Group-Theory of Sequences of Numbers.

By

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This is the extension of the *group-theory of semi-convergent series.*[1] I consider in this paper the sequences of numbers. Rearrange the terms of a sequence. Then we shall obtain a substitution. If the series formed by the differences of corresponding terms of the sequence and the rearranged one be absolutely convergent, I say as in the previous paper that the given sequence admits the substitution. The totality of all such substitutions forms a group which I call the group of the sequence. The totality of all possible substitutions is called the symmetric group. Here the necessary and sufficient conditions that a sequence admits the symmetric group are given. From the hypothesis that a sequence admits the group of another sequence, I have obtained several propositions. These are nothing but the properties of a group and its divisors where the cardinal numbers of the derived sets of both sequences are concerned.

The set of all the substitutions of the symmetric group which I call the set of the symmetric group may be ordered and its cardinal number is equal to that of the continuum and then I determine the cardinal numbers of the sets of all the groups of sequences of numbers. The set of the group of a sequence is either countable or has the cardinal number of the continuum. The condition to determine the alternate case is found. Next the sum of all the numbers of a sequence taken in order is called a series notwithstanding that it is convergent or divergent and

1. These Memoirs, A, **10**, 211 (1927).

extending the idea of the equivalent series which I have given in the previous paper, I have divided all the series having the same terms into classes. This is nothing but the extension of the idea of the co-sets of finite groups. The set of these classes may be found to have the same cardinal number as the continuum provided that the given series be convergent or in some other cases.

As applications of the group-theory, I consider the sequence of functions. Here I consider the cases that a sequence of holomorphic functions admits the symmetric group simply or uniformly in the domain. I explain briefly some relations with the normal family of functions, quasi-analytic functions and the analytic continuation, while many others are not touched. I expect further researches.

1. Consider a *sequence* of numbers or a *set* of points

$$z_1,\ z_2,\ \ldots\ldots\ldots,\ z_\nu,\ \ldots\ldots\ldots.$$

We denote it by $\{z_\nu\}$. Rearranging the terms of the sequence, we have another sequence

$$\{z_{s_\nu}\}\equiv z_{s_1},\ z_{s_2},\ \ldots\ldots\ldots,\ z_{s_\nu},\ \ldots\ldots\ldots.$$

This rearrangement defines a *substitution*

$$S\equiv\begin{pmatrix}1 & 2 & \ldots\ldots & \nu & \ldots\ldots\\ s_1 & s_2 & \ldots\ldots & s_\nu & \ldots\ldots\end{pmatrix}\equiv\begin{pmatrix}\nu\\ s_\nu\end{pmatrix}.$$

We shall correspond in the following to any sequence $\{u_\nu\}$, the *series*

$$\sum u_\nu\equiv u_1+u_2+\ldots\ldots+u_\nu+\ldots\ldots,$$

where the notation does not concern the convergency of the series on the right-hand side.

If the series $\sum(z_\nu-z_{s_\nu})$ be absolutely convergent, we say that the sequence $\{z_\nu\}$ *admits* the substitution $S=\begin{pmatrix}\nu\\ s_\nu\end{pmatrix}$. The system of all possible substitutions is the *symmetric group*[1] and the system of all substitutions admitted by a sequence forms a group of the sequence. It can easily be proved as in the previous paper.[2]

2. *The necessary and sufficient condition that the sequence $\{z_\nu\}$ admits the symmetric group is that we may put for any ν*

$$z_\nu=a_\nu+c,$$

where the series $\sum a_\nu$ is absolutely convergent and c is a constant.

This is the generalization of the like proposition with the convergent

1, 2. *Loc. cit.*, 220.

series, given in the previous paper.

If the sequence $\{z_\nu\}$ admits the symmetric group, it is clear that $\{z_\nu\}$ must be a limited sequence. (When the absolute values of all the numbers of a sequence are limited as a whole, then we say the sequence is *limited* or else *not limited.* A limited sequence is nothing but a *bounded* set of points.) Moreover the derived set of the set of points $\{z_\nu\}$ cannot have more than one limiting point. For if otherwise, let α, β be two of them such that

$$\lim_{n\to\infty} z_{\nu_n} = \alpha,$$

$$\lim_{n\to\infty} z_{\mu_n} = \beta$$

where the sequences $\{\nu_n\}$, $\{\mu_n\}$ are the partial sequences of $\{\nu\}$, then the series

$$(z_{\nu_1}-z_{\mu_1})+(z_{\mu_1}-z_{\nu_1})+\ldots\ldots+(z_{\nu_n}-z_{\mu_n})+(z_{\mu_n}-z_{\nu_n})+\ldots\ldots$$

cannot be absolutely convergent. Hence the substitution written by cycles[1] $(\nu_1\ \mu_1)\ldots\ldots(\nu_n\ \mu_n)\ldots\ldots$, the remaining integers being unaffected, is not admitted by the sequence $\{z_\nu\}$. This is contradictory to the hypothesis. We denote by c the single limiting point of the set of points $\{z_\nu\}$ and write

$$z_\nu - c \equiv a_\nu, \qquad \nu = 1, 2, \ldots\ldots$$

Then $\lim\limits_{n\to\infty} a_\nu = 0$ and the sequence $\{z_\nu - c\}$ or $\{a_\nu\}$ must admit the symmetric group and hence the series $\sum a_\nu$ must be absolutely convergent. For if the series $\sum a_\nu$ is not absolutely convergent, its real or imaginary part must not be absolutely convergent. Suppose the real part is not so. Let $\sum a_\nu$ be the real part. Then since $\sum a_\nu$ is not absolutely convergent, the sum of all the positive terms or that of all the negative terms of $\sum a_\nu$ must be divergent. Now suppose the sum of all the positive terms of $\sum a_\nu$ be divergent and denote it by $\sum b_n$. By the rearrangement of the terms of the sequence $\{a_\nu\}$, we may assume

$$b_1 \geqq b_2 \geqq \ldots\ldots \geqq b_n \geqq \ldots\ldots \geqq 0.$$

Since a_ν tends to zero, we have

$$\lim_{n\to\infty} b_n = 0,$$

and the sequence $\{b_n\}$ must admit any substitution since it is a substitution of the symmetric group. Now we can find terms b_{n_1}, b_{n_2}, b_{n_3}, ... such

1. *Loc. cit.*, 213.

that

$$\frac{1}{2}b_1 > b_{n_1},$$

$$\frac{1}{2}b_2 > b_{n_2},$$

$$\frac{1}{2}b_3 > b_{n_3},$$

$$\cdots\cdots\cdots\cdots,$$

different from one another, where we do not write on the left-hand sides of these inequalities any term which once stood on the right-hand side of these inequalities. Since

$$\begin{aligned} &|b_1 - b_{n_1}| + |b_2 - b_{n_2}| + |b_3 - b_{n_3}| + \cdots\cdots \\ &+ |b_{n_1} - b_1| + \cdots\cdots \\ &+ |b_{n_2} - b_2| + \cdots\cdots \\ &+ |b_{n_3} - b_3| + \cdots\cdots \\ &+ \cdots\cdots\cdots \\ &> \frac{b_1}{2} + \frac{b_2}{2} + \frac{b_3}{2} \\ &+ \frac{b_{n_1}}{2} + \cdots\cdots \\ &+ \frac{b_{n_2}}{2} + \cdots\cdots \\ &+ \frac{b_{n_3}}{2} + \cdots\cdots \\ &+ \cdots\cdots\cdots \\ &= \frac{1}{2}(b_1 + b_2 + \cdots\cdots + b_n + \cdots\cdots) \end{aligned}$$

and the substitution

$$\begin{pmatrix} 1 & 2 & 3 & \cdots\cdots & n_1 & \cdots\cdots & n_2 & \cdots\cdots & n_3 & \cdots\cdots \\ n_1 & n_2 & n_3 & \cdots\cdots & 1 & \cdots\cdots & 2 & \cdots\cdots & 3 & \cdots\cdots \end{pmatrix}$$

must be admitted by the sequence $\{b_n\}$, the left-hand side of the above inequality must be convergent. Therefore $\sum b_n$ must be convergent. This is contradictory to the assumption. Hence the series $\sum a_\nu$ must be absolutely convergent.

Conversely if

$$z_\nu = a_\nu + c, \qquad \nu = 1,\ 2,\ \ldots\ldots\ldots$$

and $\sum a_\nu$ be absolutely convergent, the sequence $\{a_\nu\}$ *i. e.*, $\{z_\nu - c\}$ admits cleary the symmetric group. Hence the sequence $\{z_\nu\}$ admits the symmetric group.

