On the Decomposition of Substitutions of Groups of Sequences

By

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1. Consider a sequence of numbers $z_1, z_2, \ldots\ldots, z_\nu, \ldots\ldots$ and a sequence $z_{s_1}, z_{s_2}, \ldots\ldots, z_{s_\nu}, \ldots\ldots\ldots$ whose terms are the same as those of the former but in a different order. Here we obtain an abstract idea of the substitution

$$S \equiv \begin{pmatrix} 1 & 2 & \ldots\ldots\ldots \nu & \ldots\ldots\ldots \\ s_1 & s_2 & \ldots\ldots\ldots s_\nu & \ldots\ldots\ldots \end{pmatrix} \equiv \begin{pmatrix} \nu \\ s_\nu \end{pmatrix}.$$

I consider such a substitution as the series

$$(z_1 - z_{s_1}) + (z_2 - z_{s_2}) + \cdots\cdots + (z_\nu - z_{s_\nu}) + \cdots\cdots$$

is absolutely convergent. The totality of such substitutions forms the group of the sequence. This group has been studied by the author in two papers.[1] The present paper is a continuation of them with respect to the limited sequence and it deals with the inner structure of the substitutions of the group of sequence. The derived set (nucleus) of the set of points affected by a substitution and the order of the substitutions are defined. An infinite product of substitutions is not necessarily a substitution even in the abstract sense. A rigorous condition (or definition) is necessary. The author gives the condition and has decomposed cycles and substitutions into an infinite product of cycles. Among the cycles of the infinite degree, there is a special kind of cycles, called simple cycles, which play a special rôle. If the

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1. These Memoirs, A, **10**, 211 (1927); **11**, 11 (1928).

cycle $(\ldots\ldots\beta\ \alpha\ a\ b\ c\ldots\ldots)$, the letters denoting some integers of the sequence 1, 2,......, ν,......, be simple, it has the properties that in the limit, $z_a, z_b, z_c, \cdots\cdots \to z'$ and $z_\alpha, z_\beta, \cdots\cdots \to z''$ where z', z'' are two numbers and the series $(z_a - z') + (z_b - z') + (z_c - z') + \cdots$ and $(z_\alpha - z'') + (z_\beta - z'') + \cdots$ are absolutely convergent. Any substitution of order at most a is decomposed into a product of simple cycles. Finally, if two sequences have the same group, they must admit all the same simple cycles. Though this is not a sufficient condition, yet it is closer than the condition given in a previous paper.

2. Let $z_1, z_2, \ldots\ldots, z_\nu, \ldots\ldots$ be a sequence of numbers which we denote by $\{z_\nu\}$. In this paper we consider the *limited sequence*, i. e., the sequence of which all the numbers are in absolute value less than a certain positive number. Often we consider the sequence $\{z_\nu\}$ as a set of points. Then the set is bounded. We denote the series $z_1 + z_2 + \cdots\cdots + z_\nu + \cdots\cdots$ by $\sum z_\nu$ without regard to its convergency.

Consider a substitution $S \equiv \binom{\nu}{s_\nu}$. We define a sequence $\varepsilon_1^s, \varepsilon_2^s, \ldots, \varepsilon_\nu^s, \ldots\ldots$ such that

$$\varepsilon_\nu^s = 1, \text{ if } \nu \neq s_\nu$$

and $$\varepsilon_\nu^s = 0, \text{ if } \nu = s_\nu,$$

where s is marked to make clear that these numbers correspond to the substitution S. Next we consider the set of all points z_ν for which $\varepsilon_\nu^s = 1$. We denote the set by $\{\varepsilon_\nu^s z_\nu\}$ and suppose that the set does not contain the point $\varepsilon_\nu^s z_\nu (=0)$ where $\varepsilon_\nu^s = 0$. This set is equal to the set $\{\varepsilon_\nu^s z_{s_\nu}\}$, for the numbers z_{s_ν} which we substitute are the same in the totality as those numbers z_ν for which they are substituted. The derived set of the set $\{\varepsilon_\nu^s z_\nu\}$ is called in this paper, the *nucleus of the substitution* S and it is denoted by N_S. If N_S has no element, then the substitution is called of *order zero*. Such a case occurs only when the substitution is of the finite degree, provided that the sequence is made of all different numbers. If N_S has m points, then the substitution is of *order m*. In the same way if N_S has a countably infinite number of points, then S is of *order* a. Such a substitution is one of the substitutions of the infinite order.

