$, then we have

$$\sum_{n=1}^{h_1} \mathfrak{R}(\alpha_n) > 1.$$

This is always possible, for in any angle containing the real axis within it, there is an infinite number of terms $a_{2n-1} + ia_{2n-1}^2$ whose sum is infinite From the sequence $a_{2n-1} + ia_{2n-1}^2$ $(n = 1, 2, \cdots\cdots)$ after the term α_{h_1} take $h_2 - h_1$ terms in order whose arguments are less than $\dfrac{\pi}{4 \times 2}$ so that the sum of their real parts becomes greater than unity. We denote them by $\alpha_{h_1+1}, \alpha_{h_1+2}, \cdots\cdots, \alpha_{h_2}$, then

$$\sum_{n=h_1+1}^{h_2} \mathfrak{R}(\alpha_n) > 1.$$

We continue this process. Now we have

$$0 < tan(arc\ a_n) < 1,$$
$$arc\ a_n \rightarrow 0,$$
$$\sum_{\nu=1}^{n} \mathcal{R}(a_\nu) \rightarrow \infty.$$

Excepting the terms a_1, a_2,...... we denote the remaining terms of the series F in order by β_1, β_2,...... and let $\varphi(n)$ be the number of terms a_ν standing before β_n, then for $n \rightarrow \infty$,

$$\sum_{\nu=1}^{n} \beta_\nu + \sum_{\nu=1}^{\varphi(n)} a_\nu \rightarrow 0.$$

Next we define the integer $\chi(n)$ such that

$$\sum_{\nu=\varphi(n)+1}^{\chi(n)} \mathcal{R}(a_\nu) > 1,$$

and it converges to unity for $n \rightarrow \infty$. This is always possible by the choice of a_ν and $a_\nu \rightarrow 0$. Next $\varepsilon > 0$ being given, for sufficiently great ν, by the choice of a_ν,

$$|a_\nu - \mathcal{R}(a_\nu)| < \varepsilon \mathcal{R}(a_\nu).$$

Therefore for $n \rightarrow \infty$, $\sum_{\nu=\varphi(n)+1}^{\chi(n)} a_\nu \rightarrow 1$.

Hence for $n \rightarrow \infty$

$$\sum_{\nu=1}^{n} \beta_\nu + \sum_{\nu=1}^{\chi(n)} a_\nu \rightarrow 1,$$

and we arrange in order those terms a_ν before those terms β_ν ($n = 1, 2, ...$). Thus we obtain a new series by the rearrangements of the terms of the given series. Let Q be that substitution. We shall prove the convergency of the new series.

Suppose that among k terms of the new series, there are m terms of a_ν and n terms of β_ν and that

$$\chi(n) \leqslant m \leqslant \chi(n+1).$$

Then the sum of k terms of the seties is equal to

$$\sum_{\nu=1}^{n} \beta_\nu + \sum_{\nu=1}^{\chi(n)} a_\nu + \sum_{\nu=\chi(n)+1}^{m} a_\nu$$

where we have

$$\left|\sum_{\nu=\chi(n)+1}^{m} a_\nu\right| \leqslant \sum_{\nu=\chi(n)+1}^{\chi(n+1)} |a_\nu| < 2 \sum_{\nu=\chi(n)+1}^{\chi(n+1)} \mathfrak{R}(a_\nu).$$

But $\chi(n)$ is so defined that

$$\sum_{\nu=\varphi(n)+1}^{\chi(n)} \mathfrak{R}(a_\nu) \to 1.$$

On the other hand since the original series is convergent

$$\sum_{\nu=\varphi(n)+1}^{\varphi(n+1)} \mathfrak{R}(a_\nu) \to 0.$$

Therefore $\chi(n)$ is greater than $\varphi(n+1)$. Hence by the relations

$$\sum_{\nu=\varphi(n)+1}^{\chi(n)} \mathfrak{R}(a_\nu) = \sum_{\nu=\varphi(n)+1}^{\varphi(n+1)} \mathfrak{R}(a_\nu) + \sum_{\nu=\varphi(n+1)+1}^{\chi(n)} \mathfrak{R}(a_\nu) \to 1,$$

$$\sum_{\nu=\varphi(n+1)+1}^{\chi(n+1)} \mathfrak{R}(a_\nu) = \sum_{\nu=\varphi(n+1)+1}^{\chi(n)} \mathfrak{R}(a_\nu) + \sum_{\nu=\chi(n)+1}^{\chi(n+1)} \mathfrak{R}(a_\nu) \to 1,$$

we conclude that

$$\sum_{\nu=\chi(n)+1}^{\chi(n+1)} \mathfrak{R}(a_\nu) \to 0.$$

Consequently

$$\lim_{n\to\infty} \left|\sum_{\nu=\chi(n)+1}^{m} a_\nu\right| = 0.$$

Therefore the new series converges to unity. Hence we have by the substitution Q applied to the imaginary and real parts of F,

$$fQ=0, \quad fPQ=1,$$

i. e., by PQ the series f changes its value. Consequently *all the substitutions which do not change the value of the series f can not construct a group.*