We remark that any sequence of constant terms is a special case of the above sequence.

3. By the proposition just proved we see that *if the sequence* $\{z_\nu\}$ *be not limited, it cannot admit the symmetric group. Moreover some of the unlimited sequence cannot admit any substitution of the infinite degree.* For example the sequence

$$\{\nu\} = 1,\ 2,\ \ldots\ldots,\ \nu,\ \ldots\ldots$$

can neither admit the symmetric group nor any substitution of the infinite degree.

On the other hand *any limited sequence* $\{z_\nu\}$ *must admit some substitutions of the infinite degree.* For let a be a point of the derived set of $\{z_\nu\}$, then we can find its partial sequence $\{z_{\nu_n}\}$ such that

$$\lim_{n\to\infty} z_{\nu_n} = a\ .$$

Since a is finite, there is an infinite number of pairs of terms of $\{z_{\nu_n}\}$ such that for any given positive number $\epsilon\,(<1)$,

$$|z_{\nu_m} - z_{\nu_n}| < \epsilon,$$

$$|z_{\nu_p} - z_{\nu_q}| < \epsilon^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Therefore the sequence $\{z_\nu\}$ admits the substitution of the infinite degree

$$(\nu_m\ \nu_n)(\nu_p\ \nu_q)\ \ldots\ldots\ldots\ .$$

We may suppose $m<n<p<q<\ldots\ldots$. Let us write

$$\mu_1 \equiv \nu_m,\quad \mu_2 \equiv \nu_n,\quad \mu_3 \equiv \nu_p,\quad \mu_4 \equiv \nu_q\ \ldots\ldots,$$

then we have

$$\lim_{k\to\infty} z_{\mu_k} = a,$$

and the sequence $\{z_\nu\}$ admits the substitution $(\mu_1\ \mu_2)(\mu_3\ \mu_4)\ldots$ of the infinite degree. Hence if the sequence $\{z_\nu\}$ be limited and a (finite) be one of the point of the derived set of $\{z_\nu\}$, then we can find a substitution $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\ldots$ admitted by $\{z_\nu\}$ and that $\lim\limits_{n\to\infty} z_{\nu_n} = a$.

Moreover if the derived set of any sequence $\{z_\nu\}$ contains at least a point at finiteness, then we can also find a partial sequence of $\{z_\nu\}$ whose terms tend to a. Hence we have in general the proposition:

If the derived set of any sequence $\{z_\nu\}$ *contains at least a point* a *at finiteness, then we can find a substitution* $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\ldots$ *admitted by* $\{z_\nu\}$ *such that* $\lim_{n\to\infty} z_{\nu_n}=a$.

This proposition is not true with some unlimited sequences.

But if the derived set of $\{z_\nu\}$ contains no point different from ∞, then we must have $\lim_{\nu\to\infty} z_\nu=\infty$, whatever the group admitted by the sequence $\{z_\nu\}$.

Conversely *if any sequence* $\{z_\nu\}$ *admits substitution of the form* $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\ldots$, *we may extract from the sequence* $\{z_{\nu_n}\}$ *a partial sequence which have a determinate limit.* For if the derived set of points of $\{z_{\nu_n}\}$ has a point different from ∞, then by the preceding proposition, there is a partial sequence of $\{z_{\nu_n}\}$ which converges to a finite determinate limit. If the derived set of $\{z_{\nu_n}\}$ has no point different from ∞, as we have remarked, the sequence $\{z_{\nu_n}\}$ must tend to ∞.

4. Given two sequences $\{z_\nu\}$ and $\{Z_\nu\}$, if the sequence $\{Z_\nu\}$ admits all the substitutions of the group of the sequence $\{z_\nu\}$, we say that *the sequence* $\{Z_\nu\}$ *admits the group* of the sequence $\{z_\nu\}$. In this case the group of the sequence $\{z_\nu\}$ is a divisor of the group of the sequence $\{Z_\nu\}$.

If a sequence $\{Z_\nu\}$ *admits the group of a limited sequence* $\{z_\nu\}$, *we may correspond to any point* a *of the derived set of* $\{z_\nu\}$ *one and only one point of the derived set of* $\{Z_\nu\}$ *and by this correspondence there remains no point in the derived set of* $\{Z_\nu\}$ *which does not correspond to the points of the derived set of* $\{z_\nu\}$.

For since a is finite, by a proposition of § 3, we can extract a partial sequence $\{z_{\nu_n}\}$ of $\{z_\nu\}$ such that the sequence $\{z_\nu\}$ admits the substitution $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\ldots$ and that $\lim_{n\to\infty} z_{\nu_n}=a$. By hypothesis the sequence $\{Z_\nu\}$ admits the substitution $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\ldots$. Hence by a proposition of § 3, from the sequence $\{\nu_n\}$ we can extract a partial sequence $\{\mu_m\}$, such that $\lim_{m\to\infty} Z_{\mu_m}=A$, where A is determinate. It may be supposed that there would be another partial sequence $\{\lambda_l\}$ of the same sequence $\{\nu_n\}$, such that $\lim_{l\to\infty} Z_{\lambda_l}=B$, where B is determinate and different from A. We shall prove that *under our hypothesis* B *is* *necessarily identical to A.*

For if the sequence $\{\mu_m\}$ and $\{\lambda_l\}$ have an infinite number of common elements, then clearly $A=B$. We assume that there are only a finite number of common elements, or omitting the common elements, we may assume that there are no common elements. In such a case, suppose $A\neq B$. Then since

$$\lim_{m\to\infty} Z_{\mu_m}=A, \qquad \lim_{l\to\infty} Z_{\lambda_l}=B,$$

δ being a positive number such as

$$|A-B|>\delta,$$

we may find an integer E such that

$$|Z_{\mu_m}-Z_{\lambda_l}|>\delta, \quad \textit{for any} \quad \mu_m,\ \lambda_l>E.$$

On the other hand since $\{\mu_m\}$ and $\{\lambda_l\}$ are the partial sequences of $\{\nu_n\}$, we have

$$\lim_{m\to\infty} z_{\mu_m}=\lim_{l\to\infty} z_{\lambda_l}=a.$$

Therefore for any given positive number $\epsilon(<1)$, we can find an integer G such that

$$|z_{\mu_m}-z_{\lambda_l}|<\epsilon, \quad \textit{for any} \quad \mu_m,\ \lambda_l>G.$$

Hence there are pairs of integers (μ_m, λ_l) which satisfy the simulteneous inequalities

$$|Z_{\mu_m}-Z_{\lambda_l}|>\delta,$$
$$|z_{\mu_m}-z_{\lambda_l}|<\epsilon^k,$$

where k is a positive integer. We take successively such a pair of integers for which $k=1, 2, \ldots$ We may assume, avoiding the complex notations, that for $k=1$, (μ_1, λ_1) is such a pair; for $k=2$, (μ_2, λ_2); ... Then since the series $\sum(z_{\mu_k}-z_{\lambda_k})$ is absolutely convergent, the sequence $\{z_\nu\}$ admits the substitution $(\mu_1\,\lambda_1)(\mu_2\,\lambda_2)\ldots(\mu_k\,\lambda_k)\ldots$ which by hypothesis must also be admitted by the sequence $\{Z_\nu\}$. Accordingly the series $\sum(Z_{\mu_k}-Z_{\lambda_k})$ must be absolutely convergent which is contrary to the inequality

$$|Z_{\mu_m}-Z_{\lambda_l}|>\delta.$$

Therefore the assumption $A\neq B$ is inadmissible, or we must have $A=B$. Thus it is proved that $\lim_{n\to\infty} Z_{\nu_n}=A$.

We have arrived at the result that by aid of the substitution $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\ldots$ admitted by the sequence $\{z_\nu\}$ such that $\lim_{n\to\infty} z_{\nu_n} = a$, there corresponds to a one and only one point $A = \lim_{n\to\infty} Z_{\nu_n}$.

