On the Contact Surface of Fresh- and Salt-Water under the Ground near a Sandy Sea-Shore.

By

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ABSTRACT.

The problem was first investigated theoretically from the hydrodynamical standpoint and then by model experiments to verify the theory obtained.

The result shows that the contact surface of the fresh- and salt-water under the sandy ground near the sea-shore will assume approximately a parabola, which depends only on the densities of the two kinds of water and on the level of the fresh ground-water at a place landward from the shore line, but not on the kind of the sand.

I. Introduction.

One of the remarkable phenomena connected with the underground water of a sandy coast is that, even at a place moderately distant from the shore line, if one dig a well deeper and deeper, one will at first find fresh water, but at last a layer of salt water generally. Of course, this salt water must be sea water which has infiltrated through the pervious soil. Thus we have two kinds of ground-water near a sandy sea-shore: one is the fresh-water layer which flows from inland to the sea, and the other a layer of salt water of sea-origin.

Then the following questions arise:

(1) At what depth can we expect salt-water when we dig a well near a sandy sea-shore?

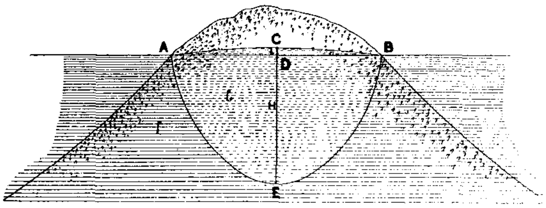
(2) At what distance from the shore line may we find by digging no infiltrating sea-water? Also,

(3) What shape has the contact surface between fresh- and salt-water under the ground?

One of the present authors (Nomitsu), taking a great interest in these problems, inquired first for literature on the subject. In magazines or books on geology and civil engineering, he found reports by several authors, for example A. Herzberg[1], P. Wintgens[2], R. d'Andrimont[3], etc. But most of these reports give only the results of measurements and do not attempt to find a quantitative law applicable to any place generally with theoretical considerations. Only the opinion of Herzberg concerning the nature of the boundary surface between fresh- and salt-water is worthy of a little attention.

After practical investigations of the underground water in Norderney Island, Herzberg assumed[4] that the layer of fresh water in such a sandy

Fig. 1.



island must be in a state of floating over the layer of salt-water, as shown in Fig. 1, and that its boundary surface would maintain hydrostatical equilibrium. So the depth of the contact surface of the fresh- and salt-water would be determined by their densities together with the height of the upper surface of the fresh-water measured from the sea-level, as if the ratio of the upper and the lower part of a floating body from water level can be determined by its specific gravity.

Thus, at the highest point (C) of underground water in Fig. 1, the

1 J. f. Gasbel. u. Wasservers., **44**, 815 (1901).
2 Beitrag zur Hydrologie von Nordholland (1911).
3 Ann. Soc. géol. d. Belgique (1902, 1903, 1905).
4 *cf.* K. Keilhack, Grundwasser- und Quellenkunde, 163 (1914); E. Prinz, Handbuch der Hydrologie, 244 (1923).

following relation must hold:

$$H\rho = (H + h)\rho_0$$

or

$$\frac{H}{h} = \frac{\rho_0}{\rho - \rho_0},$$

where $h = CD =$ the height from the sea-level to the highest point of underground-water,

$H = DE =$ the depth from the sea-level to the deepest point of fresh-water,

$\rho_0 =$ the specific gravity of the fresh water,

$\rho =$ the specific gravity of the sea water.

Then Herzberg compared this formula with the results of actual measurements carried out at Norderney Island and found a fair accordance.

The above view of Herzberg alone, however, is inadequate to answer the questions stated previously. For example, in order to determine the figure of the contact surface by Herzberg's formula only, it is necessary to observe the level of underground water at innumerable points. Hence, Nomitsu investigated the problem more fully from the theoretical stand-point and established a formula for the boundary curve, and then tried to verify it by model experiments with his students (Toyohara and Kamimoto).

II. Theory.

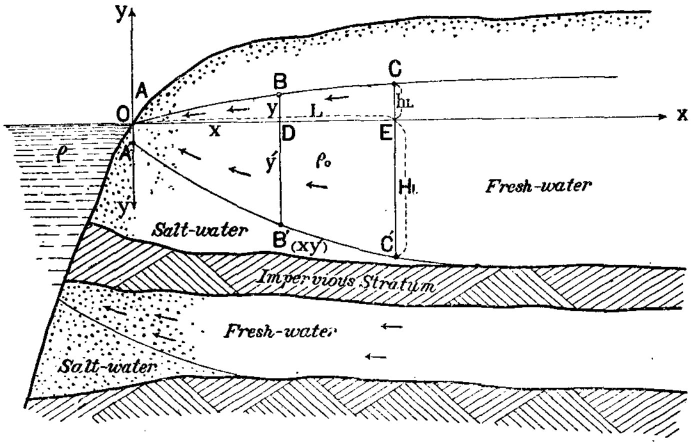
1. Long Straight Coast.

As stated before, Herzberg assumed that fresh ground water maintains a state of statical equilibrium above the salt water, but really it can not. The fresh water under a sandy sea-coast must obviously have its level higher inland and lower and lower toward the sea, and so it must be flowing seawards, though very slowly. Thus, the ground water may be in dynamical equilibrium, but not in statical equilibrium.

First let us consider a long straight coast, under which sea water percolates from only one side of the coast.

As in Fig. 2, take the origin of co-ordinates on the strand-line, and perpendicularly to the latter draw as x-axis a horizontal line landwards. As ordinate line, take y vertically upwards and y′ vertically downwards. It is required to obtain the equations to the curves ABC, and A′B′C′, which are respectively the upper and lower boundaries of the fresh water stratum.

Fig. 2.



Since the motion of the ground water is very slow, usually not greater than a few meters per day, the influence of the vertical motion upon the pressure in the ground water is negligible compared with the effect of the gravity, and the pressure at any point in the ground water can be taken as

Water pressure = The weight of the water column above the point considered,

theoretically and also empirically.

Then, by Darcy's law[1] the equation of the motion of fresh water in the horizontal direction may be written

$$u = -c\frac{dy}{dx}, \quad \text{.............................} \quad (1)$$

where

u = the horizontal velocity of motion,

y = the ordinate of the curve ABC at the place considered,

c = the "transmission constant," which depends on the nature of the soil.

It will be useful to notice here that the horizontal flow will be uniform everywhere throughout a cross-section of the stratum of fresh water as long as Darcy's law is valid.

1 Darcy's law is entirely analogous to the well-known law of Poiseuille for the flow of a liquid through a capillary tube and can be said to have been verified experimentally at least for a comparatively small pressure gradient.

