On the Space of Classes of Point Sets

By

Fumitomo Maeda

(Received November 19, 1929)

According to the definition given by Hausdorff[1], if the distance between two point sets in a metric space E be zero, then they have the same closed cover (abgeschlossene Hülle). Therefore, for the closed bounded point sets, namely the greatest point sets which have the same closed cover, the space whose elements are such point sets is a metric space, and it is compact in itself when the space E is compact in itself.

In this paper, I modify Hausdorff's definition of the distance between two point sets, such that the two point sets between which the distance is zero, have not only the same closed cover but also the same open nucleus (offene Kern). Then all such point sets may be considered as identical with respect to the distance, and I say that the aggregate of all such point sets forms a class. I prove that the space of such classes is metric, and compact in itself when the space E is compact in itself, and also that it is dense-in-itself, when the space E is dense-in-itself.

Next I classify these classes of point sets in the Euclidian space into two kinds; one is the quadrable classes in which all the point sets are quadrable and of the same measure, the other is the non-quadrable classes in which the point sets are measurable or non-measurable in the Lebesgue sense, and their exterior and interior measure can be of any values between the measure of the closed cover and that of the open nucleus. And I find that the set of quadrable classes and that of non-quadrable classes are both dense in the space of all classes.

---

1. Hausdorff: Mengenlehre, zweite Aufl. (1927), p. 145—150.

From any set function in the space of point sets, we can deduce a point function in the space of classes, such that the values of this point function at a class are the same as the values of that set function of all the point sets which belong to the class. I consider the case in which the set function is the measure of point sets, and I conclude that the point functions which correspond to the exterior and interior Peano-Jordan measure are both semi-continuous, and the point function which corresponds to the Lebesgue measure is the function obtained by extension of the point function which corresponds to the Peano-Jordan measure in the set of quadrable classes.

1. In a metric space E, I consider a point set A. Let a be any point contained in the set A, then for a point b in the space E, we denote the lower bound of the distance between the two points a and b by, $\delta(A, b)$. Then the neighbourhood of A with radius ρ may be defined as the set of points b which satisfy

$$\delta(A, b) < \rho.$$

We denote it by the symbol $U(A, \rho)$.[1]

Let B be any other point set in the space E. We denote the lower bound of ρ which satisfies

$$B \leqq U(A, \rho)$$

by the symbol $\rho(A, B)$, and similarily the lower bound of ρ which satisfies

$$B_c \leqq U(A_c, \rho)$$

by the symbol $\rho(A_c, B_c)$, where A_c and B_c denote the complementary sets of A and B respectively in the space E. We denote the greater of $\rho(A, B)$ and $\rho(A_c, B_c)$ by the symbol $s(A, B)$.

When $s(A, B) = 0$, it must follow that

$B \leqq A_\alpha$, and

$B_c \leqq A_{c\alpha}$, that is $B \geqq A_i$,

where A_α and A_i mean respectively the closed cover and open nucleus of A. Therefore

$$A_i \leqq B \leqq A_\alpha.$$

Conversely, if $A_i \leqq B \leqq A_\alpha$, it is obvious that

$$s(A, B) = 0.$$

1. I use mostly the terminology and the symbols of Hausdorff.

Since, in general $s(A, B)$ is not equal to $s(B, A)$, I define *the distance between the point sets A and B* as the greater of $s(A, B)$ and $s(B, A)$, and denote it by the symbol $r(A, B)$. Then $r(A, B)=r(B, A)$.

When $r(A, B)=0$, it must follow that

$$A_i \leqq B \leqq A \quad \text{and} \quad B_i \leqq B \leqq A_a,$$

therefore $\qquad A_a = B_a \qquad \text{and} \qquad A_i = B_i.$

Conversely, if A and B have the same closed cover and open nucleus, it is obvious that $r(A, B)=0$, and in other cases $r(A, B)\neq 0$.

2. When A, B and C are any point sets in the space E, then

$$\rho(A, B)+\rho(B, C)\geqq \rho(A, C),$$

for, if $B\leqq U(A, \rho)$ and $C\leqq U(B, \sigma)$ then $C\leqq U(A, \rho+\sigma)$.

Similarily $\qquad \rho(A_c, B_c)+\rho(B_c, C_c)\geqq \rho(A_c, C_c).$

Then from the definition of $s(A, B)$,

$$s(A, B)+s(B\ C)\geqq s(A, C);$$

similarily $\qquad s(C, B)+s(B, A)\geqq s(C, A).$

Therefore from the definition of $r(A, B)$

$$r(A, B)+r(B, C)\geqq r(A, C). \qquad (1)$$

If A, B and C be such that $r(A, B)=0$ and $r(B, C)=0$, then from (1) $r(A, C)=0$; that is, the distance between two point sets which are at zero-distance from a point set, is also zero.

If B and C be such shat $r(B, C)=0$, then from (1)

$$r(A, B)\geqq r(A, C),$$

and since

$$r(A, C)+r(C, B)\geqq r(A, B),$$

we have also

$$r(A, C)\geqq r(A, B),$$

therefore $\qquad r(A; B)=r(A, C);$

that is, two point sets between which the distance is zero are at the same distance from any other point set.

From these properties, any two point sets between which the distance is zero can be considered as identical with respect to the distance. Therefore I say that *the aggregate of all point sets, between any two of which the distance is zero, forms a class.* Since no point set ever belongs to two different classes, any class is uniquely determined by a point set which belongs to it. Therefore I denote the class determined by a point set A by the symbol $[A]$.

Since the distance between two point sets A and B which belong respectively to the classes $[A]$ and $[B]$, is invariable for any point set A in $[A]$ and B in $[B]$, we define *the distance between two classes* $[A]$ and $[B]$ as the distance between two point sets A and B, that is

$$r([A],\ [B]) = r(A,\ B),$$

which is always finite when A and B are bounded in the space E.

