On the Convergency of the Analytic Elements of Meromorphic Functions on the Circle of Convergence.

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1. Let $f(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n + \ldots\ldots$ be an analytic element convergent in the circle of convergence with the radius unity. If the function defined by $f(z)$ has a pole at $z = 1$, from the interior (in an angle) of the circle of convergence we have

$$\lim_{z \to 1} f(z) = \infty,$$

while the partial sums

$$S_n(z) \equiv a_0 + a_1 z + \ldots\ldots + a_n z^n$$

may take several values. We give at first *an example that the pole* $z = 1$ *becomes a root of some number of the partial sums.*

An example is given by the analytic element of the function

$$f(z) = \frac{1}{1 - z} + \frac{2}{(1 + z)^2}.$$

We may write about the origin

$$f(z) = 1 + z + z^2 + \ldots\ldots + z^n + \ldots\ldots$$
$$+ 2\left\{1 - 2z + 3z^2 - \ldots\ldots + (-1)^n (n+1) z^n \pm \ldots\ldots\right\}$$

The point $z = 1$ is a pole of the function, while

$$S_{2m+1}(1) = 0$$

In the example we notice that $\lim_{n \to \infty} S_n(-1) = \infty$ which may be proved genarally for meromorphic functions.

2. *Let $\alpha, \beta, \gamma, \ldots\ldots$ be the poles on the circle of convergence of the analytic element of a meromorphic function $f(z)$ within the domain of existence, then among these poles there is at least one, say α, such that*

$$\lim_{n\to\infty} S_n(\alpha) = \infty.$$

To prove it we introduce the lemma:

Put

$$s_n(z) \equiv 1 + z + z^2 + \ldots\ldots + z^n,$$

$$s_n^{(m-1)}(z) \equiv 1 + \frac{m}{1!} z + \frac{m(m+1)}{2!} z^2 + \ldots\ldots + \frac{m(m+1)\ldots\ldots(m+n-1)}{n!} z^n,$$

then for n great (m being finite) and $z_0 \neq 1$, $|z_0| = 1$, *we have*

$$\left.\begin{aligned} s_n^{(m-1)}(1) &= \lambda_n n^m \\ s_n^{(m-1)}(z_0) &= \mu_n n^{m-1} \end{aligned}\right\} \qquad (1)$$

where λ_n, μ_n are finite and do not tend to zero.

For we have the identity

$$s_n^{(m)}(z) \cdot (1-z) \equiv s_n^{(m-1)}(z) - \frac{(m+1)(m+2)\ldots\ldots(m+n)}{n!} z^{n+1}.$$

Hence at $z=1$, applying Stirling's formula we can prove the first Formula. For the second formula, differentiating $m-1$ times the identity

$$s_{n+m-1}(z) = 1 + z + z^2 + \ldots\ldots + z^{n+m-1}$$
$$\equiv (1 - z^{n+m})(1-z)^{-1},$$

we have

$$(m-1)!\ s_n^{(m-1)}(z) = (n+2)(n+3)\ldots\ldots(n+m) \times \left\{ -z^{n+1}(1-z)^{-1} + \ldots\ldots \right\}$$

where the dotted part stands for the sum of $m-1$ terms tending to zero for $n \to \infty$ provided $z \neq 1$, $|z| = 1$. Hence we have

$$s_n^{(m-1)}(z_0) = \mu_n n^{m-1}$$

where μ_n is finite and do not tends to zero.

Now we may prove the first proposition.

For simplicity suppose that the circle of convergence is the unit circle about the origin and there are only three poles α, β, γ on the circle of convergence, $|\alpha| = |\beta| = |\gamma| = 1$. $f(z)$ may be written as follows:

$$f(z) = \frac{A_p}{\left(1 - \frac{z}{\alpha}\right)^p} + \frac{A_{p-1}}{\left(1 - \frac{z}{\alpha}\right)^{p-1}} + \ldots\ldots + \frac{A_1}{1 - \frac{z}{\alpha}}$$

$$+\frac{B_q}{\left(1-\frac{z}{\beta}\right)^q}+\frac{B_{q-1}}{\left(1-\frac{z}{\beta}\right)^{q-1}}+\ldots\ldots+\frac{B_1}{1-\frac{z}{\beta}}$$
$$+\frac{C_r}{\left(1-\frac{z}{\gamma}\right)^r}+\frac{C_{r-1}}{\left(1-\frac{z}{\gamma}\right)^{r-1}}+\ldots\ldots+\frac{C_1}{1-\frac{z}{\gamma}}$$
$$+\varphi(z)$$