But to prove completely our proposition, we must show that for any two substitutions $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\ldots$ and $(\mu_1\ \mu_2)(\mu_3\ \mu_4)\ldots$ admitted by the sequence $\{z_\nu\}$ such that

$$\lim_{n\to\infty} z_{\nu_n} = \lim_{n\to\infty} z_{\mu_n} = a,$$

we have the equality

$$\lim_{n\to\infty} Z_{\nu_n} = \lim_{n\to\infty} Z_{\mu_n} = A.$$

This may be proved by a similar consideration. If the sequence $\{\nu_n\}$ and $\{\mu_n\}$ have an infinite number of common elements, then the above equalities are cleary true. If otherwise, since

$$\lim_{n\to\infty} z_{\nu_n} = \lim_{n\to\infty} z_{\mu_n} = a,$$

for any given positive number $\epsilon(<1)$, we can find for each $k=1, 2, \ldots$, a pair of integers (ν_n, μ_n) such that

$$|z_{\nu_n} - z_{\mu_n}| < \epsilon^k,$$

where ν_n and μ_n are different from each other. Avoiding the complex notations, we may write

$$|z_{\nu_1} - z_{\mu_1}| < \epsilon,$$

$$|z_{\nu_2} - z_{\mu_2}| < \epsilon^2,$$

$$\ldots\ldots\ldots\ldots\ldots,$$

Therefore the series $\sum (z_{\nu_k} - z_{\mu_k})$ is absolutely convergent. Accordingly the sequence $\{z_\nu\}$ admits the substitution $(\nu_1\ \mu_1)(\nu_2\ \mu_2)\ldots$ which by hypothesis must be admitted by the sequence $\{Z_\nu\}$. Hence as we have already proved, the sequence $Z_{\nu_1}, Z_{\mu_1}, Z_{\nu_2}, Z_{\mu_2}, \ldots\ldots$ tends to a determinate limit. So we have

$$\lim_{n\to\infty} Z_{\nu_n} = \lim_{n\to\infty} Z_{\mu_n} = A.$$

To finish our proof we must prove that there is no point of the derived set of $\{Z_\nu\}$ which does not correspond to a point of the derived set of $\{z_\nu\}$. For let A be any point of the derived set of $\{Z_\nu\}$, then there is a partial sequence $\{Z_{\nu_n}\}$ of $\{Z_\nu\}$ such that $\lim_{n\to\infty} Z_{\nu_n} = A$.

Consider the derived set of the partial sequence $\{z_{\nu_n}\}$ of $\{z_\nu\}$. Since the sequence $\{z_\nu\}$ is limited, the derived set of $\{z_{\nu_n}\}$ must have at least a point a at finiteness. Therefore A is already corresponded to a.

Thus to any point of the derived set of the bounded set $\{z_\nu\}$, we may correspond one and only one point of the derived set of $\{Z_\nu\}$ and there remains no point in the derived set of $\{Z_\nu\}$ out of the correspondence. We remark that in general the above established correspondence is not reversible. Hence *the cardinal number of the set of the bounded set $\{z_\nu\}$ is not less than that of the derived set of $\{Z_\nu\}$ where the sequence $\{Z_\nu\}$ is conditioned to admit the group of the limited sequence $\{z_\nu\}$.*

For example consider any limited sequence $\{z_\nu\}$ and a sequence $\{Z_\nu\}$ which admits the symmetric group. The sequence $\{Z_\nu\}$ admits the group of the sequence $\{z_\nu\}$ and whatever be the cardinal number of the derived set of $\{z_\nu\}$, that of the derived set of $\{Z_\nu\}$ is unity.

5. *If the sequence $\{Z_\nu\}$ admits the group of the limited sequence $\{z_\nu\}$, the sequence $\{Z_\nu\}$ must itself be limited.*

For if the sequence $\{Z_\nu\}$ is not limited, we may extract a partial sequence $\{Z_{\nu_n}\}$ of $\{Z_\nu\}$ such that $\lim_{n\to\infty} Z_{\nu_n}=\infty$. Here we may assume that the absolute values of the numbers of $\{Z_{\nu_n}\}$ are increasing with n. Consider the partial set $\{z_{\nu_n}\}$ of $\{z_\nu\}$. Since the set of points $\{z_\nu\}$ is bounded, the derived set of the partial set $\{z_{\nu_n}\}$ must have a point a at finiteness and we may extract a new partial sequence $\{z_{\mu_m}\}$ of $\{z_{\nu_n}\}$ such that

$$\lim_{m\to\infty} z_{\mu_m}=a.$$

Next we want to extract a partial sequence $\{z_{\lambda_l}\}$ of $\{z_{\mu_m}\}$, hence of $\{z_{\nu_n}\}$ such that the series $\sum(z_{\lambda_l}-z_{\lambda_{l+1}})$ shall be absolutely convergent. This is always possible. To prove it, put

$$\zeta_m \equiv z_{\mu_m} - a, \qquad m=1,\ 2,\ \ldots\ldots$$

and consider the set $\{\zeta_m\}$. The limiting point of the set is the origin of ζ-plane. Since the set $\{\zeta_m\}$ contains an infinite number of points tending to the origin, there must be at least in one of the four quadrants, an infinite number of points tending to the origin. Let it be for simplicity the first quadrant. Let ϵ be a positive number and consider the square in the first quadrant, the length of its sides being equal to ϵ and the sides being the axes of ζ-plane and their parallels. In this square there is a point of the set $\{\zeta_m\}$. Let it be

$$\zeta_{m_1} \equiv \xi_{m_1} + i\eta_{m_1},$$

then we have

$$0 \leqq \xi_{m_1} \leqq \epsilon, \qquad 0 \leqq \eta_{m_1} \leqq \epsilon.$$

Again consider the square in the quadrant, the length of its sides being equal to $\frac{\epsilon}{2}$ and the sides being the axes and their parallels. In this square there is a point of the set $\{\zeta_m\}$, different from the preceding point. Let it be

$$\zeta_{m_2} \equiv \xi_{m_2} + i\eta_{m_2},$$

then we have

$$0 \leqq \xi_{m} \leqq \frac{\epsilon}{2}, \qquad 0 \leqq \eta_{m_2} \leqq \frac{\epsilon}{2}.$$

(It would be unnecessary to notice $m_1 < m_2$ and the like in the following.) Again consider the square in the quadrant, the length of its sides being equal to $\frac{\epsilon}{2^2}$ and the sides being the axes and their parallels. There is a point in this square different from the preceding points. Let it be

$$\zeta_{m_3} \equiv \xi_{m_3} + \eta_{m_3},$$

then we have

$$0 \leqq \xi_{m_3} \leqq \frac{\epsilon}{2^2}, \qquad 0 \leqq \eta_{m_3} \leqq \frac{\epsilon}{2^2}.$$

We continue these processes indefinitely. Now we have

$$\begin{aligned}
|\zeta_{m_1} - \zeta_{m_2}| &= \sqrt{(\xi_{m_1} - \xi_{m_2})^2 + (\eta_{m_1} - \eta_{m_2})^2} \\
&< |\xi_{m_1} - \xi_{m_2}| + |\eta_{m_1} - \eta_{m_2}| \\
&< 2\left(\epsilon + \frac{\epsilon}{2}\right),
\end{aligned}$$

also

$$|\zeta_{m_2}-\zeta_{m_3}|<2\left(\frac{\epsilon}{2}+\frac{\epsilon}{2^2}\right),$$

$$\dots\dots\dots\dots\dots\dots\dots,$$

Hence

$$|\zeta_{m_1}-\zeta_{m_2}|+|\zeta_{m_2}-\zeta_{m_3}|+\cdots\cdots+|\zeta_{m_l}-\zeta_{m_{l+1}}|+\cdots\cdots$$

$$<2\epsilon+4\frac{\epsilon}{2}+4\frac{\epsilon}{2^2}+\cdots\cdots\cdots\cdots$$

$$=6\epsilon.$$

The sequence $\{\zeta_{m_l}\}$ is a partial sequence of $\{\zeta_m\}$ *i. e.*, of $\{z_{\mu_m}-a\}$. Avoiding the complex notations, we write

$$\zeta_{m_l}=z_{\lambda_l}-a, \qquad l=1,\ 2,\ \cdots\cdots\ \cdots\cdots,$$

where $\{\lambda_l\}$ is a partial sequence of $\{\mu_m\}$. Then the series

$$\sum(\zeta_{m_l}-\zeta_{m_{l+1}})=\sum(z_{\lambda_l}-z_{\lambda_{l+1}})$$

is absolutely convergent and the sequence $\{z_{\lambda_l}\}$ is a partial sequence of $\{z_{\mu_m}\}$, hence of $\{z_{\nu_n}\}$.