Now we consider the space whose elements are the classes of *bounded* point sets in the metric space E, and denote it by the symbol $\{E\}$. Then *the space* $\{E\}$ *is also a metric space,* since the real number $r([A],\ [B])$ satisfy the following axioms of the distance[1]:

(α) $r([A],\ [A]) = 0,$

(β) $r([A],\ [B]) = r([B],\ [A]) > 0$ when $[A] \neq [B],$

(γ) $r([A],\ [B]) + r([B],\ [C]) \geqq r([A],\ [C]);$

where (γ) follows from (1).

3. In § 1, we have seen that when $r(A, B)=0$, A and B have the same closed cover and open nucleus, and conversely. Therefore, *the class may also be defined as an aggregate of all point sets which have the same closed cover and open nucleus.* Therefore all point sets in $[A]$ have the same frontier $A_g = A_\alpha - A_i$.

When the space E is separable, to any closed set F and any open set O where $F \supset O$, there corresponds a class $[A]$ such that $A_\alpha = F$ and $A_i = O$.

To prove this, we consider the following point set

$$A = O + D + O_{\alpha c}(F - O)_g,$$

where D is a subset of the open set $(F-O)_i$, such that D and $(F-O)_i - D$ are both dense in $(F-O)_i$. $(F-O)_g$ means the frontier of the closed set $F-O$, and $O_{\alpha c}$ the complementary set of O_α in E.

We must prove first that such a point set D exists. Since the space E is separable, we may use the enumerable set of special neighbourhoods[2] $V_1,\ V_2, \ldots\ldots, V_n, \ldots\ldots$. Let us denote by $V_p(p=p_1, p_2, \ldots\ldots, p_n, \ldots\ldots)$ the special neighbourhoods V_n which contain points of the set $(F-O)_i$. We pick out from $V_p(F-O)_i$ two different points a_p and a'_p. Since $V_p(F-O)_i$ contains infinitely many points, we can

---

1. Hausdorff: op. cit. p. 94.
2. op. cit. p. 126.

choose a_p and a'_p such that $a_{p_m} \neq a'_{p_n}$ for any value of m and n. Therefore, we have two different sets D and D' of the points a_p and a'_p respectively, which are both dense in $(F-O)_i$. For, let x be any point of $(F-O)_i$, and U_x any neighbourhood of x; then we can find a special neighbourhood V_p such that $V_p \leqq U_x$. But V_p contains a point a_p of the set D, then U_x contains a_p; consequently D is dense in $(F-O)_i$. Similarily D' is so also. But $(F-O)_i - D \geqq D'$, therefore $(F-O)_i - D$ is dense in $(F-O)_i$. Q. E. D.

To prove $A_\alpha = F$ and $A_i = O,$

first consider A_α,

$$A_\alpha = O_\alpha \dotplus D_\alpha \dotplus \{O_{\alpha c}(F-O)_g\}_\alpha,$$

but $\quad D_\alpha = (F-O)_{i\alpha},$

and since $\quad (F-O)_g = O_\alpha(F-O)_g + O_{\alpha c}(F-O)_g,$

$$(F-O)_{g\alpha} = \{O_\alpha(F-O)_g\}_\alpha \dotplus \{O_{\alpha c}(F-O)_g\}_\alpha,$$

then $\quad (F-O)_g \leqq O_\alpha \dotplus \{O_{\alpha c}(F-O)_g\}_\alpha,$

therefore, $\quad A_\alpha \geqq O_\alpha \dotplus (F-O)_{i\alpha} \dotplus (F-O)_g,$

that is, $\quad A_\alpha \geqq F.$

But since $\quad A \subset F \quad$ therefore $\quad A_\alpha \leqq F;$

consequently $\quad A_\alpha = F.$

Next consider the inner points of A. All points of the set O are obviously inner points of A. But any point of D and $O_{\alpha c}(F-O)_g$ is not an inner point of A, since in any neighbourhood of a point of D, there exist points of $(F-O)_i - D$ which do not belong to A; and in any neighbourhood of the point of $O_{\alpha c}(F-O)_g$, there exist points of F_c or $(F-O)_i$, but the points of F_c do not belong to A, and in any neighbourhood of a point of $(F-O)_i$ there exist points of $(F-O)_i - D$. Consequently the points of O are the only inner points of A; that is

$$A_i = O.$$

Therefore the class $[A]$ is the one required which corresponds to the given closed set F and the open set O.

4. Since the space $\{E\}$ of classes is a metric space, we can use the same terminology and symbols as in the case of the metric space of point sets by changing the term point into class. A sequence of classes in the space $\{E\}$, $[A_1], [A_2], \ldots\ldots, [A_n], \ldots\ldots$ is said to have the limiting class $[A]$, if $\lim_{n\to\infty} r([A], [A_n]) = 0$, and we denote it by $[A] = \lim_{n\to\infty} [A_n]$.

If $\lim_{n\to\infty}[A_n]=[A]$, *then the closed limit* $\underline{Fl}A_n$ *is* A_α *and the open limit* $\overline{Gl}A_n$[1] *is* A_i, *where* A_n *and* A *are any point sets which belong to the classes* $[A_n]$ *and* $[A]$ *respectively.*

Proof. Hausdorff[2] proves that

when $\rho(A_n, X)\to 0$, then $X \leqq \underline{Fl}A_n$, and

when $\rho(X, A_n)\to 0$, then $X \geqq \overline{Fl}A_n$,

where X is a closed point set. Now $\lim_{n\to\infty}[A_n]=[A]$, then, since $\rho(A_n, A)=\rho(A_n, A_\alpha)$ etc.,

$$\rho(A_n, A_\alpha)\to 0, \qquad \rho(A_\alpha, A_n)\to 0,$$

$$\rho(A_{nc}, A_{c\alpha})\to 0, \qquad \rho(A_{c\alpha}, A_{nc})\to 0.$$

Therefore $A_\alpha \leqq \underline{Fl}A_n$, $\quad A_\alpha \geqq \overline{Fl}A_n$,

$A_{c\alpha} \leqq \underline{Fl}A_{nc}$, that is $A_i \geqq \overline{Gl}A_n$,

$A_{c\alpha} \geqq \overline{Fl}A_{nc}$, that is $A_i \leqq \underline{Gl}A_n$.