where A's, B's, C's are constant, A_p, B_q, C_r being different from zero and p, q, r are constant positive integers, while

$$\varphi(z)=l_0+l_1 z+\ldots\ldots+l_n z^n+\ldots\ldots$$

is a power-series convergent in the circle whose radius of convergence is greater than unity, so that its partial sums are limited for $z=1$. We may assume that p is not less than any of q, r. The partial sum $S_n(z)$ of the analytic element of $f(z)$ is given by

$$S_n(z)=A_p s_n^{(p-1)}\left(\frac{z}{\alpha}\right)+A_{p-1}s_n^{(p-2)}\left(\frac{z}{\alpha}\right)+\ldots\ldots+A_1 s_n\left(\frac{z}{\alpha}\right)$$
$$+B_q s_n^{(q-1)}\left(\frac{z}{\beta}\right)+B_{q-1}s_n^{(q-2)}\left(\frac{z}{\beta}\right)+\ldots\ldots+B_1 s_n\left(\frac{z}{\beta}\right) \quad (2)$$
$$+C_r s_n^{(r-1)}\left(\frac{z}{\gamma}\right)+C_{r-1}s_n^{(r-2)}\left(\frac{z}{\gamma}\right)+\ldots\ldots+C_1 s_n\left(\frac{z}{\gamma}\right)$$
$$+l_0+l_1 z+\ldots\ldots+l_n z^n$$

By the lemma the orders of magnitude for $n\to\infty$ of the terms standing in the first line for $z=\alpha$ are $n^p, n^{p-1},\ldots\ldots, n$ respectively. Hence their sum is of order n^p. The orders of terms standing in the second line for $z=\alpha$ are $n^{q-1},\ldots\ldots, n^0$ respectively, since $\alpha\neq\beta$ but $\left|\frac{\alpha}{\beta}\right|=1$. Hence their sum is of order $n^{q-1}(<n^p)$. In the same way for the third line its order is $n^{r-1}(<n^p)$, while $l_0+l_1\alpha+\ldots\ldots+l_n\alpha^n$ is limited. Hence we have

$$\lim_{n\to\infty} S_n(\alpha)=\infty.$$

This is clearly true when the number of poles on the circle of convergence of the series $f(z)$ is more or less than three.

In the above proof the circle of convergence is supposed to be unit circle. If its centre be at z' with the radius of convergence $\rho\neq 1$, then we have only to consider the analytic element of the meromorphic function $f(z'+\rho z)$. It has the unit circle as the circle of convergence. Thus our proposition is generally true. Q. E. D.

We remark that if $p=q\geqq r$, we have in the same way

$$\lim_{n\to\infty} S_n(\beta) = \infty.$$

3. Next we shall prove the following proposition:

The expansion into Taylor's series of the meromorphic function within the domain of existence, cannot be convergent on its circle of convergence.

To prove it we introduce the following lemma:

Let A, B, C be any constants not all zero or any functions of n tending for $n\to\infty$ to finite determinate limits not all zero and α, β, γ be three different points on the unit circle i. e., $|\alpha|=|\beta|=|\gamma|=1$, then

$$A\alpha^n + B\beta^n + C\gamma^n$$

cannot converge to zero for $n\to\infty$.

For put

$$P_n \equiv A\alpha^n + B\beta^n + C\gamma^n.$$

If A, B, C be constant, solving the equations

$$\left.\begin{aligned} P_n &= A\alpha^n + B\beta^n + C\gamma^n \\ P_{n+1} &= A\alpha^{n+1} + B\beta^{n+1} + C\gamma^{n+1} \\ P_{n+2} &= A\alpha^{n+2} + B\beta^{n+2} + C\gamma^{n+2} \end{aligned}\right\}$$

simulteneously with respect to A, B, C, we have

$$A = \frac{\beta\gamma P_n - (\beta+\gamma)P_{n+1} + P_{n+2}}{\alpha^n(\alpha-\beta)(\alpha-\gamma)}$$

and the like expression for B, C. Since $\alpha \neq \beta$, $\alpha \neq \gamma$, $|\alpha|=1$, if P_n tend to zero, A should tend to zero for $n\to\infty$; the same with B and C. This is contradictory to the hypothesis. Thus in the sequence $P_n (n=0, 1, 2, \ldots\ldots)$ there is a partial sequence whose numbers are in absolute value all greater than a positive number.