Since $\{z_{\lambda_l}\}$ is a partial sequence of $\{z_{\nu_n}\}$ which corresponds to the sequence $\{Z_{\nu_n}\}$, the sequence $\{Z_{\lambda_l}\}$ is a partial sequence of $\{Z_{\nu_n}\}$. Moreover since the absolute values of the numbers of the sequence $\{Z_{\nu_n}\}$ are assumed increasing with n and $\lim_{n\to\infty} Z_{\nu_n}=\infty$, the absolute values of the numbers of $\{Z_{\lambda_l}\}$ must be also increasing and $\lim_{l\to\infty} Z_{\lambda_l}=\infty$. On the other hand as we have proved, the series $\sum(z_{\lambda_l}-z_{\lambda_{l+1}})$ is absolutely convergent. Therefore the substitutions $(\lambda_1\lambda_2)(\lambda_3\lambda_4)\cdots\cdots$and $(\lambda_2\lambda_3)\times(\lambda_4\lambda_5)\cdots\cdots$ are admitted by the sequence $\{z_\nu\}$. Hence by hypothesis these substitutions must be also admitted by the sequence $\{Z_\nu\}$ *i. e.*, the series

$$(Z_{\lambda_1}-Z_{\lambda_2})+(Z_{\lambda_3}-Z_{\lambda_4})+\cdots\cdots$$

and

$$(Z_{\lambda_2}-Z_{\lambda_3})+(Z_{\lambda_4}-Z_{\lambda_5})+\cdots\cdots$$

are absolutely convergent. Therefore the series

$$|Z_{\lambda_1}|+|Z_{\lambda_1}-Z_{\lambda_2}|+|Z_{\lambda_2}-Z_{\lambda_3}|+\cdots\cdots$$

is also convergent. Since

$$|Z_{\lambda_{l+1}}|>|Z_{\lambda_l}|,$$

$$|Z_{\lambda_{l+1}}-Z_{\lambda_l}|>|Z_{\lambda_{l+1}}|-|Z_{\lambda_l}|,$$

$$l=1,\ 2,\cdots\cdots\cdots\cdots\cdots\cdots,$$

comparing with the above series, the series

$$|Z_{\lambda_1}|+(|Z_{\lambda_2}|-|Z_{\lambda_1}|)+(|Z_{\lambda_3}|-|Z_{\lambda_2}|)+\cdots\cdots$$

must be convergent, which is nothing but

$$\lim_{l\to\infty}|Z_{\lambda_l}|=\textit{finite and determinate.}$$

This is contradictory to $\lim_{l\to\infty} Z_{\lambda_l}=\infty$. Thus the sequence $\{Z_\nu\}$ must be limited.

In the above proof we have assumed that in the first quadrant there is an infinite number of different points of the set $\{\zeta_m\}$. But the proof is also valid when the first quadrant contains only an infinite number of coincident points of the set.

6. *If two limited sequences $\{z_\nu\}$ and $\{Z_\nu\}$ have the same group, the cardinal numbers of the derived sets of $\{z_\nu\}$ and $\{Z_\nu\}$ must be equal.*

As we have said in course of the proof of the proposition of § 4, since the sequence $\{Z_\nu\}$ admits the group of the sequence $\{z_\nu\}$, to any point a of the derived set of $\{z_\nu\}$ there is a substitution $(\nu_1\ \nu_2)(\nu_3\ \nu_4)\cdots$ admitted by the sequences $\{z_\nu\}$ and $\{Z_\nu\}$ such that

$$\lim_{n\to\infty} z_{\nu_n}=a, \qquad \lim_{n\to\infty} Z_{\nu_n}=A,$$

where the point A is one and only one point corresponding to the point a. In the present case, since the sequence $\{z_\nu\}$ admits the group of the sequence $\{Z_\nu\}$, to a point A of the derived set of $\{Z_\nu\}$, there corresponds one and only one point of the derived set of $\{z_\nu\}$ while the point a corresponds to the point A. Therefore the correspondence of a and A is one to one reversible. Moreover a is any point of the derived set of $\{z_\nu\}$ and there are no points of the derived set of $\{Z_\nu\}$ which remain out of the present correspondence. Thus we have established one to one correspondence between the elements of the derived sets of $\{z_\nu\}$ and $\{Z_\nu\}$. Therefore the derived sets of $\{z_\nu\}$ and $\{Z_\nu\}$ are equivalent to each other, or their cardinal numbers are equal.

We remark that the equality of the cardinal numbers of the derived sets of $\{z_\nu\}$ and $\{Z_\nu\}$ is not the sufficient condition of the coincidence of their group. For let

$$\{z_\nu\}=\left\{(-1)^{\nu-1}\frac{1}{\nu}\right\},\quad \{Z_\nu\}=\left\{\frac{1}{\nu^2}\right\}.$$

Their derived sets are the same point 0. But the series $\sum z_\nu$ being semi-convergent, the set $\{z_\nu\}$ can not admit the symmetric group. On the other hand the series $\sum Z_\nu$ being absolutely convergent, the sequence $\{Z_\nu\}$ admits the symmetric group (§ 2). Thus they have different groups.

7. Consider a holomorphic function $Z=f(z)$ in a domain and $\{z_\nu\}$ be a set of points within the domain such that all the points of the derived set shall be also within the domain. Suppose that the sequence $\{z_\nu\}$ admits the substitution $S\equiv\binom{\nu}{s_\nu}$. Putting

$$Z_\nu=f(z_\nu),\qquad \nu=1,2,\cdots\cdots\cdots\cdots,$$

we have

$$|Z_\nu-Z_{s_\nu}|=\frac{|z_\nu-z_{s_\nu}|}{2\pi}\left|\int_C\frac{f(z)dz}{(z-z_\nu)(z-z_{s_\nu})}\right|,$$

where the curve (C) in the domain is so drawn that it contains all the points of the derived set of $\{z_\nu\}$. By the choice of the path of integration, there is a finite number ω such that

$$|z-z_\nu|>\omega>0,\qquad \nu=1,2,\cdots\cdots\cdots\cdots\cdots.$$

Therefore the second factor on the right of the above equality is limited for all ν. Since the series $\sum(z_\nu-z_{s_\nu})$ is by hypothesis absolutely convergent, the series $\sum(Z_\nu-Z_{s_\nu})$ is so also and the sequence $\{Z_\nu\}$ admits the substitution S. Therefore the sequence $\{Z_\nu\}$ admits the group of the sequence $\{z_\nu\}$.

For example consider a linear function

$$Z=\frac{\alpha z+\beta}{\gamma z+\delta},$$

and a limited sequence $\{z_\nu\}$. If the point $z=-\dfrac{\delta}{\gamma}$ is not contained in the set $\{z_\nu\}$ or in its derived set, the sequence

$$\{Z_\nu\}=\left\{\frac{\alpha z_\nu+\beta}{\gamma z_\nu+\delta}\right\}$$

admits the group of the sequence $\{z_\nu\}$. Consider the inverse function

$$z=\frac{\delta Z-\beta}{-\gamma Z+\alpha}.$$

Since $\{z_\nu\}$ is a limited sequence, the point $Z=\dfrac{\alpha}{\gamma}$ is contained neither in the sequence $\{Z_\nu\}$ nor in its derived set (§ 5). Therefore the sequence $\{z_\nu\}$ admits the group of the sequence $\{Z_\nu\}$ *i. e.*, they have the same group. Therefore the cardinal numbers of the derived sets of $\{z_\nu\}$ and $\{Z_\nu\}$ are equal to each other. *Thus the group of a sequence does not change by a linear transformation of the numbers of the sequence provided that the pole of the transformation is not contained in the sequence or in its derived set.*

We remark that even when a sequence $\{Z_\nu\}$ admits the group of a sequence $\{z_\nu\}$, Z cannot necessarily be expressed by a holomorphic function of z provided that all the points of the set $\{z_\nu\}$ and those of its derived set be required to lie within the domain of the holomorphic function.

For let the group of the sequence $\{Z_\nu\}$ be symmetric and the derived set of $\{z_\nu\}$ contain an infinite number of points. a being any point of the derived set of $\{z_\nu\}$, there is a substitution $(\nu_1\,\nu_2)\,(\nu_3\,\nu_4)\cdots\cdots\cdots\cdots\cdots$, admitted by the sequences $\{z_\nu\}$ and $\{Z_\nu\}$ such that

$$\lim_{n\to\infty} z_{\nu_n}=a, \qquad \lim_{n\to\infty} Z_{\nu_n}=A,$$

where A is the unique limit of the sequence $\{Z_\nu\}$. Let $Z=f(z)$ be the required functional relation, then since it is supposed to be holomorphic, we have

$$f(a)=A.$$

But by assumption there is an infinite number of such point a within the domain of existence. Therefore $f(z)$ must be the constant A.