But $\underline{Fl}A_n \leqq \overline{Fl}A_n$ and $\underline{Gl}A_n \leqq \overline{Gl}A_n$,

therefore $A_\alpha = FlA_n$ and $A_i = GlA_n$,

which is as required.

Since $FlA_n = FlA_{n\alpha}$[3] and $GlA_n = GlA_{ni}$,

then $A = FlA_{n\alpha}$ and $A_i = GlA_{ni}$,

that is, *when the sequence of classes* $[A_1], [A_2], \ldots\ldots, [A_n], \ldots\ldots$ *has the limiting class* $[A]$, *then the closed limit of the sequence of corresponding closed sets* $A_{1\alpha}, A_{2\alpha}, \ldots\ldots, A_{n\alpha}, \ldots\ldots$ *is the corresponding closed set* A_α, *and similarily for the open limit of the sequence of open sets* $A_{1i}, A_{2i}, \ldots\ldots, A_{ni}, \ldots\ldots$.

5. *When the space* E *is compact in itself,* the converse theorem is true, that is, *when the closed limit* FlA_n *and the open limit* GlA_n *exist, then the sequence of classes* $[A_1], [A_2], \ldots\ldots, [A_n], \ldots\ldots$ *has the limiting class* $[A]$, *and*

$$A_\alpha = FlA_n \quad \textit{and} \quad A_i = GlA_n.$$

When E is compact in itself, it is also separable[4], then by § 3 we can always find a point set A such that its closed cover A_α and open nucleus A_i are the given closed set FlA_n and the open set GlA_n

1. Hausdorff: op cit. p. 146.
2. op. cit. p. 149; this proof holds when the point sets are not bounded.
3. op. cit. p. 148.
4. op. cit. p. 126 and p. 107.

respectively. Therefore we have only to prove that $\lim_{n\to\infty}[A_n]=[A]$.

Since $FlA_n = A_\alpha$,

then $\rho(A_n, A_\alpha)\to 0$ and $\rho(A_\alpha, A_n)\to 0$[1];

from $GlA_n = A_i$, that is $FlA_{nc} = A_{c\alpha}$,

we have $\rho(A_{nc}, A_{c\alpha})\to 0$ and $\rho(A_{c\alpha}, A_{nc})\to 0$.

But $\rho(A_n, A_\alpha)=\rho(A_n, A)$ etc., from these we have

$$\lim_{n\to\infty}[A_n] = [A].$$

From this theorem we have the following theorem: *If the space E is compact in itself, then the space $\{E\}$ is also compact in itself.*

For, let $[A_1], [A_2], \ldots\ldots, [A_n], \ldots\ldots$ be any sequence of classes. Since E is compact in itself, it is separable; therefore the sequence of point sets $A_1, A_2, \ldots\ldots, A_n, \ldots\ldots$ contain a sequence $A_{p_1}, A_{p_2}, \ldots\ldots, A_{p_n}, \ldots\ldots$ for which both the closed limit FlA_{p_n} and the open limit GlA_{p_n} exist[2]. Then by the above theorem, there is a point set A such that $\lim_{n\to\infty}[A_{p_n}]=[A]$, that is, the space $\{E\}$ is compact in itself.

6. *If the space E is dense-in-itself, then the space $\{E\}$ is also dense-in-itself.*

To prove this, I construct a point set B which does not belong to any given class $[A]$, such that $r(A, B)\leqq\rho$ for any positive value of ρ.

Let $$B = AU_c(A_g, \rho) + A_cU(A_g, \rho), \qquad (1)$$

where $U_c(A_g, \rho)$ means the complementary set in E of the neighbourhood $U(A_g, \rho)$ of the frontier of the set A; the point set B always exists, for the space E is dense-in-itself. Since A is bounded, in $A_cU(A_g, \rho)$ there is a point x such that $\delta(A, x)>\rho'$ where $0<\rho'<\rho$; therefore $r(A, B)\neq 0$, that is B does not belong to the class $[A]$.

From (1) we have

$$B_c = \{A_c \dot{+} U(A_g, \rho)\}\{A \dot{+} U_c(A_g, \rho)\}. \qquad (2)$$

But, since $U(A, \rho) = A \dot{+} U(A_g, \rho)$,

then $$B\leqq U(A, \rho); \qquad (3)$$

and since $U(A_c, \rho) = A_c \dot{+} U(A_g, \rho)$,

then $$B_c\leqq U(A_c, \rho). \qquad (4)$$

Divide A into two parts $AU_c(A_g, \rho)$ and $AU(A_g, \rho)$; from (1) $AU_c(A_g, \rho)\lt B$, and $AU(A_g, \rho)$ is in the ρ-neighbourhood of $A_cU(A_g, \rho)$,

1. Hausdorff: op. cit. p. 150.
2. op. cit. p. 147; this theorem is applied twice.

therefore

$$A \leqq U(B, \rho). \tag{5}$$

From (2) $$B_e = AU(A_g, \rho) + A_e U_e(A_g, \rho); \tag{6}$$

divide A_e into two parts $A_e U_e(A_g, \rho)$ and $A_e U(A_g, \rho)$, then $A_e U_e(A_g, \rho) \subset B_e$, and $A_e U(A_g, \rho)$ is in the ρ-neighbourhood of $AU(A_g, \rho)$, therefore

$$A_e \leqq U(B_e, \rho). \tag{7}$$

From (3), (4), (5) and (7), we have $r(A, B) \leqq \rho$. Q.E.D.

