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Informal Constructive Reverse Mathematics

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Bishop's constructive mathematics (BISH) [1, 2, 7, 28] is an informal mathematics using intuitionistic logic and assuming some function existence axioms:

the axiom of countable choice

$$\forall n \in \mathbf{N} \exists x \in X A(n, x) \rightarrow \exists f \in X^{\mathbf{N}} \forall n \in \mathbf{N} A(n, f(n));$$

the axiom of dependent choice

$$\begin{aligned} &\forall x \in X \exists y \in X A(x, y) \rightarrow \\ &\qquad \forall x \in X \exists f \in X^{\mathbf{N}}(f(0) = x \wedge \forall n \in \mathbf{N} A(f(n), f(n+1))); \end{aligned}$$

the axiom of unique choice

$$\forall x \in X \exists! y \in Y A(x, y) \rightarrow \exists f \in Y^X \forall x \in X A(x, f(x)).$$

It is a core of the varieties of mathematics in the sense that it can be extended not only to intuitionistic mathematics (INT) by adding the principle of continuous choice and the fan theorem [7, 8, 10, 11, 28], and constructive recursive mathematics (RUSS) by adding Markov's principle and the extended Church's thesis [7, 23, 28], but also to classical mathematics (CLASS) practised by most mathematicians today by adding the principle of the excluded middle and the full axiom of choice.

Bishop's constructive (forward) mathematicians have been making every effort, for a given classical theorem A, to find its constructive substitute A' such that

$$\text{BISH} \vdash A' \quad \text{and} \quad \text{CLASS} \vdash A \leftrightarrow A'$$ [1].

Of course, it happens that we can take A' as A; for examples we can prove the following classical theorems in BISH.

Theorem 1 (The completeness of R) *Every Cauchy sequence of real numbers converges.*

Theorem 2 (The constructive compactness of $[0,1]$**)** $[0,1]$ *is totally bounded and complete.*

Theorem 3 (The Baire category theorem) *The intersection of a sequence of dense open subsets of a complete metric space is dense.*

When A and A' are not equivalent in BISH, we also try to exhibit that A does not admit a constructive proof by giving a **Brouwerian counterexample** to A such that

$$\text{BISH} \vdash A \to P \quad \text{and} \quad \text{BISH} \nvdash P$$ [2]

for some principle P. The constructive compactness of $[0,1]$ is classically equivalent to the following special case of **the Bolzano-Weierstraß theorem**:

The sequential compactness of $[0,1]$. *Every sequence of* $[0,1]$ *has a convergent subsequence.*

But it is well known that the sequential compactness of $[0,1]$ entails in BISH **the limited principle of omniscience** (LPO):

$$\forall \alpha \in \mathbf{N}^{\mathbf{N}}[\exists n(\alpha(n) \neq 0) \vee \neg \exists n(\alpha(n) \neq 0)],$$

[1] In general, we may find more than one such A', say $A'_1, \ldots, A'_n$, and in this case, we try to find A'_k such that BISH $\vdash A'_k \to A'_i$ for all $i = 1, \ldots, n$. In some cases, we have to be contented ourselves with A' such that BISH $\vdash A'$, CLASS $\vdash A \to A'$, and it is strong enough for applications.

[2] Since BISH is an informal mathematics, this does not mean formal unprovability, but unacceptability, or at least high dubitation in BISH

which is an instance of **the Principle of the Excluded Middle** $P \vee \neg P$, and false both in INT and in RUSS [7, 28].

Mandelkern [24] showed its converse and proved the equivalence between the Bolzano-Weierstraß theorem and LPO in BISH, which led the subsequent research of informal constructive reverse mathematics aiming at finding a logical principle P such that

$$\mathrm{BISH} \vdash A \leftrightarrow P,$$

not only for a theorem A in CLASS but also for a theorem A in INT and in RUSS even if it is inconsistent with CLASS. This is possible because CLASS, INT and RUSS are extensions of BISH. In the rest of the paper, we will overview the results in informal constructive reverse mathematics to date; see also Mandelkern [24], Ishihara [12], Bridges, Ishihara and Schuster [4, 5], and Ishihara and Schuster [19] for informal constructive reverse mathematics with various compactness principles; Ishihara [13, 14], Bridges, Ishihara, Schuster and Vîţă [6], and Bridges, Ishihara and Schuster [5] with various continuity principles; Ishihara and Schuster [20] with Baire's theorem and its contraposition; see [27] for formal classical reverse mathematics.

The first class of theorems consists of theorems which are equivalent to LPO.

Theorem 4 *The following are equivalent in* BISH.

1. LPO.
2. $\forall x \in \mathbf{R}(0 < x \vee \neg(0 < x))$.
3. **The monotone convergence theorem** [24]. *Every bounded monotone sequence of real numbers converges.*
4. **The Bolzano-Weierstraß theorem** [24]. *Every bounded sequence of real numbers has a convergent subsequence.*
5. **The sequential compactness theorem** [19]. *Every compact metric space is sequentially compact.* [3]

[3] A metric space is *compact* if it is totally bounded and complete, and *sequentially compact* if every sequence of its elements has a convergent subsequence.