3. *The product of two substitutions of order h and k respectively is a substitution of order at most* $h+k$. For let the two substitutions be

$$S \equiv \binom{\nu}{s_\nu} \text{ of order } h,$$

$$T \equiv \binom{\mu}{t_\mu} \text{ of order } k$$

and put
$$ST = \binom{\nu}{s_\nu}\binom{\mu}{t_\mu} \equiv \binom{\nu}{x_\nu}.$$

Now suppose $s_\nu = \mu$, then we have $t_\mu = x_\nu$. To determine ε_ν^{st}, we have four cases to consider.

(1) If $\varepsilon_\nu^s = 0$, $\varepsilon_\mu^t = 0$, then $\nu = s_\nu = \mu = t_\mu = x_\nu$.

Therefore $\varepsilon_\nu^{st} = 0$.

(2) If $\varepsilon_\nu^s = 0$, $\varepsilon_\mu^t = 1$, then $\nu = s_\nu = \mu \neq t_\mu = x_\nu$.

Therefore $\varepsilon_\nu^{st} = 1$.

(3) If $\varepsilon_\nu^s = 1$, $\varepsilon_\mu^t = 0$, then $\nu \neq s_\nu = \mu = t_\mu = x_\nu$.

Therefore $\varepsilon_\nu^{st} = 1$.

(4) If $\varepsilon_\nu^s = 1$, $\varepsilon_\mu^t = 1$, then $\nu \neq s_\nu = \mu$, $\mu \neq t_\mu = x_\nu$.

Therefore $\varepsilon_\nu^{st} = 1$ or 0,

according as ν is different from or equal to t_μ.

Hence we know that any element of the set $\{\varepsilon_\nu^{st} z_{x_\nu}\}$ is an element of the set $\{\varepsilon_\mu^t z_{t_\mu}\}$ (cases (2), (4)) or an element of the set $\{\varepsilon_\nu^s z_{s_\nu}\}$ (case (3)). In case (4), we consider, as has been defined, only the elements z_{x_ν} where $\varepsilon_\nu^{st} = 1$. In this case z_{x_ν} is an element of the set $\{\varepsilon_\mu^t z_{t_\mu}\}$. Therefore the set $\{\varepsilon_\nu^{st} z_\nu\}$ is contained in the set $\{\varepsilon_\nu^s z_\nu\} + \{\varepsilon_\mu^t z_\mu\}$. Hence the nucleus N_{ST} is a subset of $N_S + N_T$. This shows that the order of the product ST is at most $h+k$.

4. In this section we shall define the product of a countably infinite number of substitutions. Let the substitutions be

$$S^{(j)} \equiv \binom{\nu}{s_\nu^j}, \; j = 1, 2, \ldots\ldots, n, \ldots\ldots$$

and let

$$S^{(1)}S^{(2)}\ldots S^{(n)} \equiv P^{(n)} \equiv \binom{\nu}{p_\nu^n}$$

where $\{p_\nu^n\}$ is a sequence obtained from the sequence $\{\nu\}$ by the successive rearrangements due to the substitutions $S^{(j)}$, $j = 1, 2, \ldots, n$ (in this order). *If for any given integer ν' (however great), we can find integers $\bar{\nu}$ and $\bar{n}$ such that for any integer $n > \bar{n}$, we have $p_j^n = p_j^{\bar{n}}$,*

$j=1, 2,\dots,\bar{\nu}$, *and the integers* $1, 2,\dots,\nu'$ *are a part of the integers* $p_j^{\bar{n}}$, $j=1, 2,\dots, \bar{\nu}$ (*consequently* $\bar{\nu}\geq\nu'$), *we may consider* $\lim_{n\to\infty} S^{(1)}S^{(2)}\dots\dots S^{(n)}$. This limit is a substitution, say $P\equiv\begin{pmatrix}\nu\\ p_\nu\end{pmatrix}$. For $\begin{pmatrix}\nu\\ p_\nu\end{pmatrix}$ is in general a substitution, if all integers p_ν standing under ν are determined *uniquely*, *not repeatedly* and *without lack*. In the present case, under our condition stated above, for any integer ν', we can find integers $\bar{\nu}$, $\bar{n}$ such that in the product $P^{(n)}$

$$p_j^n=p_j^{\bar{n}},\ j=1, 2,\dots, \bar{\nu},\quad \text{for any } n>\bar{n}.$$