7. In the above, I have looked at the general properties of the space $\{E\}$ of the classes of the bounded point sets in the metric space E. *If there is given a set function $f(A)$ in the space E, then we can always deduce from this a point function $\varphi([A])$ in the space $\{E\}$,* such that $\varphi([A])$ signifies all values of $f(A)$, A being all sets which belong to the class $[A]$, and hence in general $\varphi([A])$ is a many-valued function.

Next I consider, for example, the measure of point sets in the Euclidian space E_n of n dimensions, and I will find a point function which corresponds to it.

For this purpose, I investigate first the space $\{E_n\}$ more in detail. In § 3 I have noted that all point sets in the class $[A]$ have the same frontier A_g. If the Lebesgue measure of A_g is zero, all point sets in $[A]$ are quadrable,[1] and have the same measure as A_i and A_α. In this case I say it is *a quadrable class,* and the others are *non-quadrable classes.* I denote the set of all quadrable classes by the symbol $\{E_n^*\}$, and that of all non-quadrable classes by the symbol $\{E_n^{**}\}$.

8. *The set $\{E_n^*\}$ of all quadrable classes is dense in the space $\{E_n\}$*; that is, in any neighbourhood of a class $[A]$ there is a quadrable class.

Proof. Since the point set A is bounded, there is a closed cell which contains A. Divide this cell by n-1 dimensional planes parallel to the coordinate planes into smaller cells, the length of the diagonals of which is less than ρ, where ρ is any given number. We call the aggregate of these small cells a net, and these cells the meshes of it[2]. Let us denote the sum of the closed meshes which contain a point of A_g by S_1, and the sum of the open meshes which are contained in S_1 by S_2; then $S_1 - S_2$ is a set of points on the boundary of the meshes

1. Carathéodory: Vorlesungen über reelle Funktionen, zweite Aufl. (1927), p. 290.
2. Hobson: The theory of functions of a real variable, I, third ed. (1927), pp.64—69.

which are contained in S_1, therefore the measure of S_1-S_2 is zero.

Consider the point set

$$A^* = (A - AS_1) + (S_1 - S_2), \qquad (1)$$

then it follows that $r(A, A^*) < \rho$.

For, since $U(A, \rho) = A + U(A_g, \rho)$,

and $A - AS_1 \subset A$, $S_1 - S_2 \subset U(A_g, \rho)$,

therefore $A^* \subset U(A, \rho)$. (2)

From (1) $A_c^* = (A_c + AS_1)(S_{1c} + S_2) = A_c S_{1c} + S_2$; (3)

since $U(A_c, \rho) = A_c + U(A_g, \rho)$,

and $A_c S_{1c} \subset A_c$, $S_2 \subset U(A_g, \rho)$,

therefore $A_c^* \subset U(A_c, \rho)$. (4)

To prove $A \subset U(A^*, \rho)$, (5)

divide A into two parts $A-AS_1$ and AS_1,

but $A - AS_1 \subset A^*$ from (1),

and $AS_1 \subset U\{(S_1 - S_2), \rho\}$

therefore by (1) we have (5).

To prove $A_c \subset U(A_c^*, \rho)$, (6)

divide A_c into two parts $A_c S_{1c}$ and $A_c S_1$,

but $A_c S_{1c} \subset A_c^*$ from (3),

and $A_c S_1 \subset U(S_2, \rho)$,

therefore by (3) we have (6).

From (2), (4), (5) and (6) we conclude that

$$r(A, A^*) \subset \rho.$$

Next consider the frontier of A^*. From (1),

$$A_\alpha^* = A^* \quad \text{and} \quad A_i^* = A - AS_1,$$

then $A_g^* = A_\alpha^* - A_i^* = S_1 - S_2$,

therefore the measure of A_g^* is zero, that is A^* is quadrable.

Therefore the quadrable class $[A^*]$ is in the ρ-neighbourhood of the class $[A]$. Q. E. D.

From the above proof, when the class $[A]$ is quadrable, it also follows that *the set $\{E_n^*\}$ is dense-in-itself,* since it is obvious that A^* does not belong to the class $[A]$.

9. *The set $\{E_n^{**}\}$ of all non-quadrable classes is dense in the space $\{E_n\}$*; that is, in any neighbourhood of a class $[A]$, there is a non-quadrable class.

To prove this, we use instead of $U(A_g, \rho)$ in

$$B = AU_c(A_g, \rho) + A_cU(A_g, \rho)$$

of § 6, the set $V(A_g, \rho)$ of rational points[1] which are contained in $U(A_g, \rho)$, and construct

$$A^{**} = AV_c(A_g, \rho) + A_cV(A_g, \rho), \qquad (1)$$

then $r(A, A^{**}) \leqq \rho$; the proof is almost the same as that of § 6, hence I omit it.

Next consider the frontier of A^{**}, from (1)

$$A_\alpha^{**} \leqq A_\alpha + A_{ic}U_\alpha(A_g, \rho),^2$$

since $$A_c^{**} = AV(A_g, \rho) + A_cV_c(A_g, \rho)$$

as in (6) of § 6, then

$$A_{c\alpha}^{**} \leqq A_\alpha U_\alpha(A_g, \rho) + A_{ic},$$

therefore $$A_g^{**} = A_\alpha^{**}A_{c\alpha}^{**} \leqq U_\alpha(A_g, \rho). \qquad (2)$$

To show that $$A_g^{**} \geqq U_\alpha(A_g, \rho), \qquad (3)$$

divide $U_\alpha(A_g, \rho)$ into two parts $AU_\alpha(A_g, \rho)$ and $A_cU_\alpha(A_g, \rho)$. In any neighbourhood of a point of $AU_\alpha(A_g, \rho)$, there are points of $AV_c(A_g, \rho)$ and $AV(A_g, \rho)$, that is, the points of A^{**} and A_c^{**}. Hence $A_g^{**} \geqq AU_\alpha(A_g, \rho)$. Similarily $A_g^{**} \geqq A_cU_\alpha(A_g, \rho)$; therefore we have (3).