8. We consider in the following the set of all the substitutions of

a group. Omitting the phraseology "of all the substitutions" we say the above set, *the set of the group*. For example the set of the symmetric group is the set of all substitutions of the symmetric group.

The set of the symmetric group may be ordered (geordnet).

For consider any two substitutions written in the normal form (*i. e.*, the integers in the upper line of the substitutions are in the natural order):

$$S \equiv \begin{pmatrix} 1 & 2 \cdots\cdots\cdots & \nu \cdots\cdots\cdots \\ s_1 & s_2 \cdots\cdots\cdots & s_\nu \cdots\cdots\cdots \end{pmatrix},$$

$$T \equiv \begin{pmatrix} 1 & 2 \cdots\cdots\cdots & \nu \cdots\cdots\cdots \\ t_1 & t_2 \cdots\cdots\cdots & t_\nu \cdots\cdots\cdots \end{pmatrix}.$$

The equality of the two substitutions is by definition that for any given positive integer N however great, we have the relations

$$s_\nu = t_\nu, \qquad \nu = 1,\ 2, \cdots\cdots\cdots,\ N.$$

If the condition be not fulfilled, we say they are different. If the two substitutions are different, there is a positive integer μ such that

$$s_\mu \neq t_\mu.$$

Anyhow *two substitutions are either equal or different.* Of-course the set of a group does not contain equal substitutions as the group itself does not so.

Now we prove the above proposition. Let S and T be any two elements of the set of the symmetric group, then since these substitutions are different, there is an integer μ such that

$$s_\mu \neq t_\mu.$$

Comparing all the two integers standing in the same vertical line of the finite sequences:

$$s_1,\quad s_2, \cdots\cdots\cdots,\ s_{\mu-1},$$
$$t_1,\quad t_2, \cdots\cdots\cdots,\ t_{\mu-1}.$$

If there be a pair of integers $(s_\lambda,\ t_\lambda)$ such that $s_\lambda \neq t_\lambda$, we may write μ instead of λ, so that we may assume

$$s_1 = t_1,\quad s_2 = t_2, \cdots\cdots\cdots, s_{\mu-1} = t_{\mu-1},\quad s_\mu \neq t_\mu.$$

If $s_\mu < t_\mu$, we order S before T, *i. e.*, $S < T$. If $s_\mu > t_\mu$, we order T before S, *i. e.*, $T < S$. Thus the set of the symmetric group is ordered. We notice that the set has the first element, namely

$$E\equiv\begin{pmatrix}1 & 2\cdots\cdots n\cdots\cdots\\ 1 & 2\cdots\cdots n\cdots\cdots\end{pmatrix}.$$

For consider any element of the set

$$S\equiv\begin{pmatrix}1 & 2\cdots\cdots \nu\cdots\cdots\\ s_1 & s_2\cdots\cdots s_\nu\cdots\cdots\end{pmatrix}.$$

Since s_1 is a positive integer, it is not less than unity. If $s_1>1$, we have $E<S$, or else let s_ν be the first integer different from μ. Then since

$$s_1=1,\quad s_2=2,\cdots\cdots,\quad s_{\mu-1}=\mu-1,\quad s_\mu\neq\mu,$$

we must have $s_\mu>\mu$. Therefore again $E<S$.

9. *The cardinal number of the set of the symmetric group is equal to* $\mathfrak{c}$, *where the german letter* $\mathfrak{c}$ *signifies the cardinal number of the continuum.*

For consider a substitution of the group. It is nothing but a permutation of a countably infinite number of elements *i.e.*, a distribution (Belegung) of a countably infinite number of elements. Therefore the set of the symmetric group is equivalent to a proper subset of the distribution-set (Belegungsmenge) of countably infinite number of elements on a countably infinite number of elements which is equivalent to the continuum. The subset is proper, since for the substitution, the distributed elements must be different from one another. We shall prove in the following that the continuum is equivalent to a proper subset of the set of the symmetric group.

Consider the continuum formed by all the points of a square drawn in z-plane. We can denote all the rational points of the continuum by $z_1, z_2,\cdots\cdots\cdots, z_\nu,\cdots\cdots\cdots$ which form a limited sequence $\{z_\nu\}$. The derived set of $\{z_\nu\}$ is nothing but the continuum. Now consider any point of the continuum. As we have proved (§ 3), there is a substitution $(\nu_1\,\nu_2)\times(\nu_3\,\nu_4)\cdots\cdots\cdots$ admitted by the sequence $\{z_\nu\}$ such that

$$\lim_{n\to\infty} z_{\nu_n}=\alpha.$$

We correspond the point α to the substitution $(\nu_1\,\nu_2)\,(\nu_3\,\nu_4)\cdots\cdots\cdots$. Let β be another point different from the point α of the continuum. In the same way we may correspond the point β to a substitution, say $(\mu_1\,\mu_2)\,(\mu_3\,\mu_4)\cdots\cdots$ admitted by the sequence $\{z_\nu\}$. It is clear that these two substitutions are different, or else since

$$\lim_{n\to\infty} z_{\nu_n}=\alpha,\qquad \lim_{m\to\infty} z_{\mu_m}=\beta,$$

α and β must be equal. Thus we may correspond any point of the continuum to a substitution of the group of the sequence $\{z_\nu\}$.

Since the derived set of $\{z_\nu\}$ contains more than one point, the group of the sequence $\{z_\nu\}$ cannot be the symmetric group (§ 2). It is a divisor of the latter. Hence the continuum is equivalent to the proper subset of the set of the symmetric group. This shows the equality of the cardinal numbers of the continuum and of the set of the symmetric group.

10. I have defined in the previous paper[1] the equivalency of two convergent series of real numbers. In the following we shall extend this idea. Let $\sum u_\nu$, $\sum v_\nu$ be two series of complex numbers (convergent or divergent). If the series $\sum(u_\nu - v_\nu)$ are absolutely convergent, we say that the two series are *equivalent* and by symbol we write $\sum u_\nu \sim \sum v_\nu$.

Two series which are equivalent to a series are equivalent to each other. The proof is the same with the convergent series.

Let $S = \binom{\nu}{s_\nu}$ be a substitution admitted by a sequence $\{z_\nu\}$, then by definition $\sum z_\nu \sim \sum z_{s_\nu}$. We write for simplicity

$$\sum z_{s_\nu} \equiv \sum z_\nu S.$$

If $\sum z_\nu$ be convergent, it has the same sum with $\sum z_\nu S$. But in the general case, we cannot say about their convergency, though we can prove that

$$\sum (z_\nu - z_{s_\nu}) = 0,$$

quite in the same way as in the previous paper.[2]

Now we can extend the idea of the class of the convergent series. Let $\sum z_\nu$ be any series (convergent or divergent) with a group Γ, then the series transformed of $\sum z_\nu$ by any substitutions form a *class i. e.*, the class in which is contained the series $\sum z_\nu$, is the totality of all the transformed series of $\sum z_\nu$ which are equivalent to $\sum z_\nu$. Let $P \equiv \binom{\nu}{p_\nu}$ be any substitution not contained in the group Γ. The series $\sum z_\nu$ and $\sum z_\nu P$ are not of-course equivalent. But we have

$$\sum z_\nu P \sim \sum z_\nu PS,$$

i. e., the group of the series $\sum z_\nu P$ is also Γ[3]. Now denoting the symmetric group by $\sum$, we may divide its elements into classes by the

1. *Loc. cit.*, 236. 2. *Loc. cit.*, 215. 3. *Ibid.*, 235.

idea of the co-sets. Since P is not contained in Γ, all the products of P with all the elements of Γ (we denote them by $P\Gamma$), are quite different from the elements of Γ. Let Q be another substitution contained neither in Γ nor in $P\Gamma$, then all the substitutions of $Q\Gamma$ are quite different from those of Γ and $P\Gamma$. We continue this process and obtain the sets $P\Gamma$, $Q\Gamma$,........ Therefore we may write

$$\sum = \Gamma + P\Gamma + Q\Gamma + \cdots\cdots\cdots\cdots .$$

Let the class of the series $\sum z_\nu$ be denoted by C_E, that of $\sum z_\nu P$ by C_P, that of $\sum z_\nu Q$ by C_Q, ..., then *all the series which have the same terms as those of the series* $\sum z_\nu$ *(but different by the order) can be divided into classes* C_E, C_P, C_Q.................

