On Paradoxes and Definitions.

By

Toshizô Matsumoto.

(Received June 5, 1928)

1. When a thing has two names, it must be made clear that the two names signify the one and the same thing. If the two names given to a thing should not at once make clear that they signify the one and the same thing, some confusion would occur. Such confusion is often met in our daily life. Quite in the same way, if we give two definitions to a mathematical thing, we must make sure that the definitions point out the one and the same thing. No contradiction will occur if and when such verification is possible. When such verification is possible, we shall say that the two definitions are *congruent.*

The locus of points equidistant from a fixed point in a plane is called a circle. (Def. A) The circle may also be defined as the intersection of a sphere and a plane. (Def. B) Thus we have given two definitions to the circle. Now we must prove that the definitions A and B are congruent. To do it we prove under the definition A that the intersection of a sphere and a plane is a circle and conversely under the definition B that the locus of points equidistant from a fixed point in a plane is a circle. If it be done, the figures defined by the definitions A and B are identical.

So far as I know, it seems that in several antinomies or paradoxes in Cantor's theory of sets, such verifications are often forgotten. In this paper I intend to discuss from the point of view of the definitions, those antinomies given in Frenkel's books[1].

1. Einleitung in die Mengenlehre (1923).
Zehn Vorlesungen uber die Grundlegung der Mengenlehre (1927).

2. Richard's paradox. All the decimal numbers which may be defined by a finite number of words (including signs) are countably infinite in number. But the decimal number a defined by Cantor's diagonal method applied to those decimal numbers already defined is not contained in them. Hence the decimal number a must not be defined by a finite number of words. But the number a is already completely defined. This is a contradiction.

According to Russel, the definition of the decimal number a is not *predicative*. That which is defined by all elements of a set cannot be an element of the set. It explains that the decimal number a is not contained in the set of the decimal numbers used to define a.

According to my consideration, if the decimal number a be contained in the set, the number a will have two definitions. For since we have assumed that the number a is contained in the set whose decimal numbers are already defined, it has a definition. (Def. A) On the other hand the number a has been defined by the diagonal method. (Def. B) Consequently we must prove that those two definitions are congruent. If these definitions be congruent, the decimal number a must be contained in the original set of the decimal numbers. If otherwise we cannot identify the number a with a number of the set. Cantor's proof on the power of the continuum says that the definitions A and B are not congruent. Thus the decimal number a is not contained in the original set of the decimal numbers.

In the above paradox, the newly defined decimal number a is not contained in the original set. But paradoxes may occur in some cases where the newly defined thing belongs to the original set.

Let a, b, c,.........be positive integers which can be defined by one hundred words. Then there is necessarily a minimum integer, Let a be the minimum integer. Thus the integer a is defined by less than one hundred words. This is a contradiction. Let us consider the definitions more closely.

Since the integer a is an element of the set of positive integers defined by a hundred words, the integer a was already defined. (Def. A)

Next, we have defined the integer a as the minimum of the original set of positive integers. (Def. B) Thus two definitions are given to an integer a. If these definitions be congruent, assuredly there is a contradiction. But a close inspection of these two definitions will show us that they are not congruent. To prove it, at first efface the definition A, then the definitions B loses its sense. For without the aid of the definition A, the element a of the set a, b, c,......... is undefined. Consequently, the word minimum among elements partly defined and partly undefined loses its meaning. Therefore the definition B comes to contain two unknowns namely a and *minimum*. Thus without the definition A, the definition B does not define completely the minimum integer a.

After Poincaré's consideration[1], the integer a with the definition B will cease to be the positive integer definable by exactly one hundred words. Therefore the classification of the positive integers definable by a hundred words will be changed. Then I think the integer a will leave the original set. But if it do so, the number a will lose definition A. Therefore, a will become meaningless and the essential point of the explanation of our paradox seems to be the incompleteness of definition B.