Consequently from (2) and (3)

$$A_g^{**} = U_\alpha(A_g, \rho),$$

but the measure of $U_\alpha(A_g, \rho)$ is not zero, and A^{**} is not quadrable. Therefore in the ρ-neighbourhood of the class $[A]$ there is a non-quadrable class $[A^{**}]$. Q. E. D.

From the above proof, as in § 8, it also follows that *the set* $\{E_n^{**}\}$ *is dense-in-itself.*

10. In § 7, I have said that to any set function $f(A)$ in the space E, there corresponds a point function $\varphi([A])$ in the space $\{E\}$. Here I shall consider first the Peano-Jordan measure of point sets.

We denote the exterior and interior Peano-Jordan measure of the set A by the symbols $j_e(A)$ and $j_i(A)$ respectively. Since

$$j_e(A) = m(A_\alpha) \qquad \text{and} \qquad j_i(A) = m(A_i)^3$$

where $m(A)$ means the Lebesgue measure of A, *all point sets which*

1. That is, the points all coordinates of which are rational numbers.
2. Since $(A+B)_\alpha = A_\alpha + B_\alpha$ and $(AB)_\alpha \leqq A_\alpha B_\alpha$.
3. Schlesinger u. Plessner; Lebesgueshe Integrale und Fouriersche Reihen, (1926), p. 63.

belong to the same class have the same exterior and interior Peano-Jordan measure. Therefore the point functions which correspond to them, which we denote by the symbols $\zeta_e([A])$ and $\zeta_i([A])$ respectively, are *one-valued* in $\{E_n\}$.

Consider any sequence of classes $[A_1], [A_2], \ldots\ldots, [A_n], \ldots\ldots$ which has the limiting class $[A]$, that is, $\lim_{n\to\infty}[A_n]=[A]$. Then for any given number ρ there exists a number n', such that $A_n \subset U(A, \rho)$ for all values of $n>n'$. If we take a sequence $\rho_1, \rho_2, \ldots\ldots, \rho_m, \ldots\ldots$, with zero as its limit, then

$$\mathop{\mathrm{Lim}}_{m\to\infty} U(A, \rho_m) = A_\alpha,$$

therefore $$\overline{\lim_{n\to\infty}}\, j_e(A_n) \leqq m(A_\alpha),$$

that is, $$\overline{\lim_{n\to\infty}}\, \zeta_e([A_n]) \leqq \zeta_e([A]). \qquad (1)$$

Therefore $\zeta_e([A])$ *is upper semi-continuous in* $\{E_n\}$ *at the class* $[A]$[1].

Similarily from $\quad A_{nc} \subset U(A_c, \rho)$,

we have $$\underline{\lim_{n\to\infty}}\, j_i(A_n) \geqq m(A_i),$$

that is $$\underline{\lim_{n\to\infty}}\, \zeta_i([A_n]) \geqq \zeta_i([A]). \qquad (2)$$

Therefore $\zeta_i([A])$ *is lower semi-continuous in* $\{E_n\}$ *at the class* $[A]$.

If the class $[A]$ is quadrable, then $\zeta_e([A])=\zeta_i([A])$, and from (2), since $\zeta_i([A_n])\leqq\zeta_e([A_n])$,

$$\underline{\lim_{n\to\infty}}\, \zeta_e([A_n]) \geqq \zeta_e([A]),$$

therefore with (1),

$$\left.\begin{aligned} &\lim_{n\to\infty} \zeta_e([A_n]) = \zeta_e([A]) \\ \text{Similarily}\quad &\lim_{n\to\infty} \zeta_i([A_n]) = \zeta_i([A]). \end{aligned}\right\} \qquad (3)$$

Therefore $\zeta_e([A])$ *and* $\zeta_i([A])$ *are continuous in* $\{E_n\}$ *at the quadrable class* $[A]$.

Since the set $\{E_n^*\}$ of the quadrable classes is dense in the space $\{E_n\}$, $\zeta_e([A])$ *and* $\zeta_i([A])$ *are point-wise discontinuous in* $\{E_n\}$[2].

It is obvious that $\zeta_e([A])$ *and* $\zeta_i([A])$ *are discontinuous in* $\{E_n\}$ *at the non-quadrable classes,* for, if not, they must have the same values at the classes of $\{E_n^{**}\}$, since they are continuous in $\{E_n\}$ and have the same values at the set $\{E_n^*\}$ which is dense in $\{E_n\}$[3]. Therefore,

---

1. Hahn: Theorie der reellen Funktionen, I, (1921), p. 152.
2. op. cit. p. 203.
3. op. cit. p. 133.

the set $\{E_n^{**}\}$ of non-quadrable classes is of the first category in $\{E_n\}$[1].

The point function which corresponds to the Peano-Jordan measure $j(A^*)$, is defined only in the set $\{E_n^*\}$ of quadrable classes, and when we denote it by $\zeta([A^*])$,

$$\zeta([A^*]) = \zeta_e([A^*]) = \zeta_i([A^*])$$

at the quadrable class $[A^*]$. Since (3) holds when all sets A_n are quadrable, $\zeta([A^*])$ *is continuous in* $\{E_n^*\}$. It will be seen in § 15 that $\zeta_e([A])$ and $\zeta_i([A])$ are the upper and the lower boundery function respectively of $\zeta([A^*])$ in $\{E_n^*\}$[2].

11. Next we consider the Lebesgue measure. All point sets which belong to the quadrable class are measurable and have the same measure. But for the non-quadrable classes it is complicated.

In the non-quadrable class $[A^{**}]$, there exist measurable or non-measurable point sets whose exterior and interior measure are any values between $m(A_a^{**})$ and $m(A_i^{**})$, including these two values.