If we denote by Q the quantity of flow per unit time through a cross-section BB' whose breadth is b and area $A=b\times BB'$, then we have

$$Q = nA.u = n.b\,(y + y')\,u, \qquad (2)$$

where

$$y = DB, \qquad y' = DB',$$

and n = "voids" or fraction of the free space unoccupied by the sand grains in the whole volume of the soil.

After the steady state is attained, the equation of continuity for fresh water gives

$$Q = \text{Const. everywhere} \qquad (3)$$

Next, as to the salt water under the ground the circumstances are entirely different from those of fresh water. In the steady state, the salt water may be considered to be in statical equilibrium, i. e., its motion is nil, because as long as the percolation from the sea continues, the amount of the salt water under the fresh increases gradually, causing the boundary between the two kinds of water to rise higher and higher, and the position of the boundary can not be fixed. Thus a relation similar to the equation of Herzberg[1] will apply to all sections, namely

$$\rho \times DB' = \rho_0 \times BB',$$

or

$$\rho\cdot y' = \rho_0\,(y + y').$$

$$\therefore \quad \left.\begin{aligned} y' &= \frac{\rho_0}{\rho-\rho_0}\cdot y \\ \text{and} \quad (y + y') &= \frac{\rho}{\rho-\rho_0}\cdot y \end{aligned}\right\} \qquad (4)$$

where ρ and ρ_0 denote the densities of the salt- and fresh-water respectively. On account of this relation (4), if we know either one of the upper and the lower boundaries of the fresh water, the other also can be determined immediately.

Now, to solve the above differential equations, we first substitute equation (1) in (2) and get

$$\frac{Q}{A} = \frac{Q}{b(y+y')} = nu = -k\frac{dy}{dx}, \qquad (5)$$

where $k = nc$ is the so-called "soil constant" which represents the

1 Loc. Cit.

hydrological nature of the soil, that is, the quantity of water which flows through a unit cross-section of the soil in a unit time when the gradient of the head of water is unity.

Equation (5) combined with equation (4) gives

$$y \cdot dy = -\frac{Q}{kb} \cdot \frac{\rho - \rho_0}{\rho} \cdot dx \qquad (6)$$

Keeping equation (3) in mind, we integrate this and obtain the equation to the curve ABC:

$$y^2 = C - \frac{2Q}{kb} \cdot \frac{\rho - \rho_0}{\rho} \cdot x, \qquad (7)$$

where C is the integration constant.

The curve is thus a parabola, and so it can be completely determined if the level of the ground-water at any two points is actually observed. For instance, suppose that the level of the ground water at a place, distant L from the coast-line, has been measured and we have

$$y = EC = h_L \quad \text{at} \quad x = L.$$

Then the integration constant in eq. (7) must be

$$C = h_L^2 + \frac{2Q}{kb} \cdot \frac{\rho - \rho_0}{\rho} \cdot L \qquad (8)$$

Moreover, suppose that by measuring the ground-water just near the coast-line we find that

$$\text{at } x = o,$$
$$y' = H_0,$$

and
$$y = h_0 = \frac{\rho - \rho_0}{\rho_0} \cdot H_0.$$

The latter, being substituted in equation (7), gives

$$C = h_0^2. \qquad (9)$$

Hence finally the equation to the curve ABC may be written as:

$$y^2 = h_0^2 + \frac{h_L^2 - h_0^2}{L} \cdot x \qquad (10)$$

Here we shall also notice an important relation between k and Q; namely from eqs. (8) and (9),

$$\frac{k}{|Q|} = \frac{2L}{b(h_L^2 - h_0^2)} \cdot \frac{\rho - \rho_0}{\rho}, \qquad (11)$$

where $|Q|$ means the absolute value of Q.

This relation will be useful for the calculation of either the soil constant k or the quantity of flow Q, provided that one of them is known.

Lastly, the equation to the curve A′B′C′ is obtained by substitution of eq. (4) into eq. (10):

$$y'^2 = H_0^2 + \frac{H_L^2 - H_0^2}{L} x, \qquad (12)$$

where we put

$$\left. \begin{aligned} H_0 &= \frac{\rho_0}{\rho - \rho_0} h_0 \\ H_L &= \frac{\rho_0}{\rho - \rho_0} h_L \end{aligned} \right\}, \qquad (13)$$

and these correspond to the depths of the lower boundary of the fresh water at $x = o$ and $x = L$ respectively.

The velocity of flow u is obtained from eqs. (2) and (4):

$$u = \frac{1}{y} \cdot \frac{Q}{nb} \cdot \frac{\rho - \rho_0}{\rho}. \qquad (14)$$

Thus, when steady flow has been attained, u varies inversely as y, and so, strictly speaking, it is not admissible to consider $y = y' = o$ at the shore line as in the figure of Herzberg, because it makes the velocity there infinitely great.

But since h_0^2 and H_0^2 will obviously be very small compared with h_L^2 and H_L^2 respectively, so except in the vicinity of the shore line we may write eqs. (10) and (12) in the forms

$$y^2 \doteqdot \frac{h_L^2}{L} \cdot x \qquad (10_a)$$

and

$$y'^2 \doteqdot \frac{H_L^2}{L} \cdot x, \qquad (12_a)$$

where

$$H_L = \frac{\rho_0}{\rho - \rho_0} . h_L.$$

Thus it is very useful to notice here that, in the above degree of approximation, if we merely fix the level difference h_L at the two ends of the distance L, the intervening curve will take a definite form independent of the size of the grains and the other nature of the soil.

The relation between the soil const. and the quantity of the flow also may be written approximately as

$$\frac{k}{|Q|} \doteqdot \frac{2L}{bh_L^2} \cdot \frac{\rho - \rho_0}{\rho} \doteqdot \frac{2L}{bH_L^2} \qquad (11_a)$$

2. Sandy island of circular form.

In a small sandy island, the sea water percolates under ground from all sides of the island, and over it the fresh water floats as in the conception of Herzberg. When the island is a long narrow one like Norderney, the condition of the ground water will also be just similar to that in the previous case.

Now consider here a circular island of radius R, and suppose that all its fresh groundwater is supplied from a small central region. Then at any section of radius r from the center we must have

$$2y \cdot dy = -\frac{Q}{\pi k} \cdot \frac{\rho-\rho_0}{\rho} \cdot \frac{dr}{r} \quad \ldots\ldots(15)$$

instead of eq. (6).