6. **The pseudo Heine-Borel theorem** [26]. *Every sequence of closed sets of a compact metric space with the finite intersection property has nonempty intersection.*

The second class of theorems consists of theorems which are equivalent to **the weak limited principle of omniscience** (WLPO):

$$\forall \alpha \in \mathbf{N}^{\mathbf{N}}[\neg\exists n(\alpha(n) \neq 0) \vee \neg\neg\exists n(\alpha(n) \neq 0)].$$

WLPO is an instance of the Principle of the Excluded Middle, weaker than LPO, and false both in INT and in RUSS [7, 28].

Theorem 5 *The following are equivalent in* BISH.

1. WLPO.

2. $\forall x \in \mathbf{R}(\neg(0 < x) \vee \neg\neg(0 < x))$.

3. **The existence of a discontinuous function** [29]. *A discontinuous function from* $\mathbf{N}^{\mathbf{N}}$ *into* $\mathbf{N}$ *exists.* [4]

Theorem 6 *The following are equivalent in* BISH.

1. $\neg$WLPO.

2. **The nondiscontinuity theorem** [14]. *Every mapping of a complete metric space into a metric space is nondiscontinuous.* [5]

The third class of theorems consists of theorems which are equivalent to **the lesser limited principle of omniscience** (LLPO):

$$\forall \alpha\beta \in \mathbf{N}^{\mathbf{N}}[\neg(\exists n(\alpha(n) \neq 0) \wedge \exists n(\beta(n) \neq 0)) \rightarrow \neg\exists n(\alpha(n) \neq 0) \vee \neg\exists n(\beta(n) \neq 0)].$$

LLPO is an instance of **De Morgan's low** $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$, weaker than WLPO, and false both in INT and in RUSS [7, 28].

[4] A function f between metric spaces is *discontinuous* if there exist $\delta > 0$ and a sequence $\{x_n\}$ converging to a limit x such that $d(f(x_n), f(x)) \geq \delta$ for all n.

[5] A function f between metric spaces is *nondiscontinuous* if $x_n \rightarrow x$ and $d(f(x_n), f(x)) \geq \delta$ for all n imply $\delta \leq 0$.

Theorem 7 *The following are equivalent in* BISH.

1. LLPO.
2. $\forall x \in \mathbf{R}(\neg(0 < x) \vee \neg(x < 0))$.
3. $\forall xy \in \mathbf{R}(xy = 0 \to x = 0 \vee y = 0)$.
4. *For all* $x, y \in \mathbf{R}$ *with* $\neg(x < y)$, $\{x, y\}$ *is closed subset of* $\mathbf{R}$ [15].
5. **The weak König lemma** (WKL) [12]. *Every infinite tree has an infinite path.* [6]
6. **The intermediate value theorem** [7]. *If* $f : [a, b] \to \mathbf{R}$ *is a uniformly continuous function, and* y *is a real number such that* $f(a) < y < f(b)$, *then there exists* x *in* $[a, b]$ *such that* $f(x) = y$.
7. **The minimum principle** [12, 19]. *Every uniformly continuous real function* f *on a compact metric space* X *attains its minimum.* [7]
8. **The pseudo Heine-Borel theorem for zero sets** [8] [12, 19]. *Every sequence of zero sets of a compact metric space with the finite intersection property has nonempty intersection.*
9. **The Hahn-Banach theorem** [12]. *Every bounded linear functional* f *on a subspace of a separable normed space* E, *whose kernel is located in* E [9], *has an extension* g *with* $\|f\| = \|g\|$.

The fourth class of theorems consists of theorems which are equivalent to **the fan theorem** (FAN):

Every detachable bar is uniform.[10]

[6] A (binary) *tree* is a subset T of $\mathbf{2}^{<\mathbf{N}}$ such that it is *detachable* from $\mathbf{2}^{<\mathbf{N}}$ in the sense that for each $n \in \mathbf{2}^{<\mathbf{N}}$, either $n \in T$ or $n \notin T$, and it is closed under restriction, that is if $n \in T$ and $m \preceq n$, then $m \in T$. A tree T is *infinite* if for each k there exists $n \in \mathbf{2}^k$ such that $n \in T$, and $\alpha \in \mathbf{2}^{\mathbf{N}}$ is an *infinite path* in T if $\forall k(\overline{\alpha}k \in T)$.

[7] That is there exists x in X such that $f(x) \le f(y)$ for all $y \in X$.

[8] A subset S of a metric space X is a *zero set* if there is a pointwise continuous function $f : X \to \mathbf{R}$ such that $S = \{x \in X \mid f(x) = 0\}$.

[9] A subset S of a metric space X is said to be *located* in X if $d(x, S) := \inf\{d(x, y) : y \in S\}$ exists for each x in X.

[10] A subset B of $\mathbf{2}^{<\mathbf{N}}$ is called a *bar* if for each $\alpha \in \mathbf{2}^{\mathbf{N}}$ there exists $n \in \mathbf{N}$ such that $\overline{\alpha}n \in B$. A bar B is *uniform* if there exists k such that for each $\alpha \in \mathbf{2}^{\mathbf{N}}$ $\exists i \le k(\overline{\alpha}i \in B)$.