To prove this, for brevity, let A^{**} be a point set in the space E_2 of two dimensions. Consider the frontier A_g^{**} of A^{**} and divide it into its open nucleus A_{gi}^{**} and its rand A_{gr}^{**}, such that

$$A_g^{**} = A_{gi}^{**} + A_{gr}^{**},$$ [3]

then A_{gr}^{**} is a non-dense closed plane point set. Therefore there exists an enumerable set S of rectangles which is dense in E_2, such that all points of A_{gr}^{**} are classified into three kinds:

(1) points on a boundary of one or more of the rectangles,

(2) points of accumulation of the set of such points,

(3) points which lie in a linear interval which is the limit of a sequence of the rectangles[4].

But the points of the third kind can be put into the first two kinds by the following modification of the set S of rectangles. If there exists a linear interval, such that all points x of A_{gr}^{**} on it are not the points of accumulation of the set of the points of the first kind, then there is a neighbourhood of the linear interval in which no point of A_{gr}^{**} exists except the points on the linear interval. Therefore a rectangle R can be constructed which has the linear interval as one side, and in its interior no point of A_{gr}^{**} exists. Then some rectangles of the set S,

1. Hahn: op. cit. p. 204.
2. op. cit. p. 121.
3. Hausdorff: op. cit. p. 110.
4. Hobson: op. cit. p. 127.

which have common points with the rectangle R are divided into smaller rectangles by the sides of the rectangle R and its elongations. Now by using the rectangle R instead of such rectangles as are in R, and other rectangles which are outside of R, we obtain a new set of rectangles, for which the points x are of the first kind.

By this process, all points of the third kind can be put into the first or second kind, that is, there exists an enumerable set S' of rectangles which is dense in E_2, such that every point of A_{gr}^{**} belongs to one of the two kinds :

(1) points on a boundary of one or more of the rectangles, and

(2) the other points which are the points of accumulation of the set of the points of the first kind.

We denote the sets of all points of the first kind in A_{gr}^{**} by the symbol $A_{gr\rho}^{**}$, and that of the second kind by $A_{gr\eta}^{**}$, then

$$A_{gr}^{**} = A_{gr\rho}^{**} + A_{gr\eta}^{**}.$$

12. Now we construct a point set which belongs to the class $[A^{**}]$. Consider the point set

$$A = A_i^{**} + DA_{gi}^{**} + P_c A_{gr\rho}^{**} + TA_{gr\eta}^{**},$$

where D is a point set which is with its complementary set D_c dense in E_2, and P is the sum of the closed rectangles of the set S' which contain the points of the open set A_i^{**}, and T is any point set, such that $TA_{gr\eta}^{**}$ denotes any part of $A_{gr\eta}^{**}$.

Then $\quad A_\alpha = A_{i\alpha}^{**} \dotplus (DA_{gi}^{**})_\alpha + (P_c A_{gr\rho}^{**})_\alpha \dotplus (TA_{gr\eta}^{**})_\alpha,$

but $\quad (DA_{gi}^{**})_\alpha = A_{gi\alpha}^{**},$

and since $\quad A_{gr\rho}^{*} = PA_{gr\rho}^{**} + P_c A_{gr\rho}^{**},$

$\quad A_{gr\rho\alpha}^{**} = (PA_{gr\rho}^{*})_\alpha \dotplus (P_c A_{gr\rho}^{**})_\alpha,$

that is $\quad A_{gr}^{**} \subset A_{i\alpha}^{**} \dotplus (P_c A_{gr\rho}^{**})_\alpha, \quad$ for $\quad P_\alpha \leqq A_{i\alpha}^{**},$

therefore $\quad A_\alpha \geqq A_{i\alpha}^{**} \dotplus A_{gi\alpha}^{**} + A_{gr}^{**},$

that is $\quad A_\alpha \geqq A_\alpha^{**}$;

but since $\quad A \subset A_\alpha^{**},$

$\quad A_\alpha \leqq A_\alpha^{**}$;

therefore $\quad A_\alpha = A_\alpha^{**}.$

Next consider the inner point of A. Obviously the points of A_i^{**} are the inner points of A ; but any point of DA_{gi}^{*}, $P_c A_{gr\rho}^{**}$ and $TA_{gr\eta}^{**}$ is not the inner point of A, since in any neighbourhood of the point of DA_{gi}^{**}, there exist points of $D_c A_{gi}^{**}$; and in any neighbourhood of the point of $P_c A_{gr\rho}^{**}$ there exist points of a rectangle which does not belong to P, that is the points of A_{gi}^{**} or A_{ac}^{**}, hence there exist points which do not belong to A. Finally in any neighbourhood of $TA_{gr\eta}^{**}$ there exists a point of $PA_{gr\rho}^{**}$ or $P_c A_{gr\rho}^{**}$ but a point of $PA_{gr\rho}^{**}$ does not belong to A, and in any neighbourhood of $P_c A_{gr\rho}^{**}$, we have seen that there is a point which does not belong to A. Consequently in any neighbourhood of the point of $TA_{gr\eta}^{**}$, there exists a point which does not belong to A. Therefore

$$A_i = A_i^{**}.$$

Consequently A belongs to the class $[A^{**}]$.