Starting from this differential equation, we shall get the following results:

Equation for the soil constant.

$$\frac{k}{Q} = \frac{1}{\pi(h_L^2-h_0^2)} \frac{\rho-\rho_0}{\rho} \log\frac{R}{R_L} \cdot \quad (16)$$

Equation for the upper boundary of the fresh-water.

$$y^2 = h_0^2 + (h_L^2 - h_0^2)\frac{\log R/r}{\log R/R_L} \cdot \quad \ldots\ldots\ldots(17)$$

Equation for the lower boundary of the fresh-water.

$$y'^2 = H_0^2 + (H_L^2 - H_0^2)\frac{\log R/r}{\log R/R_L} \quad \ldots\ldots(18)$$

Here we put

h_0 = value of y at the coast line ($r = R$),

h_L = value of y at $r=R_L=R-L$, L being the distance from the coast line,

$H_0 = \frac{\rho_0}{\rho-\rho_0} h_0$ = value of y' at $r=R$,

$H_L = \frac{\rho_0}{\rho-\rho_0} h_L$ = value of y' at $r=R_L$.

As the first approximation, h_0^2 and H_0^2 may be neglected compared with h_L^2 and H_L^2 respectively.

It is easily seen that each of the above equations just coincides with the corresponding one in the foregoing article, if we restrict ourselves to the coastal region only where L is very small compared with R.

3. Horizontal limit of the percolation of sea water.

If the pervious character of the ground were extended to an in-definite depth, we would always reach salt-water by sufficiently deep boring at any place however distant from the coast line. But in the case of real coasts, the thickness of the sandy stratum is not very great, and at a certain depth there is an impervious layer, clayey or rocky for example, and consequently the sea water can not be seen beyond a certain distance landwards from the coast line. This horizontal limit of the percolation of sea water can be estimated if we know the thickness of the pervious stratum. For, let the thickness be H, then it is necessary only to calculate the coastal distance x corresponding to $y'=H$ by eq. (12) or (18).

We must add here one more point to notice. In reality, the ground generally consists of many strata which are alternately pervious and impervious. All the foregoing discussions relate only to the uppermost sandy stratum, but for the lower pervious strata also, similar relations may be obtained with ease.

III. Remarks on the theory.

The above theory, of course, is not absolutely perfect, but only approximately so. Many causes of deviation from reality can be considered, and the chief of them are as follows.

(1) *The local difference of the nature of soil.*

We assumed that the stratum of sand is homogeneous and k is constant throughout the ground, but evidently this is not the case for the natural soil. If there is a part having a much smaller value of k, the level of ground water will be abnormally high just behind that place and abnormally low just before it. If k is very large at some locality, the deviation of the water level will be reversed. Consequently the lower boundary of the fresh-water will also deviate a little from the theoretical curve.

(2) *Effect of the vertical motion in the vicinity of the strand line.*

Since in places pretty far from the strand line the vertical motion of ground-water is, of course, very small compared with the horizontal motion and also with the gravitational acceleration g, it is very proper to neglect the effect of vertical motion. In places very near the strand line, however, the vertical motion of the lower part of fresh-water is rather great, so that it is too rough an approximation to suppose that the hydraulic pressure is equal to the statical one. Hence, in the vicinity of the strand line, we must expect some error in our theoretical equation.

(3) *Confusion of water near the contact surface of the fresh- and salt-water.*

We supposed for simplicity that the fresh- and salt-water do not intermingle and do not impede to slide each other. Strictly speaking, however, this is impossible practically. The current of the lowest part of the fresh water is impeded by the salt water and will have a velocity less than that which ought properly to be produced by the pressure gradient, so that the horizontal current will not be uniform everywhere on an intersectional surface. Similarly, the salt-water layer can not be in a state of absolute rest even if it has assumed a steady state. Near the contact surface, agitated by the moving fresh water, the salt water will tend to be transported seawards, although it may be very very slowly. By way of compensation, sea-water in the lower part will continue to infiltrate landwards.

Thus, the contact surface of the fresh- and salt-water will be disturbed and many small irregularities will be produced in the curve.

(4) *Effect of the diffusion of salt water.*

Our theory was built on the assumption that the infiltration of salt water into the sand stratum was produced by hydraulic pressure only. But really the diffusion of salt-water also comes into play.

At the contact surface, salt diffuses into the fresh-water in addition to the mixing of water itself from the previous cause (3). So the real boundary surface will somewhat deviate from our theoretical one, and the transition to the fresh from the salt water must be gradual and vague. If the diffusion velocity were tolerably great compared with that of the fresh-water current, the boundary curve would be so much disturbed that our theory would be dealt a fatal blow. Therefore the question of diffusion velocity is extremely important for us, and so we made experimental investigations on this point too, and some of them will be described later.

According to our experiments, fortunately the diffusion velocity is negligibly slow compared with the velocity of the ground-water. In addition to that, the nearer the water comes to the strand, the more compactly the stream lines are concentrated, and so even a rather great amount of diffusion at a place far from the shoreline will correspond to only a slight amount near the strand. Thus, we know that the effect of the diffusion is comparatively small.

At any rate the diffusion in the sand stratum is itself of scientific interest, and moreover it will not be very difficult to treat mathematically the effect of diffusion on the contact surface, so we are intending to treat separately this special problem of diffusion in detail in the near future.

(5) *Effect of the adsorption of salt-water by the sand stratum.*

Another important factor we must consider is the adsorption of salt-water by sand. Will not the salinity of sea-water change during the infiltration through sand? Will there not be a great change in the density of salt-water after passing a long distance through the subsoil of a sandy coast, however small the adsorption may be?

This question seems difficult to answer at first sight. As we stated in (3), however, the salt-water underground is not absolutely at rest even in the steady state but infiltrates slowly from the lower part of the sand stratum and flows out again towards the sea along the contact surface with the fresh water. Thus it continues to circulate perpetually, though very very slowly. Besides this, the action of diffusion exists. For these reasons, the adsorption of salt in a real coast by the sand stratum must have already reached the saturation point in the thousands of years during which it has been going on, and may be put aside from the present problem.

For our model experiments, however, the time of operation was comparatively short, and the question of adsorption is not so simple as it is under natural conditions. Moreover the adsorption of salt by sand has its own interest, so that we have studied it as a special problem separately, the results being ready for publication[1]. According to the result, contrary to our first expectation, the salinity of the salt-water generally increases a little when the water has passed through a stratum of sand, but the amount is so small that we may neglect it in the model experiments described in the present paper.