This is an extension of the proposition given in the previous paper.[1]

11. Let the series $\sum x_\nu$ of real numbers be semi-convergent, then we can determine the set of the classes.

Since by hypothesis the series $\sum x_\nu$ is semi-convergent, we may rearrange its terms so that the sum of the new series becomes any number. The rearrangement is nothing but a substitution, say $S = \binom{\nu}{s_\nu}$. Therefore the sum of the series $\sum x_\nu S$ is a *function* of the substitution S. We write it as follows:

$$\sum x_\nu S = \Phi(S),$$

then by the property of the group Γ, we have

$$\sum x_\nu S = \sum x_\nu S\Gamma = \Phi(S\Gamma).$$

Thus to a class C_S, there corresponds one and only one number $\sum z_\nu S$. It may occur that two or more classes correspond to a number. But at least one class corresponds to one number. Now let C_S, C_T, ... be the classes, each corresponding to a point (number) of a segment of x-axis. Then the set of classes $\{C_S, C_T, \cdots\cdots\}$ corresponding to different numbers of the segment is equivalent to the continuum, while the set of classes is clearly a subset of all the classes $\{C_E, C_P, C_Q, \cdots\cdots\}$. Therefore the cardinal number of the set of all the classes is not less than $\mathfrak{c}$. On the other hand the set $\{E, P, Q, \cdots\cdots\}$ is a subset of the set of the symmetric group whose cardinal number is equal to $\mathfrak{c}$ (§ 9). Since the sets $\{E, P, Q, \cdots\cdots\}$ and $\{C_E, C_P, C_Q, \cdots\cdots\}$ are equivalent, the cardinal number of all the classes is not greater than that of the continuum. Therefore we have the proposition:

If a series of real numbers be convergent, then the set of all the

1. *Ibid.*, 237.

classes into which are divided the series transformed of the given series by all the substitutions of the symmetric group has the cardinal number $\mathfrak{c}$.

If the given series be absolutely convergent, the class is only one.

If the given series be divergent and if one of the transformed series be convergent, the above proposition is also valid.

If two series of complex numbers are equivalent, their real part must be also equivalent. But the converse is not generally true, since their imaginary parts may not be equivalent. Hence the number of different classes into which are divided the real part of the series $\sum z_\nu$ and all the transformed series of the real part by the symmetric group is not greater than the number of classes into which are divided the series $\sum z_\nu$ and all its transformed ones. Hence by aid of the preceding proposition, we have the proposition: *When the real part of a series or one of its transformed series of complex numbers is convergent, then the set of all the classes into which are divided the series transformed of the given series by all the substitutions of the symmetric group, has the cardinal number* $\mathfrak{c}$.

It is clear that we may state the same proposition with respect to the imaginary part of the given series, since we have only to multiply all the terms of the series by the imaginary unit.

12. *If the derived set of* $\{z_\nu\}$ *has a point* a *at finiteness, then the cardinal number of the set of the group of* $\{z_\nu\}$ *is equal to that of the continuum.*

For let $\{z_{\nu_\mu}\}$ be a partial sequence of $\{z_\nu\}$ tending to a, then we have

$$\lim_{\mu\to\infty}(z_{\nu_\mu}-a)=0.$$

Put for simplicity

$$\zeta_\mu\equiv z_{\nu_\mu}-a,\qquad \mu=1,\ 2,\cdots\cdots\cdots,$$

such that

$$\lim_{\mu\to\infty}\zeta_\mu=0.$$

From the sequence $\{\zeta_\mu\}$, we may extract a partial sequence $\{\zeta_{\mu_m}\}$ such that the series $\sum\zeta_{\mu_m}$ becomes absolutely convergent. Put again

$$\zeta_{\mu_m}\equiv Z_m,\qquad m=1,\ 2,\cdots\cdots\cdots.$$

Then the sequence $\{Z_m\}$ admits the symmetric group, since the series $\sum Z_m$ is absolutely convergent. Therefore the set of the group of $\{Z_m\}$ has the cardinal number $\mathfrak{c}$ (§ 9). Let $\{Z_m'\}$ be the result of a substitution operated on $\{Z_m\}$ and denote the corresponding letters in the same way. Then the series $\sum(Z_m - Z_m')$ is absolutely convergent and

$$Z_m - Z_m' = \zeta_{\mu_m} - \zeta'_{\mu_m}$$

$$= z_{\nu_\mu}^{(m)} - z'^{(m)}_{\nu_\mu}, \qquad m = 1, 2, \cdots\cdots\cdots,$$

where by $z_{\nu_\mu}^{(m)}$ we designate the right-hand side of the equality

$$\zeta_\mu = z_{\nu_\mu} - a$$

when the suffix of the left-hand side is μ_m. Therefore the series $\sum(z_{\nu_\mu}^{(m)} - z'^{(m)}_{\nu_\mu})$ is absolutely convergent. Thus to a substitution of the symmetric group, there corresponds a substitution of the group of $\{z_\nu\}$. So that the set of the group of the sequence $\{z_\nu\}$ has the cardinal number $\mathfrak{c}$.

As a special case *the cardinal number of the set of the group of a semi-convergent series is equal to* $\mathfrak{c}$, since its terms converge to zero. Moreover even when the given series is divergent, if there be a convergent series tronsformed of the given series by a substitution, as we have proved above, the group of the transformed series has the same cardinal number $\mathfrak{c}$. For as we have said, the group of the transformed series is identical with that of the given series (§ 10). Therefore *in such a case the set of the group of the given series has also the cardinal number* $\mathfrak{c}$.

Concluding these propositions, *if the set of the group of a sequence has a cardinal number different from* $\mathfrak{c}$, *then the derived set of the sequence must have only one point namely* ∞.

13. For example consider a sequence $\{z_\nu\}$ whose numbers satisfy the conditions

$$|z_{\nu+1}| - |z_\nu| > g > 0, \quad \nu = 1, 2, \cdots\cdots\cdots\cdots\cdots,$$

where g is a positive number. The limit of the sequence is infinite. We prove in the following that *the set of the group of this sequence is countably infinite.*

For let $S = \binom{\nu}{s_\nu}$ be any substitution admitted by the sequence, then the number of pairs of integers (ν, s_ν) such that $\nu - s_\nu \neq 0$, *i. e.*, the degree of the substitution must be finite. If otherwise the series $\sum (z_\nu - z_{s_\nu})$ has an infinite number of elements (different from zero) where by hypothesis the series is absolutely convergent. Since

$$|z_{\nu+1}| - |z_\nu| > g, \qquad \nu = 1, 2, \cdots\cdots\cdots,$$

a fortiori $\quad |z_\nu - z_{s_\nu}| > g,$

provided $\nu \neq s_\nu$. This proves the divergency of the series $\sum |z_\nu - z_{s_\nu}|$.

Or the substitution must have a finite degree. The substitution of the least degree is the cycles of two elements (of the second degree). Any cycle of a finite number of elements is admitted by our sequence. The cycles of two elements are

$$\begin{array}{l} (1\ 2),\ (1\ 3),\ (1\ 4), \cdots\cdots, \\ \qquad\qquad (2\ 3),\ (2\ 4), \cdots\cdots, \\ \qquad\qquad\qquad\quad (3\ 4), \cdots\cdots, \\ \qquad\qquad\qquad\quad \cdots\cdots\cdots\cdots. \end{array}$$

This set is a subset of the product of two countable sets. Hence it is also countable.

In general noticing that any substitution of m th degree may be expressed, according to the case, as either a cycle of m elements or as a product of some number of cycles (the elements of one cycle being different from those of any other), where the total number of the elements is m; the set of substitutions of m th degree is a subset of the product of m countable set. Therefore the set of all the substitutions of the finite degree is the sum of a countably infinite number of countable sets. Therefore it is also countable; so that the group of our series is countable.

We remark that the conditions imposed upon our example may be replaced by

$$|z_\nu| \sim |z_\mu| > g > 0, \quad \nu \neq \mu, \quad \nu, \mu = 1, 2, \cdots\cdots.$$

For it, rearrange the numbers of the sequence $\{z_\nu\}$, so that the new sequence shall be monotone in absolute value. This is possible since we are assuming that the limit of the sequence $\{z_\nu\}$ is ∞. The rearrangement is a substitution. Denoting the new sequence by $\{z'_\mu\}$, we have clearly

$$|z'_{\mu+1}| - |z'_\mu| > g > 0, \qquad \mu = 1, 2, \cdots\cdots\cdots.$$

Therefore as we have proved, the set of the group of $\{z'_\mu\}$ is countable. On the other hand we know that a sequence and its transformed one have always the same group (§ 10). Hence *if a sequence* $\{z_\nu\}$ *satisfies the condition*

$$|z_\nu|\sim|z_\mu|>g>0, \quad \nu\neq\mu, \quad \nu,\ \mu=1,\ 2,\cdots\cdots,$$

then the set of its group is countable.