13. Next we shall consider the measure of A. The set $A_{gr\rho}^{*}$ is the set of points on the boundary of enumerable rectangles of the set S', hence

$$m(A_{gr\rho}^{**}) = 0,$$

and since the closed set A_{gr}^{**} is measurable, the set $A_{gr\eta}^{**}$ is also measurable and

$$m(A_{gr\eta}^{**}) = m(A_{gr}^{**}).$$

Since the open sets A_i^{**} and A_{gi}^{**} are measurable,

$$\left.\begin{aligned} m_e(A) &= m(A_i^{**}) + m_e(DA_{gi}^{**}) + m_e(TA_{gr\eta}^{**}) \\ m_i(A) &= m(A_i^{**}) + m_i(DA_{gi}^{**}) + m_i(TA_{gr\eta}^{**}) \end{aligned}\right\}^{1} \qquad (1)$$

The exterior and interior measure of DA_{gi}^{**} may be of any values β_1 and β_2 such that $m(A_{gi}^{**}) \geqq \beta_1 \geqq \beta_2 \geqq 0$. For, we divide the space E_2 into three parts H_1, H_2 and H_3 by the two lines $x = x_1$ and $x = x_2$ parallel to the x-axis such that

$$H_1 (x \leqq x_1), \qquad H_2 (x_1 < x < x_2), \qquad H_3 (x_2 \leqq x),$$

and let x_1 and x_2 be such that

$$m(H_1 A_{gi}^{**}) = \beta_2, \qquad m(H_2 A_{gi}^{**}) = \beta_1 - \beta_2, \qquad m(H_3 A_{gi}^{**}) = m(A_{gi}^{**}) - \beta_1.^{2}$$

Denote by R the set of all rational points in E_2, that is, the points whose coordinates are rational numbers, and its complementary set by R_c; and consider the non-measurable set Ω, which has with its complementary set Ω_c, the whole space E_2 as its same-measure cover, and its interior measure is zero[3]. The sets Ω and Ω_c are dense in E_2. For if the set Ω is not dense, then there exists an open set O such

---

1. Carathéodory: op. cit. p. 273,
2. op. cit. p. 288.
3. op. cit. p. 354.

that ΩO is empty, but by the definition of the same-measure cover[1]

$$m(O) = m_e(\Omega O)$$

which is absurd. Similarily for Ω_c.

Since R and R_c are also dense in E_2, the point set

$$R_c H_1 + \Omega H_2 + R H_3$$

is with its complementary sets dense in E_2, therefore we can use this point set for D. Then

$$D A_{gi}^{**} = R_c H_1 A_{gi}^{**} + \Omega H_2 A_{gi}^{**} + R H_3 A_{gi}^{**}.$$

Since $H_1 A_{gi}^{**}$, $H_2 A_{gi}^{**}$ and $H_3 A_{gi}^{**}$ are measurable,

$$m_e(D A_{gi}^{**}) = m(R_c H_1 A_{gi}^{**}) + m_e(\Omega H_2 A_{gi}^{**}) + m(R H_3 A_{gi}^{**}),$$

$$m_i(D A_{gi}^{**}) = m(R_c H_1 A_{gi}^{**}) + m_i(\Omega H_2 A_{gi}^{**}) + m(R H_3 A_{gi}^{**}).$$

But $m(R_c H_1 A_{gi}^{**}) = m(H_1 A_{gi}^{**}) = \beta_2,$

$m_e(\Omega H_2 A_{gi}^{**}) = m(H_2 A_{gi}^{**}) = \beta_1 - \beta_2,$

$m_i(\Omega H_2 A_{gi}^{**}) = 0,$

$m(R H_3 A_{gi}^{**}) = 0;$

therefore $m_e(D A_{gi}^{**}) = \beta_1$

and $m_i(D A_{gi}^{**}) = \beta_2$. Q. E. D.

Similarily, the exterior and interior measure of $T A_{gr\eta}^{**}$ may be any values γ_1 and γ_2 such that $m(A_{gr\eta}^{**}) \geqq \gamma_1 \geqq \gamma_2 \geqq 0$. For this purpose, consider two sets T_1 and T_2, which have no common point, such that

$$m(T_1 A_{gr\eta}^{**}) = \gamma_2 \quad \text{and} \quad m(T_2 A_{gr\eta}^{**}) = \gamma_1 - \gamma_2,$$ [2]

and let $T = T_1 + \Omega T_2,$

then $T A_{gr\eta}^{**} = T_1 A_{gr\eta}^{**} + \Omega T_2 A_{gr\eta}^{**};$

since $T_1 A_{gr\eta}^{**}$ and $T_2 A_{gr\eta}^{**}$ are measurable

$$m_e(T A_{gr\eta}^{**}) = m(T_1 A_{gr\eta}^{**}) + m_e(\Omega T_2 A_{gr\eta}^{**}),$$

$$m_i(T A_{gr\eta}^{**}) = m(T_1 A_{gr\eta}^{**}) + m_i(\Omega T_2 A_{gr\eta}^{**}),$$

but $m_e(\Omega T_2 A_{gr\eta}^{**}) = \gamma_1 - \gamma_2$

and $m_i(\Omega T_2 A_{gr\eta}^{**}) = 0,$

therefore $m_e(T A_{gr\eta}^{**}) = \gamma_1$

and $m_i(T A_{gr\eta}^{**}) = \gamma_2$. Q. E. D.

Since $m(A_i^{**}) + m(A_{gi}^{**}) + m(A_{gr\eta}^{**}) = m(A_\alpha^{**}),$

---

1. Carathéodory: op. cit. p. 279.
2. op. cit. p. 288.

from (1) $m_e(A)$ and $m_i(A)$ may be any values between $m(A_\alpha^{**})$ and $m(A_i^{**})$, such that

$$m(A_i^{**}) \leqq m_i(A) \leqq m_e(A) \leqq m(A_\alpha^{**}).$$

Therefore we have proved the theorem of § 11.

14. The above proof may be extended to the space E_n of n dimensions. Therefore the point functions which correspond to the exterior and interior Lebesgue measure of any point sets in the space E_n are the same as the point function which corresponds to the Lebesgue measure of measurable point sets in the space E_n. I denote this by the symbol $\mu([A])$.

Then *at a quadrable class* $[A^*]$, $\mu([A^*])$ *is one-valued, and*

$$\mu([A^*]) = \zeta([A^*]);$$

at a non-quadrable class $[A^{**}]$, $\mu([A^{**}])$ *is infinitely many-valued, and the aggregate of all such functional values at* $[A^{**}]$ *is a closed linear interval* $(m(A_i^{**}), m(A_\alpha^{**}))$.