IV. Model experiments.

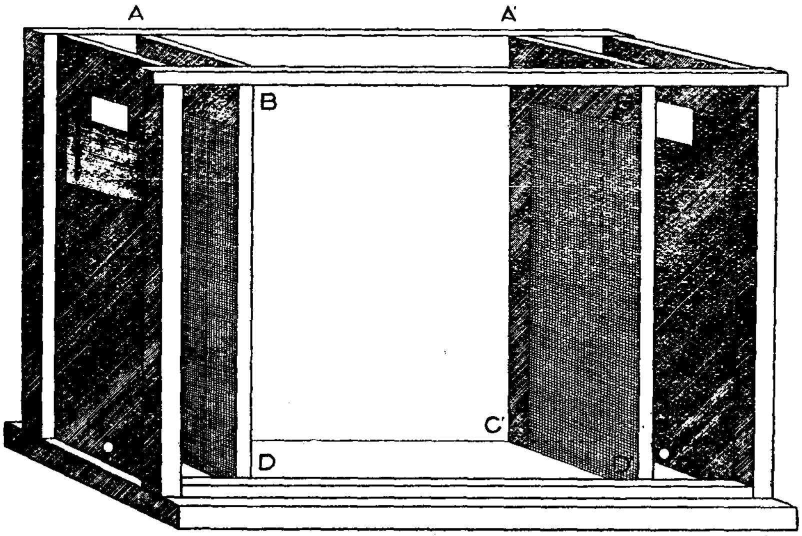
1. Apparatus for experiments.

We constructed two models for experiments, a larger and a smaller. Both are rectangular wooden boxes as shown in Fig. 3. The interior of each box is divided by two sheets of wire-gauze, ABCD and A′B′C′D′, into three compartments, the middle being the compartment for sand, the left for fresh-water and the right for salt-water respectively. The front side BDB′D′ is of glass plate, through which the interior can be seen.

In order to keep the head of water in the two side compartments constant during the experiments, rectangular holes, E and E', were made

1 It will appear in the next number of these memoirs under the title "On the Adsorption of NaCl by Sand."

Fig. 3.



in the ends of the boxes, and over each hole we put a metal plate whose upper edge formed a sharp knife-edge. By displacing the plate up or down, the heights of the edges could be adjusted at will, the edge of the fresh-water compartment, FG, always being a few mm. higher than that of the salt-water compartment, $F'G'$.

The little holes H, H', near the bottom of the ends are for the purpose of drawing out the water from the compartments whenever necessary.

Now the upper boundary of the fresh-water layer in the sand compartment may be seen directly through the front glass. But the curvature will be so flat that, in order to make clear the mathematical nature of the curve as large a model as possible is required. On the other hand, however, when the model is too large water will easily leak out from the rim in which the glass plate is put, and it will be very difficult to prevent it. So, we used our larger model whose inside dimensions were

length 150 cms , breadth 14·8 cms., height 120 cms.,

and distance between the two sheets of wire-gauze (sand compartment) 115 cms.

Secondly, since the contact surface of the fresh- and salt-water is invisible, a particular device is required to make it visible. For this purpose we form on the front glass plate a thin gelatine film which is coloured with red silver-chromate, and which will turn white just at the part where the salt-water touches it. In this case, however, we are obliged to change the glass plate for each experiment, so that the larger model is obviously inconvenient. In addition to this, the lower contact curve will be very conspicuous, the curvature being so large that it is unnecessary to use the larger model. Hence, for determining the lower curve, we used the smaller model whose inside dimensions were

length 42 cms., breadth 9 cms., height 30 cms.

2. Method of experiments and difficulties in them.

To carry out an experiment, fill up the middle compartment of the model with sand. Put in the box some fresh-water from both compartments, right and left, and leave it several minutes. When the water level has come to rest, take it as a datum level of reference and read off, and then draw out all the water from the model.

Now, continuously pour fresh-water into the left and salt-water into the right compartment. The rate of pouring is so adjusted that a little water is always flowing out over the drain-edges, FG and $F'G'$, and therefore both fresh- and salt-water, the head being always kept constant, percolate into the sand room. The difference in the height of the two edges FG and $F'G'$ is previously adjusted in such a way that the end of the contact surface of the fresh- and salt-water will finally reach to about the left end of the bottom of the sand compartment. For this, by the equation

$$h_L = \frac{\rho - \rho_0}{\rho_0} H_L,$$

the level difference h_L must be calculated beforehand using the given densities of the fresh- and salt-water. In order to know the density ρ of the salt-water used, we measure it directly by a hydrometer on the one hand and also calculate it from its temperature and salinity by Knudsen's table on the other hand, and then take the mean of the two results.

After the water has been poured in for 2 or 3 hours, the upper and the lower boundary of the fresh-water in the sand compartment will become fixed in a steady state. But carefully leave it one or two hours more.

Now, in the case of smaller model, we may stop the experiment as it stands, but in the case of the larger, we must read off the heights of the upper surface from the datum level.

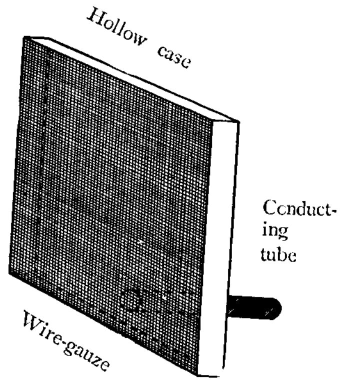
To see whether the height of the water surface may or not show any difference at the face of glass plate and at the interior of the sand stratum, a number of glass tubes were inserted as pressure guages along both the face of glass plate and the middle line of the sand compartment. We found that the two coincided well with each other.

Besides the general disturbances described already in III, there occur some difficulties peculiar to our model experiments:

(a) *Fresh-water percolating into the salt-water compartment.*

As some fresh-water percolates into the upper part of the salt-water compartment, there is ground for anxiety about its influence on the concentration of the salt-water. At first, we tried putting a water receiver (Fig. 4) at the upper part of the right end of the sand compartment, intending to draw out the fresh-water before it entered the next compartment. But later we perceived that the percolation of water through the sand is so small that, even if the fresh-water be left free to flow directly into the next compartment without any artificial device, it will soon glide out over the salt-water constantly overflowing from the opening and leave almost no influence on the concentration of the salt-water in the compartment. This was also ascertained by floating a sensitive hydrometer in the compartment.

Fig. 4.



(b) *Salt-water percolating into the fresh-water compartment.*

When the left end of the contact surface of the fresh- and salt-water falls within the bottom of the sand compartment, we feel no anxiety about this point. But, according to the difference in the head of the water in the two side-compartments and the concentration of the salt-water, the contact surface may reach the fresh-water compartment. In such a case, unless we devise some proper scheme, some salt-water will get mixed with the fresh-water and return again to the upper part of the sand compartment, which is fatal in experiments with the smaller model.

In order to avoid this danger, we tried to extract the salt-water by a pipe through the hole *H*, but the result was not very good. We soon learned, however, that the best method was rather to put sand in the fresh-water compartment up to the height to which the salt-water would reach, thus preventing the mixing of fresh- and salt-water.