When A, B, C are variable with n, we may put

$$\begin{aligned} A &= A_0 + a_n, \\ B &= B_0 + b_n, \\ C &= C_0 + c_n \end{aligned}$$

where A_0, B_0, C_0 are the limits of A, B, C respectively for $n\to\infty$ which are not all zero and the functions a_n, b_n, c_n tend to zero for $n\to\infty$.

Writing

$$\begin{aligned} Q_n &\equiv A_0\alpha^n + B_0\beta^n + C_0\gamma^n, \\ R_n &\equiv a_n\alpha^n + b_n\beta^n + c_n\gamma^n, \end{aligned}$$

we have

$$P_n = Q_n + R_n.$$

Since $|\alpha|=|\beta|=|\gamma|=1$, R_n tends to zero for $n\to\infty$. Therefore if P_n

tend to zero for $n\to\infty$, Q_n should be so also. This is against the foregoing proof. Hence the lemma is completely proved.

This lemma can clearly be extended for any number of points on the unit circle.

To prove our proposition, consider the expansion of the meromorphic function $f(z)$ into Taylor's series about the origin within the domain of existence. Then we may prove that the series cannot be convergent for any point z_0 on the circle of convergence.

For if the series $f(z_0)$ be convergent, then we should have by Abel's theorem

$$\lim_{z\to z_0} f(z)=f(z_0), \quad \textit{finite and determinate.}$$

On the other hand if z_0 is a pole of the function $f(z)$, we have for the series $f(z)$

$$\lim_{z\to z_0} f(z)=\infty$$

These are contradictory. Therefore we have only to prove the case where z_0 is any point on the circle of convergence different from the poles.

Now suppose for simplicity that the circle of convergence is the unit circle on which there are only three poles α, β, γ of $f(z)$ where z_0 is different from them.

Using the same notations as § 2 for the partial sum $S_n(z)$ of the series $f(z)$, we have the expression (2). Put

$$\frac{z_0}{\alpha}\equiv\alpha', \quad \frac{z_0}{\beta}\equiv\beta', \quad \frac{z_0}{\gamma}\equiv\gamma'.$$

Then since $\alpha, \beta, \gamma, z_0$ are the points on the unit circle and different from one another, the points α', β', γ' are also on the unit circle and different from one another and different from 1. Now (2) may be written as follows:

$$\begin{aligned} S_n(z_0)=&A_p s_n^{(p-1)}(\alpha')+A_{p-1}s_n^{(p-2)}(\alpha')+\dots\dots+A_1 s_n(\alpha')\\ &+B_q s_n^{(q-1)}(\beta')+B_{q-1}s_n^{(q-2)}(\beta')+\dots\dots+B_1 s_n(\beta')\\ &+C_r s_n^{(r-1)}(\gamma')+C_{r-1}s_n^{(r-2)}(\gamma')+\dots\dots+C_1 s_n(\gamma')\\ &+l_0+l_1 z_0+\dots\dots+l_n z_0^n \end{aligned}$$

where the last sum converges for $n\to\infty$.

If $p>q$, $p>r$, $p>1$, then using the second relation of (1), by the same reasoning as § 2, it follows that

$$\lim_{n\to\infty} S_n(z_0)=\infty.$$

If $p=1$, there is only a pole α. Therefore we have

$$S_n(z_0)=A_1 s_n(\alpha')+l_0+l_1 z_0+\dots\dots+l_n z_0^n$$

Since $\alpha' \neq 1$, $|\alpha'| = 1$, $S_n(z_0)$ is indeterminate for $n \to \infty$, so that the series $f(z_0)$ cannot be convergent.

There remains the cases where $p = q$ or $p = q = r$. The reasoning being quite the same, we shall prove the case $p = q = r$.