If there be no positive number g, the affair is quite different. By the same reasoning as above, we may always assume that the sequence $\{z_\nu\}$ is monotone in absolute value. Now we prove that *if the inferior limit of* $|z_{\nu+1}|-|z_\nu|$ *is zero, the set of the group of the sequence* $\{z_\nu\}$ *whose derived set has only a point* ∞ *has the cardinal number* $\mathfrak{c}$.

In this case the group of the sequence has substitutions of the infinite degree. But *all the cycles admitted by the sequence must be of degree finite.* For suppose that the group has a cycle of the infinite degree such as

$$\begin{pmatrix}\cdots\cdots\beta\cdots\cdots a\cdots\cdots a\cdots\cdots b\cdots\cdots\\ \cdots\cdots a\cdots\cdots a\cdots\cdots b\cdots\cdots c\cdots\cdots\end{pmatrix}=(\cdots\cdots\beta\, a\, a\, b\, c\cdots\cdots\cdots).$$

Then the series

$$(z_a-z_b)+(z_b-z_c)+\cdots\cdots\cdots\cdots\cdots$$
$$+(z_\alpha-z_a)+(z_\beta-z_\alpha)+\cdots\cdots\cdots\cdots\cdots$$

is absolutely convergent, *a fortiori* the series

$$(z_a-z_b)+(z_b-z_c)+\cdots\cdots\ \cdots\ \cdots\cdots$$

is absolutely convergent. Denoting the general term by (z_k-z_l), we must have

$$\lim_{l\to\infty}\{(z_a-z_b)+(z_b-z_c)+\cdots\cdots+(z_k-z_l)\}$$
$$=\lim_{l\to\infty}(z_a-z_l)\ :\ \textit{finite and determinate.}$$

Therefore $\quad \lim_{l\to\infty} z_l\ :\ $ *finite and determinate.*

This is contrary to the hypothesis that the derived set of $\{z_\nu\}$ has only a point ∞.

Any substitution may be represented by a product of cycles, any one of which has all different elements from those of the other.[1] In our present case any substitution of the group is equal to a product of cycles of

1. *Loc. cit.*, 213.

degree finite. Though the substitutions of our group have a very simple form, yet the set of the group will be proved to have the cardinal number $\mathfrak{c}$. Since by hypothesis

$$\lim_{\nu\to\infty} |z_{\nu+1}| - |z_\nu| = 0,$$

for a given positive number $\epsilon(<1)$ and a positive integer k, we may find an integer ν such that

$$|z_{\nu+1}| - |z_\nu| < \epsilon^k.$$

Avoiding the complex notations, without loss of generality we may assume $2\nu-1=k$. Now changing $k=1, 2, \cdots\cdots\cdots$, we have

$$|z_2| - |z_1| < \epsilon,$$
$$|z_4| - |z_3| < \epsilon^2,$$
$$\cdots\cdots\cdots\cdots\cdots.$$

Therefore the series $\sum(z_{2k}-z_{2k-1})$ is obsolutely convergent, or the sequence $\{z_\nu\}$ admits a countably infinite number of the transpositions of quite different elements. Hence the group of sequence $\{z_\nu\}$ contains any number of those transpositions, while the set of the product of these tramspositions is equivalent to the set of all the subsets of a countably infinite number of elements. The cardinal number of such a set is $\mathfrak{c}$. Therefore the set of the group of $\{z_\nu\}$ has the cardinal number $\mathfrak{c}$. We may express this in general form as follows : *If the derived set of* $\{z_\nu\}$ *has only a point* ∞ *and the inferior limit of all the differences* $|z_\nu| \sim |z_\mu|$, $(\nu \neq \mu)$ *be zero, the set of the group of* $\{z_\nu\}$ *has the cardinal number* $\mathfrak{c}$.

An example is given by the sequence $\{\log\nu\}$ where

$$\lim_{\nu\to\infty}(\log(\nu+1) - \log\nu) = 0.$$

Another example is given by $\{1, 1, 2, 2, \cdots n, n, \cdots\}$ where $z_{2\nu}-z_{2\nu-1}=0$.

Concluding the results of this section, we have the proposition :

The set of the group of any sequence of numbers $\{z_\nu\}$ *whose derived set has only a point* ∞ *is countable or of cardinal number* $\mathfrak{c}$ *according as the inferior limit of all the differences* $|z_\nu| \sim |z_\mu|$, $(\nu \neq \mu)$ *is different from or equal to zero.*

Concluding the result of these two sections, we have the proposition :

The set of the group of every sequence is countable or has the cardinal number $\mathfrak{c}$. *The cardinal number of the set of the group of every semi-convergent series is* $\mathfrak{c}$.

Thus when a subset of the set of the symmetric group coincides with a set of the group of a sequence, the subset is countable or of cardinal number $\mathfrak{c}$. For the general case it is known difficult to determine it.

As an application of the substitutions of the infinite degree we shall consider in the following the sequence of functions.

14. Given a sequence of functions $\{f_\nu(z)\}$ holomorphic in a domain and a substitution $S=\binom{\nu}{s_\nu}$, we say the sequence of functions *admits* the substitution provided that the series $\sum\{f_\nu(z)-f_{s_\nu}(z)\}$ be absolutely convergent for each point z of the domain. If the series be convergent absolutely and uniformly for all points z in the domain, we say the sequence of functions *admits* the substitutions *uniformly* in the domain.

If the sequence $\{f_\nu(z)\}$ admits the symmetric group uniformly in the domain, then there exists a function $f(z)$ holomorphic in the domain such that the series $\sum|f_\nu(z)-f(z)|$ converges in the domain.

For consider the series

$$f(z)=f_1(z)+\{f_2(z)-f_1(z)\}+\cdots\cdots+\{f_\nu(z)-f_{\nu-1}(z)\}+\cdots\cdots.$$

Since the sequence admits the symmetric group uniformly in the domain, it admits *a fortiori* the substitutions

$$(1\ \ 2)(3\ \ 4)\cdots\cdots\cdots\cdots\cdots,$$
$$(2\ \ 3)(4\ \ 5)\cdots\cdots\cdots\cdots\cdots$$

uniformly. Or the series

$$|f_2(z)-f_1(z)|+|f_4(z)-f_3(z)|+\cdots\cdots\cdots,$$
$$|f_3(z)-f_2(z)|+|f_5(z)-f_4(z)|+\cdots\cdots\cdots$$

are uniformly convergent. Therefore the series $f(z)$ is uniformly convergent. Hence $f(z)$ is holomorphic in the domain. Since the sequence $\{f_\nu(z)\}$ admits the symmetric group uniformly and

$$\lim_{\nu\to\infty}f_\nu(z)=f(z),$$

the series $\sum|f_\nu(z)-f(z)|$ converges (§ 2) in the domain.

Conversely if there is a holomorphic function $f(z)$ such that the series $\sum|f_\nu(z)-f(z)|$ converges uniformly in the domain, for any substitution $S=\binom{\nu}{s_\nu}$, the series $\sum|f_\nu(z)-f(z)|$ is also uniformly convergent. Therefore the series $\sum|f_\nu(z)-f(z)|$ is also uniformly convergent. Therefore the series $\sum|f_\nu(z)-f_{s_\nu}(z)|$ converges uniformly, or the sequence $\{f_\nu(z)\}$ admits the symmetric group uniformly in the domain. From this it follows that $f(z)$ is the limiting function of the sequence.

By aid of the above considerations, it is clear that even when a sequence converges uniformly to a function holomorphic in the domain, it may not admit the symmetric group. For example consider the sequence formed by

$$f_\nu(z)=\frac{\nu z}{\nu+z}+(-1)^\nu\frac{1}{\nu}$$

in the domain $|z|<1$. $f_\nu(z)$ converges uniformly to z for $\nu\to\infty$. The sequence $\{f_\nu(z)\}$ cannot admit the symmetric group, since the series

$$\sum\left|\frac{2nz}{2n+z}+\frac{1}{2n}-\left(\frac{(2n-1)z}{2n-1+z}-\frac{1}{2n-1}\right)\right|$$

is not convergent.