(c) *Effect of the frame of wire-gauze.*

The wire-gauze itself has no effect upon the water-percolation into the sand compartment, but the frame to which it is attached requires to be tolerably thick to give it strength, so that the percolation of the water is somewhat disturbed.

On account of this effect, we must expect some discrepancy near the ends between the theory and our experimental results.

3. Results of the experiments.

(A). *Larger-model experiments (Determination of the upper boundary).*

With the larger model we experimented on three cases of different salinities resembling the real sea-water. Each case was repeated five times, and the data given below are the mean of them.

Exp. 1. Salt water: temp. = $4^\circ.6$ *C.* $\rho = 1{\cdot}024$.
Fresh water: temp. = $5^\circ.7$ *C.* $\rho_0 = 1{\cdot}000$.

The difference in the head of water in the two side-compartments was regulated so as to be about 2 cms[1]. The level in the sand compartment was as follows.

x (cm)	0	10	20	30	40	50	60	70	80	90	100	110
y (cm)	-	0·50	0·80	1·00	1·20	1·35	1·45	1·55	1·65	1·75	1·85	1·93

The method of least square gives from these data the following empirical formula

$$y^2 = 0{\cdot}0350x + 0{\cdot}03,$$

while the theoretical equation calculated by eq. (10) is

$$y^2 = 0{\cdot}0346x$$

Exp. 2. Salt water: temp. = $5^\circ.6$ *C.* $\rho = 1{\cdot}029$.
Fresh water: temp. = $6^\circ.6$ *C.* $\rho_0 = 1{\cdot}000$.

$$h_L = 2\ 02 \text{ cms.}$$

x (cm)	0	10	20	30	40	50	60	70	80	90	100	110
y (cm)	-	0·65	0·95	1·20	1·35	1·50	1·60	1·65	1·75	1·85	1·92	2·02

∴ Theoretical curve: $y^2 = 0{\cdot}0367\,x + 0{\cdot}01.$
Empirical formula: $y^2 = 0{\cdot}0372\,x + 0{\cdot}15.$

1 This can not be used as h_L. There must be a sudden reduction in head as the fresh-water enters the sand-stratum and the conditions are not the same as in the real coast. Hence the head at that end must be taken just inside the sand itself.

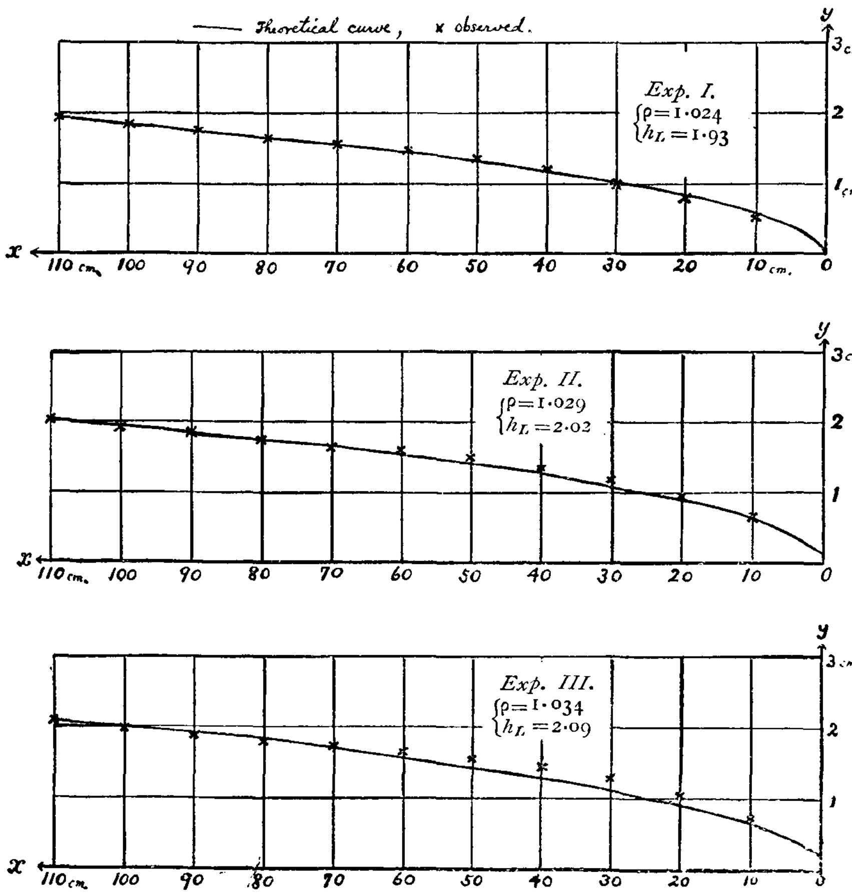
Exp. 3. Salt water: temp. = $4^\circ{\cdot}6$ *C.* $\rho = 1{\cdot}034$.
Fresh water: temp. = $5^\circ{\cdot}6$ *C.* $\rho_0 = 1{\cdot}000$.
$h_L = 2{\cdot}09$ cms.

x (cm)	0	10	20	30	40	50	60	70	80	90	100	110
y (cm)	—	0·70	1·05	1·30	1·45	1·60	1·70	1·76	1·80	1·90	1·98	2·09

Hence, theor. eq. $y^2 = 0{\cdot}0397x + 0{\cdot}04$;
emp. eq. $y^2 = 0{\cdot}0388x + 0{\cdot}35$.

All the above results are shown graphically in Fig. 5.

Fig. 5.



(B). *Smaller-model experiments* (*Determination of the lower boundary*).

We experimented on the following three cases:

	$\rho_0 = 1$	$L = 26{\cdot}5$ cms.	
Exp. I.	$\rho = 1{\cdot}024$	$h_L = 0{\cdot}5$ cm.	Water temp. $= 9^\circ C.$
Exp. II.	$\rho = 1{\cdot}029$	$h_L = 0{\cdot}6$ cm.	Water temp. $= 8^\circ C.$
Exp. III.	$\rho = 1{\cdot}037$	$h_L = 0{\cdot}7$ cm.	Water temp. $= 10^\circ C.$

Fig. 6 and Pl. XVI show examples of the appearance when the salt-water infiltrates into the sand layer gradually with time, and the actual boundary finally obtained. Comparison of the actual curves with those theoretical ones calculated by substituting the above values of ρ and h_L in the formula (12) are given in Figs. 7, 8, 9.

To express these relations numerically, read off the value of y' per 2 cms. of x in the experimental curve, and find its empirical formula by the method of least square, and then compare it with the equation of the theoretical curve. The results are as follows.

Fig. 6.