In this case we may write

$$S_n(z_0) - \varphi_n(z_0) = s_n^{(p-1)}(\alpha')\,(A_p + a_n) + s_n^{(p-1)}(\beta')\,(B_p + b_n) + s_n^{(p-1)}(\gamma')\,(C_p + c_n)$$

where

$$\varphi_n(z_0) \equiv l_0 + l_1 z_0 + \ldots\ldots + l_n z_0^n,$$

$$a_n \equiv \frac{A_{p-1} s_n^{(p-2)}(\alpha') + \ldots\ldots + A_1 s_n(\alpha')}{s_n^{(p-1)}(\alpha')},$$

$$b_n \equiv \frac{B_{p-1} s_n^{(p-2)}(\beta') + \ldots\ldots + B_1 s_n(\beta')}{s_n^{(p-1)}(\beta')},$$

$$c_n \equiv \frac{C_{p-1} s_n^{(p-2)}(\gamma') + \ldots\ldots + C_1 s_n(\gamma')}{s_n^{(p-1)}(\gamma')}.$$

Therefore by the lemma of § 2, a_n, b_n, c_n may be made as small as we please provided n be made sufficiently great. They are equal to zero if $p = 1$.

Now suppose that the sequence of numbers

$$S_n(z_0) - \varphi_n(z_0), \qquad n = 0, 1, 2, \ldots\ldots$$

converge to a finite determinate limit, then it is necessary that at least

$$S_n(z_0) - \varphi_n(z_0) - \left\{ S_{n-1}(z_0) - \varphi_{n-1}(z_0) \right\}$$

tend to zero. By the relation

$$s_n^{(p-1)}(z) \equiv s_{n-1}^{(p-1)}(z) + \frac{p(p+1)\ldots\ldots(p+n-1)}{n!} z^n,$$

the above difference is equal to

$$(a_n - a_{n-1}) s_{n-1}^{(p-1)}(\alpha') + (b_n - b_{n-1}) s_{n-1}^{(p-1)}(\beta') + (c_n - c_{n-1}) s_{n-1}^{(p-1)}(\gamma') + \frac{p(p+1)\ldots\ldots(p+n-1)}{n!} \Big\{ (A_n + a_n)\alpha'^n + (B_n + b_n)\beta'^n + (C_n + c_n)\gamma'^n \Big\}$$

which by the assumption tends to zero for $n \to \infty$.

By the second formula of (1), $s_{n-1}^{(p-1)}(\alpha')$, $s_{n-1}^{(p-1)}(\beta')$, $s_{n-1}^{(p-1)}(\gamma')$ are all of order $(n-1)^{p-1}$. By Stirling's formula $\dfrac{p(p+1)\ldots\ldots(p+n-1)}{n!}$ is also of order $(n-1)$.$^{p-1}$ Hence dividing the above expression by $\dfrac{p(p+1)\ldots\ldots(p+n-1)}{n!}$ we have for the convergency of the sequence of numbers $S_n(z_0) - \varphi_n(z_0)$, the necessary consequence:

$$\lim_{n\to\infty}\left\{(A_p+a_n)\alpha'^n+(B_p+b_n)\beta'^n+(C_p+c_n)\gamma'^n\right\}=0.$$

Considering A_p+a_n, B_p+b_n, C_p+c_n as A, B, C respectively in the above lemma, they satisfy the conditions required in it, since A_p, B_p, C_p are constants not zero and a_n, b_n, c_n tend to zero (or zero if $p=1$) for $n\to\infty$. Moreover by the definition of α', β', γ', they are three different points on the unit circle. Therefore by the lemma the limit written above can not be true. Consequently $\lim_{n\to\infty} S_n(z_0)$ is not finite and determinate.

Our proof can clearly be applied for any number of poles on the circle of convergence, provided the circle lies within the domain of existence of the meromorphic function.

In the above demonstration the radius of convergence is supposed to be unity. If it be $\rho \neq 1$, then instead of $f(z)$ we have only to consider Taylor's expansion of the meromorphic function $f(\rho z)$ whose radius of convence is unity. Moreover to treat Taylor's expansion of $f(z)$ about $z=z'$, we have only to consider that of the meromorphic function $f(z'+z)$ about $z=0$. Hence our proposition is completely proved.

As the necessary consequence, *if an analytic element of a function converges at a point on its circle of convergence which is within the domain of existence of the function, then the function is not meromorphic.*

Combining with the result of § 2, *the analytic element of a meromorphic function is either infinite or indeterminate on its circle of convergence which is within the domain of existence.*

Hence *if the analytic element of a meromorphic function be convergent at a point on its circle oj convergence, then there is at least an essential singular point of the function on the circle since the circle must touch the boundary of the domain of existence of the meromorphic function.*

As a special case, for *rational functions* the last proposition is unnecessary to be taken into consideration.