Therefore in any of the products $P^{(n)}$, $n>\bar{n}$, the integers standing under $1, 2,\dots, \nu'$ are always $p_j^{\bar{n}}$, $j=1, 2,\dots,\nu'$ respectively. Write for simplicity

$$p_j\equiv p_j^{\bar{n}},\ j=1, 2,\dots, \nu',$$

then in the limit $\lim_{n\to\infty} P^{(n)}$, the integers standing under $1, 2,\dots, \nu'$ are uniquely determined. Since ν' is arbitrarily taken into consideration, the sequence $\{p_\nu\}$ standing under $\{\nu\}$ in the infinite product P is uniquely determined. Moreover there is no repetition. For suppose $p_\lambda=p_\mu$ $(\lambda<\mu)$ in the sequence $\{p_\nu\}$, then, for $\nu'=\mu$ we may find two integers $\bar{\nu}$, $\bar{n}$ such that

$$p_j^n=p_j^{\bar{n}},\ j=1, 2,\dots,\bar{\nu} \text{ for any } n>\bar{n}.$$

Therefore none of the substitutions $S^{(n)}$ $(n>\bar{n})$ affects the first μ integers $(\mu\leq\bar{\nu})$ standing in the lower row of $P^{(\bar{n})}$. Hence we have

$$P^{(\bar{n})}=\begin{pmatrix}1 & 2\dots\dots\lambda\ \dots\dots\mu\ \dots\dots\\ p_1 & p_2\dots\dots p_\lambda\dots\dots p_\mu\dots\dots\end{pmatrix},$$

where $p_\lambda=p_\mu$. But this is clearly impossible. Therefore in the infinite product P, the sequence $\{p_\nu\}$ is not repeatedly determined. Moreover $\{p_\nu\}$ is precisely the rearrangement of the sequence $\{\nu\}$ without any lack. For consider any integer x and again apply the condition stated above for $\nu'=x$, then x should be found in the integers $p_j^{\bar{n}}$, $j=1, 2,\dots, \bar{\nu}$, not affected by any of the substitution $S^{(n)}$, $n>\bar{n}$. Q. E. D.

For example, consider the cycle

$$C\equiv(\dots\dots 4\ \ 2\ \ 1\ \ 3\ \ 5\dots\dots)=\begin{pmatrix}1 & 2 & 3 & 4 & 5\dots\dots\\ 3 & 1 & 5 & 2 & 7\dots\dots\end{pmatrix}.$$

Then

$$C^n=\begin{pmatrix} 1 & 2 & \cdots\cdots\cdots\cdots \\ 2n+1 & & \cdots\cdots\cdots\cdots \end{pmatrix}.$$

For $\nu'=1$, we cannot determine $\bar{\nu}$, $\bar{n}$ such that the first $\bar{\nu}$ elements in the lower row of C^n shall be certain fixed integers for any $n>\bar{n}$. Thus $\lim_{n\to\infty} C^n$ does not satisfy the condition.

We remark that even though all substitutions $S^{(n)}$ are admitted by the sequence $\{z_\nu\}$, it is not known a priori whether the infinite product P of the substitutions is admitted or not by $\{z_\nu\}$. Hence in the latter case the infinite product is only an abstract substitution but not a substitution in our proper sense.

We are accustomed to write the upper row of any substitution in the natural order of integers in the form

$$S=\begin{pmatrix} \nu \\ s_\nu \end{pmatrix}=\begin{pmatrix} 1 & 2 & 3 & \cdots\cdots\nu\cdots\cdots \\ s_1 & s_2 & s_3 & \cdots\cdots s_\nu\cdots\cdots \end{pmatrix}.$$

But this is merely a convention and we may write the upper row of the substitution in any manner. Therefore if an infinite product of substitutions is proved to be equal to S when the upper rows of all the substitutions are written in one and the same manner, the equality is also valid when the upper rows are written in the natural order of integers.