We remark that our sequence has the same group with the sequence $\left\{\frac{(-1)^\nu}{\nu}\right\}$. For ν, μ being any two different positive integers, we have

$$f_\nu(z)-f_\mu(z)=\frac{\nu z}{\nu+z}+\frac{(-1)^\nu}{\nu}-\left\{\frac{\mu z}{\mu+z}+\frac{(-1)^\mu}{\mu}\right\}$$

$$=(-1)^\nu\frac{\mu-(-1)^{\mu-\nu}\nu}{\nu\mu}\left\{1+(-1)^\nu\frac{\nu-\mu}{\mu-(-1)^{\mu-\nu}\nu}\;\frac{\nu\mu z^2}{(\nu+z)(\mu+z)}\right\}.$$

Hence we have

$$|f_\nu(z)-f_\mu(z)|=\left|\frac{\mu-(-1)^{\mu-\nu}\nu}{\nu\mu}\right|K(z)$$

$$=\left|\frac{(-1)^\nu}{\nu}-\frac{(-1)^\mu}{\mu}\right|K(z),$$

where $|K(z)|$ lies between two numbers not zero. Therefore the two sequences have the same group.

15. *If the sequence $\{f_\nu(z)\}$ admits the symmetric group for any point z in the domain, then there exists a function $f(z)$ limited in the domain such that the series $\sum|f_\nu(z)-f(z)|$ converges for any point z of the domain, and conversely. $f(z)$ is the limiting function of the sequence.*

This propsition differs from that of the preceding section by the condition of the uniform convergency. In the present case, since the convergency is not uniform, the limiting function is not known to be holomorphic.

For example consider the real sequence $\{x^\nu\}$ in the interval $0 \leqq x \leqq 1$. This sequence does not converge uniformly in the interval though the limiting function is limited in it. This sequence admits clearly the symmetric group for any point of the interval since for any point of $0 \leqq x \leqq 1$, the series $\sum x^\nu$ is absolutely convergent, while for $x=1$ the sequence is formed by the same number unity.

The above proposition does not contradicts to Stieltjes' theorem on the uniform convergency of a limited sequence of functions holomorphic only *within* the domain *i. e.*, excepting the points of the boundary of the domain.

16. Many theorems are related with our group-theory. In concluding the present paper, I shall give briefly some relations other than those with which we have met already.

Being given a sequence of functions $\{f_\nu(z)\}$ holomorphic in a domain, the necessary and sufficient condition that the sequence is a normal family is that any partial sequence $\{f_{\nu_n}(z)\}$ extracted from the given sequence admits substitions such as

$$(\mu_1\ \mu_2)(\mu_3\ \mu_4)\cdots\cdots\cdots\cdots\cdots,$$
$$(\mu_2\ \mu_3)(\mu_4\ \mu_5)\cdots\cdots\cdots\cdots\cdots$$

uniformly in the domain where the sequence $\{\mu_m\}$ is a partial sequence of $\{\nu_n\}$. Here we except the case where the limiting function of any partial sequence becomes ∞.

The sufficient condition is easy to prove. To prove the necessary condition, suppose that $\{f_{\lambda_l}(z)\}$ be a partial sequence of $\{f_{\nu_n}(z)\}$ such that

$$\lim_{l\to\infty} f_{\lambda_l}(z) = f(z),$$

uniformly where the limiting function $f(z)$ is holomorphic in the domain. Then we have

$$f(z) = f_{\lambda_l}(z) + \{f_{\lambda_2}(z) - f_{\lambda_1}(z)\} + \{f_{\lambda_3}(z) - f_{\lambda_2}(z)\} + \cdots\cdots$$

uniformly in the domain. As Mr. Borel remarked somewhere we may change the series by some suitable condensations of terms into an absolutely convergent series without affecting the other properties. By the condensations of terms, we shall obtain an absolutely convergent series such as

$$f(z)=f_{\mu_1}+\{f_{\mu_2}(z)-f_{\mu_1}(z)\}+\cdots\cdots\cdots\cdots,$$

where $\{\mu_m\}$ is clearly a partial sequence of $\{\lambda_l\}$, hence of $\{\nu_n\}$. Now the series

$$|f_{\mu_2}(z)-f_{\mu_1}(z)|+|f_{\mu_4}(z)-f_{\mu_3}(z)|+\cdots\cdots\cdots$$

and

$$|f_{\mu_3}(z)-f_{\mu_2}(z)|+|f_{\mu_5}(z)-f_{\mu_4}(z)|+\cdots\cdots\cdots$$

are convergent. That is there are two substitutions

$$(\mu_1\ \mu_2)(\mu_3\ \mu_4)\cdots\cdots\cdots\cdots\cdots$$

and

$$(\mu_2\ \mu_3)(\mu_4\ \mu_5)\cdots\cdots\cdots\cdots\cdots$$

admitted by the sequence $\{f_{\nu_n}(z)\}$ uniformly in the domain.

Let $\{A_\nu\}$ be a sequence of positive numbers and C_A be a class of functions $f(x)$ defined in an interval such that

$$\sqrt[\nu]{\left|\frac{f^{(\nu)}(x)}{A_\nu}\right|}$$

is less than a constant independent of ν and x in the interval. After Mr. Denjoy each function of the class C_A is determined uniquely by $f(x)$ and all its derivatives at a point of the interval provided that the series $\sum\frac{1}{\sqrt[\nu]{A_\nu}}$ is divergent.

If it be so, consider a function of the class C_A, then we have

$$|f^{(\nu)}(x)|<K^\nu A_\nu$$

$$=\left(K\sqrt[\nu]{\frac{A_\nu}{A'_\nu}}\right)^\nu A'_\nu,$$

where K is a constant independent of ν and x and $\{A'_\nu\}$ is the same sequence with $\{A_\nu\}$ but different by the order of its terms, *i. e.*, a sequence transformed of $\{A_\nu\}$ by a certain substitution, say S. Now suppose that the sequence $\{A_\nu\}$ admits the substitution S, then by definition, the series $\sum|A_\nu-A'_\nu|$ is convergent. Therefore $|A_\nu-A'_\nu|$ tends to zero for $\nu\to\infty$. On the other hand we have

$$|A_\nu-A'_\nu|=A'_\nu\left|1-\frac{A_\nu}{A'_\nu}\right|$$

$$>A'_\nu \left|1-\sqrt[\nu]{\frac{A_\nu}{A'_\nu}}\right|.$$

Therefore assuming that all the numbers of the sequence $\{A_\nu\}$ are greater than a positive number, we have

$$\lim_{\nu\to\infty}\sqrt[\nu]{\frac{A_\nu}{A'_\nu}}=1.$$

Hence there is a constant K' such that for any ν

$$K\sqrt[\nu]{\frac{A_\nu}{A'_\nu}}<K'.$$

Therefore we have

$$|f^{(\nu)}(x)|<(K')^\nu A'_\nu,$$

or

$$\sqrt[\nu]{\left|\frac{f^{(\nu)}(x)}{A'_\nu}\right|}<K',$$

where K' is independent of ν and x. Hence the function belongs also to the class $C_{A'}$ and conversely. Hence in our case the calsses C_A and $C_{A'}$ are identical, or *the class* C_A *of such functions is an invariant of the group of the sequence* $\{A_\nu\}$. It is clear that the series $\sum\frac{1}{\sqrt[\nu]{A_\nu}}$ and $\sum\frac{1}{\sqrt[\nu]{A'_\nu}}$ are divergent at the same time.

If a sequence of numbers $\{v_n\}$ *admits the group of the sequence of numbers* $\{u_n\}$ *where the series* $\sum u_n$ *is conv.rgent. then the radius of convergence of the series* $\sum v_n z^n$ *is not less than unity.*

For by §§ 4, 5 since the derived set of $\{u_n\}$ has one point 0, that of $\{v_n\}$ must also be unique and finite. Therefore we have

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|v_n|}\leqq 1.$$

This proves the propositon.

For a power-series $f(z)$, let

$$u_n\equiv\frac{f^{(n)}(0)}{n!},\quad v_n\equiv\frac{f^{(n)}(z_0)}{n!}$$

where z_0 is a point in the circle of convergence of the series $\sum u_n z^n$. When the conditions of the above propositon are satisfied and the radius of convergence of the series $\sum u_n z^n$ is unity, the series $f(z)$ is continuable on the straight line joining the origin to the point z_0.