15. We have seen in § 10, that the one-valued function $\zeta([A^*])$ is a continuous function in the domain $\{E_n^*\}$, which is dense-in-itself, but not closed. Since all classes in $\{E_n^{**}\}$ are the classes of accumulation of the set $\{E_n^*\}$, at a non-quadrable class, there is an aggregate of functional limits[1] of $\zeta([A^*])$ which is certainly a closed set. Therefore we may define a new function $\nu([A])$ for the space $\{E_n\}$, in the following manner: At each quadrable class $[A^*]$, let $\nu([A^*])=\zeta([A^*])$, and at each non-quadrable class $[A^{**}]$, attribute to $\nu([A^{**}])$ the values contained in the aggregate of functional limits of $\zeta([A^*])$ at that class $[A^{**}]$. This new function defined for the extended domain $\{E_n\}$ is called by Hobson[2] *the function obtained by extension of* $\zeta([A^*])$.

To see the relation between this function $\nu([A])$ and the function $\mu([A])$ defined in § 14, it is necessary to find the aggregate of functional limits of $\zeta([A^*])$ at any non-quadrable class $[A^{**}]$. It is obvious from (1) and (2) of § 10, that all values of functional limits of $\zeta([A^*])$ at $[A^{**}]$ are contained in the closed interval $(m(A_i^{**}), m(A_\alpha^{**}))$. But conversely, it may be proved that *any value in the closed interval* $(m(A_i^{**}), m(A_\alpha^{**}))$ *may be a functional limit of* $\zeta([A^*])$ *at* $[A^{**}]$.

To prove this, divide the space E_n into two parts H_1 and H_2 by an $n-1$ dimensional plane $x=x_1$, such that

1. Hobson: op. cit. p. 298.
2. op. cit. p. 322.

$$H_1(x \leqq x_1), \qquad H_2(x_1 < x),$$

and $$m(H_1 A_g^{**}) = \beta, \qquad m(H_2 A_g^{**}) = m(A_g^{**}) - \beta,$$

where β is any value which satisfies $m(A_g^{**}) \geqq \beta \geqq 0$. Now modify the net used in § 8, so that $x = x_1$ is one of the parallel $n-1$ dimensional planes used for the net. Let us denote the sum of the closed meshes of the net which contain a point of $H_1 A_g^{**}$ by S_1, and the sum of the open meshes in S_1 by S_2; similarily the sum of the closed meshes which contain a point of $H_2 A_g^{**}$ by T_1, and the sum of the open meshes in T_1 by T_2.

Construct the point set

$$A_\rho^* = \{A^{**} - A^{**}(S_1 + T_1)\} + \{S_2 + (T_1 - T_2)\},$$

then it may be proved as in § 8, that

$$r(A^{**}, A_\rho) < \rho,$$

ρ being any given number, and that A_ρ^* is quadrable. If we take a sequence $\rho_1, \rho_2, \ldots\ldots, \rho_n, \ldots\ldots$, with zero as its limit, then $[A_{\rho_1}^*]$, $[A_{\rho_2}^*], \ldots\ldots, [A_{\rho_n}^*], \ldots\ldots$ is a sequence of quadrable classes which has the non-quadrable class $[A^{**}]$ as its limiting class.

Since $$\operatorname*{Lim}_{n\to\infty} \{A^{**} - A^{**}(S_1 + T_1)\} = A_i^{**},$$

$$\lim_{n\to\infty} m\{A^{**} - A^{**}(S_1 + T_1)\} = m(A_i^{**});$$

and $$m\{S_2 + (T_1 - T_2)\} = mS_1,$$

but $$\operatorname*{Lim}_{n\to\infty} S_1 = H_1 A_g^{**},$$

therefore $$\lim_{n\to\infty} m\{S_2 + (T_1 - T_2)\} = m(H_1 A_g^{**}) = \beta.$$

Consequently $$\lim_{n\to\infty} m(A_{\rho_n}^*) = m(A_i^{**}) + \beta,$$

that is, $m(A_i^{**}) + \beta$ is a functional limit of the sequence $\zeta([A_{\rho_1}^*])$, $\zeta([A_{\rho_2}^*]), \ldots\ldots, \zeta([A_{\rho_n}^*]), \ldots\ldots$, but β being any value between $m(A_g^{**})$ and zero, $m(A_i^{**}) + \beta$ signifies any value between $m(A_i^{**})$ and $m(A_a^{**})$, including these two values.

Therefore the aggregate of functional limits of $\zeta([A^*])$ at $[A^{**}]$ is the closed interval $(m(A_i^{**}), m(A_a^{**}))$, which is the aggregate of functional values of $\mu([A])$ at $[A^{**}]$. Hence we conclude that *the point function $\mu([A])$ defined in the space $\{E_n\}$, which corresponds to the Lebesgue measure of point sets is the function obtained by extension of the point function $\zeta([A^*])$, defined in the set $\{E_n^*\}$, which corresponds to the Peano-Jordan measure of point sets.* If we use the term pointwise discontinuity for the many-valued function, it follows that $\mu([A])$

is point-wise discontinuous in $\{E_n\}$[1], whose measure of discontinuity is $m(A_a)-m(A_i)$; but this may be proved directly from (3) in § 10, since

$$\zeta_i([A]) \leqq \mu([A]) \leqq \zeta_e([A]),$$

$\mu([A])$ is continuous in $\{E_n\}$ at the quadrable class $[A^*]$.

Since $\zeta_e([A]) = m(A_a)$ and $\zeta_i([A]) = m(A_i)$,

it follows also that $\zeta_e([A])$ *and* $\zeta_i([A])$ *are the upper and the lower boundary functions respectively of* $\zeta([A^*])$ *in* $\{E_n^*\}$, as has been already noted at the end of § 10.

---

1 Hobson: op. cit. p. 322, this proof holds for metric space.