3. To prove the existence of a root of the algebraic equation $f(x) = 0$, we employ the property that $|f(x)|$ becomes minimum at a point of any circle of the complex plane. In the polemic between Poincaré and Zelmero, the latter noticed that the definition of the above minimum is not predicative. Poincaré so refuted it that instead of the minimum of $|f(x)|$, we have only to consider its lower limit for all rational values of the variable x. It is clear that the minimum of $|f(x)|$ and its lower limit for the rational values of x are equal. Now let a be the minimum of $|f(x)|$. (Def. A) a may be defined also as the lower limit of $|f(x)|$ for any rational x. (Def. B) Hence the definition A and B are congruent. As we have seen above, in Richard's paradox, the definitions A and B were not congruent, while in the present case, they are congruent. Consequently the impredicative definition is not necessarily contradictory. On the other hand, it is known that the term *impredicative* is contradictory to itself. For any logical criticism, it seems reasonable to avoid such a word which is contradictory to itself.

4. Tertium non datur. It has been explained by the intuitionists represented by Brower that many mathematical theorems are related to *tertium non datur*. By tertium non datur, I think the existence of the thing with a given property E will be ascertained, while the possibility of its constructing is not proved. In mathematics it is generally difficult to construct a thing having a given property E. If the existence and the possibility of its constructing be the same, among theorems which are considered to have been rigorously proved, there are many uncertain ones. De Loor gave the following example:

Let k be the number of the decimal places of π from which for the first time seven 7 continuously occur. Next, if such an integer k does not exist, put $\rho \equiv \pi$ or else put $\rho \equiv \pi + 10^{-k}$ or $\rho \equiv \pi - 10^{-k}$ according as

---

1. Dernières Pensées, 104.

k is odd or even. Then the discriminant D

$$D = -108(\pi^6 - \rho^6)$$

of the equation

$$x^3 - 3\pi^2 x + 2\rho^3 = 0$$

is not known to be zero or not. Hence by aid of tertium non datur, the discriminant cannot determine the multiplicity of the roots of the equation.

We have defined ρ as π, $\pi + 10^{-k}$ or $\pi - 10^{-k}$ according as the case may be. (Def. B) But we do not know what number ρ is. If we assume ρ to be a definite number, that number is already defined, (Def. A), since we are working under the hypothesis that all numbers are well defined. Consequently we must prove the congruence of these two definitions A and B. Before we prove it, it is a matter of-course that we may not discuss the value of the discriminant D.

The mathematics of to-day cannot determine what number ρ is. In other words, we cannot prove whether the definitions A and B are congruent or not. As we have said above tertium non datur ascertains that ρ is *a number*. That ρ is a number is nothing but that ρ is *a constant*.

The example where the discriminant of an equation cannot determine the multiplicity of its roots may be given independently of tertium non datur. Consider the quadratic equation

$$x^2 + ax + b = 0,$$

where a and b are certain numbers (i. e. constants). It cannot be settled whether this equation has a double root or not.

Brower has pointed out the uncertainty of the theorem determining the ordering of the set of all real numbers[1]. Let the decimal expansion of π be

$$\pi = 3 \cdot d_1 d_2 \ldots d_m d_{m+1} \ldots d_{m+9} \ldots,$$

where $d_1 = 1$, $d_2 = 4, \ldots\ldots$ Suppose for the first time

$$d_m = 0,\ d_{m+1} = 1, \ldots\ldots,\ d_{m+9} = 9.$$

Now put

$$c_\nu \equiv \left(-\frac{1}{2}\right)^m, \quad \text{if} \quad \nu \geq m,$$

1. Crelle J. 3, **154** (1924).

or

$$c_\nu = \left(-\frac{1}{2}\right)^\nu, \quad \text{if} \quad \nu < m.$$

Then $c_1, c_2, c_3, \ldots\ldots$ tend to a real number r, though we do not know whether $r > 0$, $r < 0$ or $r = 0$. Therefore, real numbers are not ordered.

By tertium non datur m is an integer yet undetermined. Hence m is a constant. In other words, tertium non datur ascertains only that m is a constant. Consequently it is natural that r may not be ordered. But if r be a definite number, it will have two definitions, since we are assuming that all numbers are well defined. Therefore we should prove their congruence.

Here I want to insert Borel's interesting example[1].