FAN is a contrapositive form of WKL, weaker than LLPO [12] and hence than WKL [18], accepted in INT, and false in RUSS [7, 28].

Theorem 8 *The following are equivalent in* BISH.

1. FAN.
2. **The uniform continuity theorem** [28, 30]. *Every pointwise continuous mapping from* $\mathbf{2}^{\mathbf{N}}$ *into* $\mathbf{N}$ *is uniformly continuous.*
3. **The Heine-Borel theorem for cozero sets** [11] [19]. *Every cover of a compact metric space by a sequence of cozero sets has a finite subcover.*

The fifth class of theorems consists of theorems which are equivalent to **Markov's principle** (MP):

$$\forall\alpha \in \mathbf{N}^{\mathbf{N}}[\neg\neg\exists n(\alpha(n) \neq 0) \rightarrow \exists n(\alpha(n) \neq 0)].$$

MP is an instance of **the double negation elimination** $\neg\neg P \rightarrow P$, weaker than LPO, rejected in INT, and accepted in RUSS [7, 28].

Theorem 9 *The following are equivalent in* BISH.

1. MP.
2. $\forall x \in \mathbf{R}(\neg\neg(0 < x) \rightarrow 0 < x)$.
3. **The strong extensionality theorem** [3, 15]. *Every mapping between metric spaces is strongly extensional.* [12]

The sixth class of theorems consists of theorems which are equivalent to **weak Markov's principle** (WMP):

$$\begin{aligned}\forall\alpha \in \mathbf{N}^{\mathbf{N}}[\forall\beta \in \mathbf{N}^{\mathbf{N}}(\neg\neg\exists n(\beta(n) \neq 0) \vee \neg\neg\exists n(\alpha(n) \neq 0 \wedge \beta(n) = 0))\\ \rightarrow \exists n(\alpha(n) \neq 0)].\end{aligned}$$

WMP is weaker than MP, provable both in INT and in RUSS [16, 22].

[11] A subset S of a metric space X is a *cozero set* if there is a pointwise continuous function $f : X \rightarrow \mathbf{R}$ such that $S = \{x \in X \mid f(x) \neq 0\}$.

[12] A mapping f between metric spaces is ***strongly extensional*** if $f(x) \neq f(y)$ implies $x \neq y$.

Theorem 10 *The following are equivalent in* BISH.

1. WMP.

2. $\forall x \in \mathbf{R}[\forall y \in \mathbf{R}(\neg\neg(0 < y) \vee \neg\neg(y < x)) \to 0 < x]$.

3. **The strong extensionality theorem for complete spaces** [14]. *Every mapping from a complete metric space into a metric space is strongly extensional.*

4. **The sequential continuity theorem** [14]. *Every nondiscontinuous mapping from a complete metric space to a metric space is sequentially continuous.* [13]

The seventh class of theorems consists of theorems which are equivalent to **the disjunctive version of Markov's principle** ($\mathrm{MP}^{\vee}$):

$$\forall \alpha\beta \in \mathbf{N}^{\mathbf{N}}[\neg(\neg\exists n(\alpha(n) \neq 0) \vee \neg\exists n(\beta(n) \neq 0)) \to \neg\neg\exists n(\alpha(n) \neq 0) \vee \neg\neg\exists n(\beta(n) \neq 0)].$$

$\mathrm{MP}^{\vee}$ is an instance of De Morgan's Low, weaker than MP and than LLPO, rejected in INT, accepted in RUSS, and, together with WMP, implies MP [16].

Theorem 11 *The following are equivalent in* BISH.

1. $\mathrm{MP}^{\vee}$.

2. $\forall x \in \mathbf{R}[\neg\neg(x \neq 0) \to \neg\neg(0 < x)) \vee \neg\neg(x < 0)]$.

3. *For all $x, y \in \mathbf{R}$ with $\neg\neg(x < y)$, $\{x, y\}$ is closed subset of* $\mathbf{R}$.

The eighth class of theorems consists of theorems which are equivalent to **the boundedness principle** (BD-N):

Every countable pseudobounded subset of $\mathbf{N}$ *is bounded,*

where a subset A of $\mathbf{N}$ is said to be *pseudobounded* if for each sequence $\{a_n\}$ in A, $a_n < n$ for all sufficiently large n. BD-N is weaker than LPO and provable both in INT and in RUSS [14].

[13] A mapping f between metric spaces is *sequentially continuous* if $x_n \to x$ implies $f(x_n) \to f(x)$.

Theorem 12 *The following are equivalent in* BISH.

1. BD-N.

2. **The pointwise continuity theorem** [14]. *Every sequentially continuous mapping from a separable metric space into a metric space is pointwise continuous.*

3. **Banach's inverse mapping theorem** [17]. *If T is a bounded one-one linear mapping of a separable Banach space E onto a Banach space F, then T^{-1} is bounded.*

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