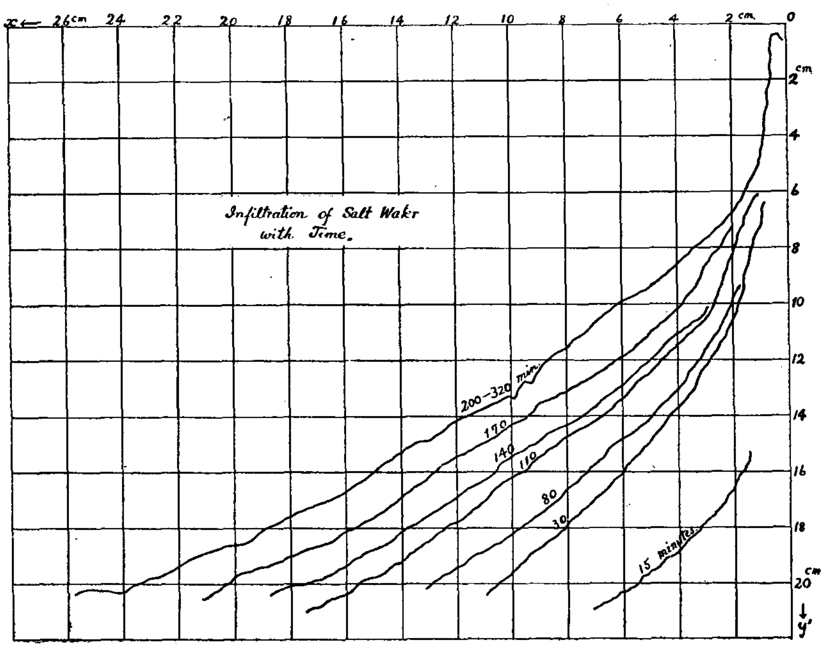


Fig. 7.

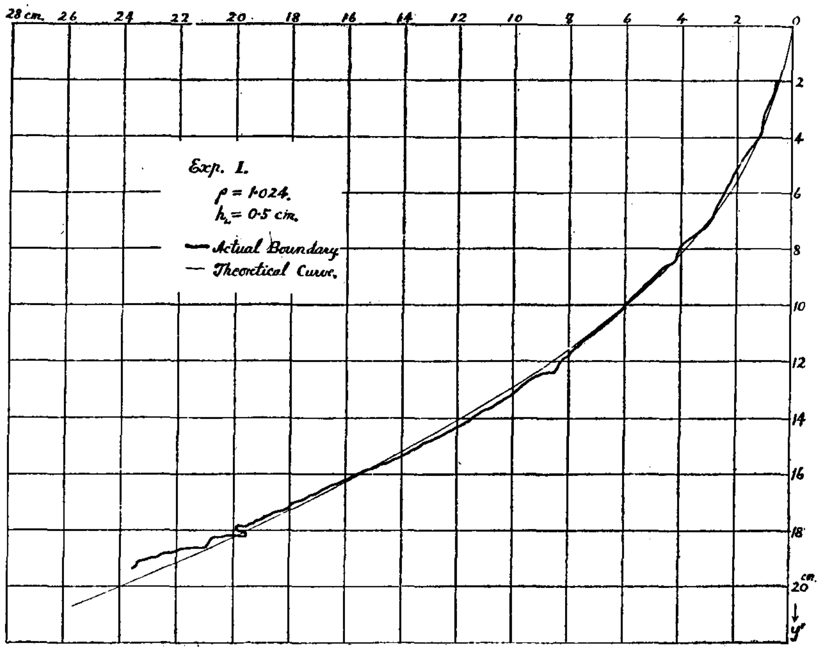

28 cm. 26 24 22 20 18 16 14 12 10 8 6 4 2 0
2
4
6
8
10
12
14
16
18
20 cm.
y'
Exp. I.
ρ = 1·024.
h_L = 0·5 cm.
Actual Boundary
Theoretical Curve.

Fig. 8.

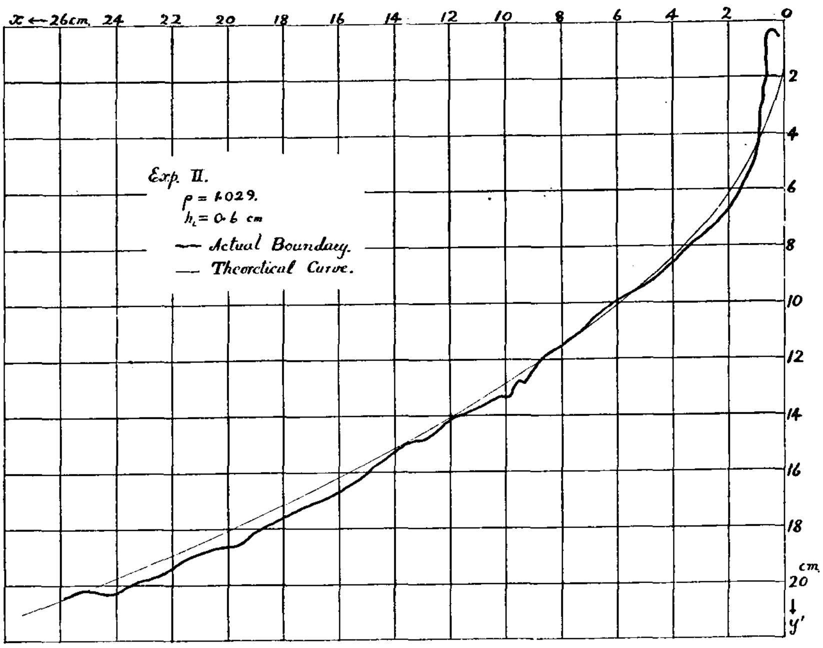
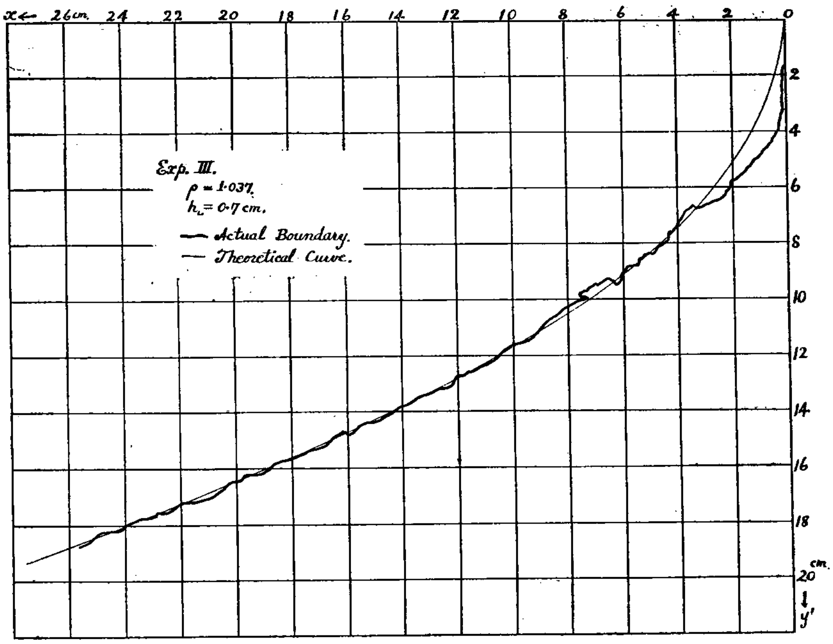