Consider a sequence of numbers $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ such that its general term a_n is

$$a_n = 0 \quad \text{or} \quad a_n = 1.$$

Put

$$x_n \equiv 1 + |a_1 - a_2| + \ldots\ldots + |a_{n-1} - a_n|,$$

$$x \equiv \lim \frac{1}{x_n}.$$

Next, we define a function $y(x)$ such that

$$y = 1, \quad \text{if} \quad x \neq 0,$$

or

$$y = 0, \quad \text{if} \quad x = 0.$$

Then though the series

$$y + a_1 y + (a_2 - a_1) y + \ldots\ldots + (a_n - a_{n-1}) y + \ldots\ldots$$

is convergent, its sum is zero if the sequence $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ tends to no limit, while the sum will be 1 or 2 according as the limit of the sequence be 0 or 1.

The contradiction occurs since there is an infinite number of constants $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$

5. Russell's antinomy. According to Russell all sets of elements are divided into two classes by the condition whether a set contains itself as an element or not. All sets none of which contains itself as an

1. Méthode et problèmes de théories des fonctions, 17 (1922).

element, form a set M by Cantor's definition. This set M is contradictory. For we can neither assume that the set M contains itself as an element nor that the set M does not contain itself.

Now in general consider a set N which contains itself as an element. We may take the set into consideration since it has been defined by a certain definition. (Def. B) On the other hand, since the set N is contained in itself as an element, and all elements are well defined, the element N has already been defined. (Def. A) Therefore the set N has two definitions A and B. We must prove their congruence.

Let us return to Russell's antinomy. We have defined the set M. (Def. B) If the set M contains itself as an element, then M is already defined. (Def. A) On the contrary, even when the set M does not contain itself as an element, yet the set M has already been defined, (Def. B), since we are assuming that all sets none of which contains itself as an element are already well defined. In any case we have two definitions A and B for the set M. How can we prove the congruence of these two definitions?

Let us compare this antinomy with Richard's paradox explained in § 2. When the definition B loses its meaning without the definition A, it seems that the thing defined by the definition B exists *relative* to the thing defined by the definition A. The relativity of higher infinities has been remarked by Skolem[1]. But here we cannot give the same explanation to Russell's antinomy, since we are considering *all possible sets* none of which contains itself as an element and no sets higher than those are allowed to be considered.

Relativity in mathematics means nothing but that mathematics undergoes *creative evolution*. It is known that the cardinal number of the set of all possible sets is contradictory. But if we could conceive all possible sets, we might consider their totality. The totality is a newly created idea. This is the stand-point of the axiomatists of to-day.

The theory of sets is strongly explained by Hilbert[2] to be the study of actual infinity, ∞. The infinitesimal, its sign being o, has already been closely studied. o has another meaning, i. e., *nothing*. Everyone knows o, nothing, after which the idea of the infinitesimal has been evoluted by the discovery of the infinitesimal calculus. At the same time the potential ∞ i. e., $\lim x \rightarrow \infty$ has become known. To speak from my

1. Wissenschaftliche Vorträge, fünften Kongress der Skandinavischen Mathematiker, 222 (1922).
2. Math. Annalen, 161, **95** (1926).

present state of recognition, I can scarcely understand the actual ∞ in so far as it does not mean *all* things. Russell's antinomy relates to *all* sets. If the theory of sets be the study of *all*, it is not weakened by Russell's antinomy. On the contrary, it is a theme of our study. Russell's antinomy loses its meaning by the axiomatic study of the theory of sets. By axioms I understand those propositions which enable us to objectify the objects of individual persons. *All sets* in Russell's antinomy is not yet completely objectified.

6. Finally I wish to remark briefly on the relation of the tertium non datur to the equally like occurrence of events in probability. Suppose a coin is tossed. The event that the head will appear is equally likely as the event that the tail will appear. To assume that the head or the tail must appear is only reasonable from the point of view of experiments. But if we desire the theory of probability to be regarded as a mathematical theory, then since exact mathematics is independent of any experiments, we must consider the following possibilities :

(1) The head will appear.

(2) The tail will appear.

(3) No reason to ascertain (1) and (2).

Only in such a case, events are said to occur equally likely and their probability is $\frac{1}{2}$. This idea is different from those of Jacob Bernoulli, Laplace and v. Kries[1].

---

1. Cf. Czuber, Wahrscheinlichkeitsrechnung, 1. Bd.