x ← 26 cm. 24 22 20 18 16 14 12 10 8 6 4 2 0
2
4
6
8
10
12
14
16
18
20 cm.
y'
Exp. II.
ρ = 1·029.
h_L = 0·6 cm
Actual Boundary.
Theoretical Curve.

Fig. 9.



Exp. I. From the actual curve we have:

x (cm)	2	4	6	8	10	12	14	16	18	20	22
y' (cm)	5·0	7·8	10·0	11·8	13·2	14·4	15·4	16·2	17·1	17·9	18·7

∴ Emp. eq. $y'^2 = 3{\cdot}62 + 16{\cdot}13\,x$

Theor. eq.

Approximate formula $y'^2 = \left(\dfrac{1}{1{\cdot}024 - 1}\right)^2 \cdot \dfrac{(0{\cdot}5)^2}{26{\cdot}5} \cdot x$

$= 16{\cdot}64\,x.$

Accurate formula $y'^2 = 3{\cdot}62 + 16{\cdot}50\,x.$

The next table shows the corresponding values of y' calculated by the empirical and the theoretical (approximate) equation respectively.

	x	2	4	6	8	10	12	14	16	18	20	22	24	26
y'	Theor.	5·55	8·16	9·99	11·54	12·89	14·13	15·26	16·31	17·33	18·21	19·13	19·98	20·78
	Emp.	5·96	8·25	10·02	11·49	12·81	14·04	15·13	16·16	17·12	18·06	18·92	19·75	20·57

Exp. II. The actual curve gives:

x (cm)	0	2	4	6	8	10	12	14	16	18	20	22	24
y' (cm)	0·3	6·7	8·6	9·9	11·5	13·3	14·1	15·2	16·5	17·4	18·4	19·2	20·0

∴ Emp. eq. $y'^2 = 4{\cdot}10 + 16{\cdot}56\,x.$

Theor. eq.

$$\begin{cases} \text{Approximate formula} & y'^2 = 16{\cdot}15\,x. \\ \text{Accurate formula} & y'^2 = 4{\cdot}85 + 16{\cdot}00\,x. \end{cases}$$

These equations give the following corresponding numbers.

	x	2	4	6	8	10	12	14	16	18	20	22	24
y'	Theor.	5·68	8·04	9·84	11·36	12·70	13·92	15·03	16·06	17·04	17·97	18·85	20·49
	Emp.	6·10	8·39	10·15	11·70	13·04	14·28	15·36	16·40	17·38	18·30	19·18	20·05

Exp. III. From the actual curve we read:

x (cm)	0	2	4	6	8	10	12	14	16	18	20	22	24
y' (cm)	1·6	5·8	7·4	9·0	10·4	11·6	12·8	13·8	14·9	15·7	16·5	17·2	18·1

Thus, emp. equation: $y'^2 = 2{\cdot}18 + 13{\cdot}49\,x,$

and as the theoretical equation

$$\begin{cases} \text{approximate formula} & y'^2 = 13{\cdot}51\,x, \\ \text{accurate formula} & y'^2 = 2{\cdot}18 + 13{\cdot}43\,x. \end{cases}$$

The comparison of these equations is given below.

	x	2	4	6	8	10	12	14	16	18	20	22	24
y'	Theor.	5·20	7·35	9·00	10·39	11·62	12·73	13·75	14·70	15·59	16·43	17·23	18·01
	Emp.	5·40	7·49	9·12	10·49	11·70	12·81	13·82	14·76	15·65	16·49	17·29	18·06

All the above experimental results are sufficient to be considered as pretty good verifications of our theory.

V. Accessory experiments.

1. Verification of our theory by the measurement of soil constant.

In our model experiment, as the salt-water can not mix with the fresh in the fresh-water compartment, the difference between the quantity of water pouring into the fresh-water compartment and that overflowing from the side opening must be the quantity of fresh-water infiltrating through the sand layer.

In the experiment with the larger model, we measured the quantity of water pouring into the fresh-water compartment several times before and after the experiment, and also measured several times the quantity of water overflowing when a state of steadiness had been reached, and thus determined the quantity of fresh-water passing into the sand compartment.

Substituting this value in formula (11), we calculated the soil constant of the sand.

Larger-model exp. 1.

$L = 110$ cms. $b = 14{\cdot}8$ cms.

$\rho = 1{\cdot}024$ $\rho_0 = 1{\cdot}000$

$h_L = 1{\cdot}93$ cms.

Quantity of water pouring in in 1·5 min. = 278 c.c.

Quantity of water overflowing in 3 min. = 300·5 c.c.

∴ $Q = 1{\cdot}42$ c.c. per sec.

and

$$k = \frac{2QL}{b h_L^2} \frac{\rho - \rho_0}{\rho} = 0{\cdot}132 \text{ cm/sec.}$$

Larger-model exp. 2.

Quantity of water pouring in in 1·5 min. = 256 c.c.

Quantity of water overflowing in 3 min. = 278 c.c.

∴ $Q = 1{\cdot}30$ c.c. per sec.

And $L = 110$ cms. $b = 14{\cdot}8$ cms. $h_L = 2{\cdot}02$ cms.

$\rho = 1{\cdot}029.$ $\rho_0 = 1{\cdot}000.$

∴ $k = 0{\cdot}134$ cm./sec.

Larger-model exp. 3.

Quantity of water pouring in in 2 min. = 339 c.c.

Quantity of water overflowing in 3 min. = 325 c.c.

∴ $Q = 1{\cdot}02$ c.c. per sec.

And also

$L = 110$ cms. $b = 14{\cdot}8$ cms. $h_L = 2{\cdot}09$ cms.

$\rho = 1{\cdot}034$ $\rho_0 = 1{\cdot}000$

∴ $k = 0{\cdot}114$ cm/sec.

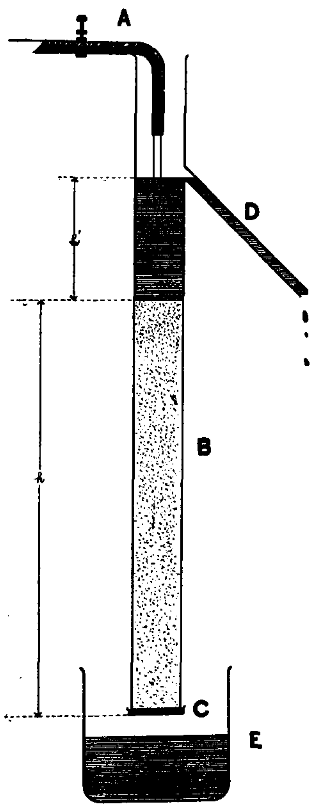
The mean of the above three experiments gives

$k = 0{\cdot}127.$

On the other hand, the soil constant of the sand can be measured directly by the following simple method. Construct an apparatus as in

Fig. 10. Pour water constantly but slowly from the upper end of the glass cylinder (B) containing sand. The head of water in the cylinder will be kept always constant automatically by means of the side tube (D). The water having passed through the sand finally falls from the wire-gauze (C) into the receiver (E). For such an experiment, it is easily seen that the following relation must hold:—

Fig. 10.



$$\frac{V}{t} = Q = k.S.\frac{h+h'}{h},$$

or
$$k = \frac{Q}{S}\cdot\frac{h}{h+h'},$$

where

V = Volume of water falling into the receiver in time t,
k = Soil constant,
S = Sectional area of the cylinder,
h = Height of the sand column only,
h' = Height of water column above the sand.

We used the same sand (dia 1·0—0·5 mm.) as was employed in the previous model experiments. The dimensions of the cylinder were

dia. = 1·5 cms. h = 50 cms.
h' = 15 cms.

And as the mean of repeated measurement several times we got

V = 127 c.c. in time t = 7 min.

Substitution of these data in the above formula gives

k = 0·132,

which is very near to the value of k already obtained indirectly by the model experiments.

Thus we have here another verification of our theory of the underground water near a sea-shore.

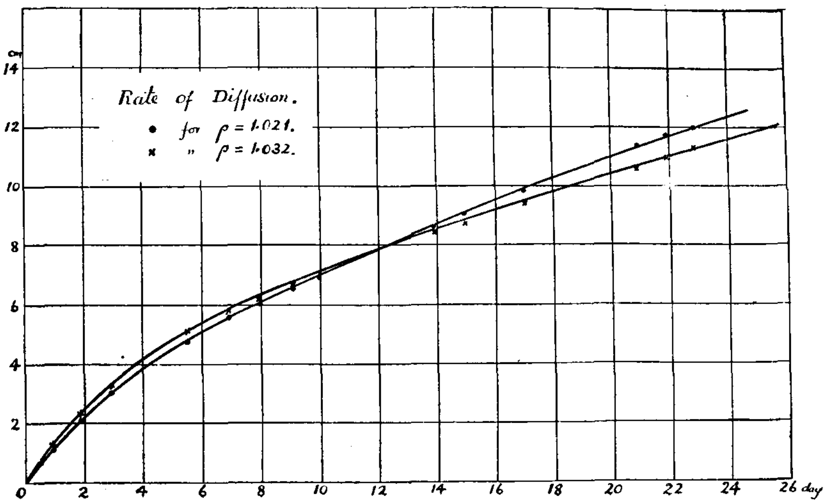
2. Diffusion of salt in sand.

To estimate the effect of diffusion of salt-water in our model experiment, we performed the following experiment:—

Take a glass tube of about 3 cm. radius, and over its internal wall form a film of silver chromate. Put in it the same sand as used in the previous experiments, and set it vertically. Then fill up the tube with fresh-water except only a small part near the lower end where salt-water is put in.

No.	Date	Time Interval	Tube I		Tube II.	
			Reading	Diff.	Reading	Diff.
	day h		cm		cm	
1	1 3·5 *Pm.*	(start)	6·5	—	2·0	—
		h		cm		
2	2 9·0 *Am.*	17·5	7·6	1·1	3·2	1·2(max.)
3	3 9·0 〃	24·0	8·7	1·1	4·3	1·1
4	4 1·5 *Pm.*	28·5	9·6	0·9	5·2	1·1
5	7 9·0 *Am.*	67·5	11·1	1·5	7·0	1·8
6	8 1·0 *Pm.*	28·0	11·8	0·7	7·6	0·6
7	9 1·0 〃	24·0	12·5	0·7	8·1	0·5
8	10 1·0 〃	24·0	12·9	0·4	8·7	0·6
9	11 2·0 〃	25·0	13·4	0·5	9·2	0·5
10	15 2·0 〃	96·0	15·0	1·6	10·4	0·8
11	16 0·0 〃	22·0	15.6	0·6	10·7	0·3
12	18 3·0 〃	51·0	16·4	0·8	11·4	0·7
13	22 1·0 〃	94·0	18·1	1·7	12·6	1·2
14	23 3·0 〃	26·0	18·4	0·3	12·8	0·2
15	24 1·0 〃	. 26·0	18·6	0·2	13·4	0·6

Fig. 11.



Then, a definite boundary can be seen and it will shift upward a little every day. If we read the position of the boundary line once every day, we can estimate the diffusion velocity.

The result is as given in the annexed table and Fig. 11.

The initial concentration of the salt-water was such that

$$\rho = 1{\cdot}032 \text{ for tube I, and} \qquad \rho = 1{\cdot}021 \text{ for tube II.}$$

Even the first day when diffusion was most active, the shifting velocity of the boundary line amounted only to

$$\frac{1{\cdot}2}{17{\cdot}5} = 0{\cdot}07 \text{ cm/hour} = 1{\cdot}65 \text{ cm/day.}$$

From this we see that the diffusion effect must be negligible in all the previous model-experiments, as they finished within 3 or 4 hours. For, even if the fresh-water layer were at rest, the diffusion distance of the salt water would be only 2 or 3 mm. In reality the fresh-water was flowing with a mean velocity of the order

$$\bar{v} = k\,\frac{h_L}{L} = 0{\cdot}00294 \text{ cm/sec. (by exp. 2),}$$

and so it required only a time of 2 hours or more to pass over the whole sand compartment of length 26·5 cm. Hence the effect of diffusion on the boundary of the fresh- and salt-water would be about 1 mm. only.

VI. Conclusion.

Summarizing all the above, we may say that our theory has been verified at least by the laboratory work. The equations for the upper and lower boundaries of the fresh ground-water and also for the soil constant, (10), (11) and (12), all proved a fair coincidence with the results of the model experiments.

We hope to have an opportunity of confirming our theory at real sea-coasts another day.