5. *Any substitution may be represented as a product of cycles.* For let $S=\begin{pmatrix} \nu \\ s_\nu \end{pmatrix}$ be a substitution. Quite in the same way as in any substitution of the finite degree, we may decompose S into a product of cycles C_1, C_2,······. The only difference is that the cycles may be generalised cycles, i. e. consisting of an infinite number of elements,[1] and the number of cycles may be infinite. We assume that C_1 is the cycle containing the element 1 as the least element. It is the middle element if C_1 be of the infinite degree. C_2 is the cycles containing as the middle element the least integer in the sequence $\{\nu\}$ but different from any element of C_1. The other cycles are defined in the same manner. We notice that if the degree of a cycle be finite, we do not use the term " middle element."

Suppose there is an infinite number of cycles. Since the cycles C_1, C_2,...... have no common elements, it is easy to verify that the product C_1C_2......... satisfies the condition of § 4. For consider an

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1. These Memoirs, A, **10**, 213.

integer ν'. Then 1, 2,..., ν' are contained (only once) in some of C_1, C_2,..., C_ν', since the least element of C_j is at least j ($j=1, 2,...,\nu'$). Let $\bar{n}$ be the greatest element among the elements anteceding 1, 2,..., ν' in the cycles which contain them, then $\bar{n} \geqq \nu'$. Put

$$C_1C_2\cdots C_{\bar{n}} \equiv T \equiv \begin{pmatrix} 1 & 2 & \cdots\cdots\nu'\cdots\cdots\bar{n}\cdots\cdots \\ t_1 & t_2 & \cdots\cdots t_{\nu'}\cdots\cdots t_{\bar{n}}\cdots\cdots \end{pmatrix};$$

then among $t_1, t_2,..., t_{\bar{n}}$, the elements 1, 2,..., ν' are contained. Since C_1, C_2,... have no common elements, $C_{\bar{n}+1}$, $C_{\bar{n}+2}$, ... do not affect the elements $t_1, t_2,...,t_{\bar{n}}$. Therefore taking $\bar{\nu}=\bar{n}$, for any $n>\bar{n}$, the first $\bar{\nu}$ elements in the lower row of the substitution $C_1C_2\cdots\cdots\cdots C_n$ remain constant and among them 1, 2,..., ν' are contained. Thus the product $C_1C_2\cdots C_n\cdots$ satisfies our condition for the product.

In the decomposition of a substitution we usually omit the identical cycles, the cycles whose degree is unity. The identical cycle say (ν) occurs only when ε_ν^{δ} are zero. Now for the decomposition

$$S=C_1C_2\cdots\cdots C_n\cdots\cdots,$$

we can easily verify that

$$N_S \geqq N_{C_1}+N_{C_2}+\cdots\cdots+N_{C_n}+\cdots\cdots,$$

since, as we have proved for the product of two substitutions, we have for any n, $N_S \geqq N_{C_n}$. The equality occurs when the number of the cycles is finite. For let z' be any point of the nucleus N_S, then about z' there must be an infinite number of elements of some one of the cycles. Therefore z' is a point of the nucleus of the cycle.

6. Let us consider a cycle of the infinite degree

$$C \equiv (\cdots\cdots\cdots\nu_\beta\ \nu_\alpha\ \nu_a\ \nu_b\ \nu_c\cdots\cdots\cdots)$$

where ν_a is the middle element. To avoid complexity of symbols, we put

$$\zeta_1 \equiv z_{\nu_a},\quad \zeta_2 \equiv z_{\nu_b},\ \zeta_3 \equiv z_{\nu_c},\cdots\cdots\cdots,$$

$$\zeta_{-1} \equiv z_{\nu_\alpha},\ \zeta_{-2} \equiv z_{\nu_\beta},\ \cdots\cdots\cdots\cdots\cdots,$$

then $$C=(\cdots\cdots\cdots-2\ \ -1\ \ \ 1\ \ \ 2\ \ \ 3\cdots\cdots\cdots).$$

Since we are dealing with the substitution admitted by the sequence $\{z_\nu\}$, the series

$$(\zeta_1-\zeta_2)+(\zeta_2-\zeta_3)+ \cdots\cdots\cdots$$
$$+(\zeta_{-1}-\zeta_1)+(\zeta_{-2}-\zeta_{-1})+\cdots\cdots$$

is absolutely convergent. Therefore

$$\lim_{n\to\infty}\{(\zeta_n-\zeta_{n+1})+\cdots\cdots+(\zeta_{n+p-1}-\zeta_{n+p})\}$$
$$=\lim_{n\to\infty}(\zeta_n-\zeta_{n+p})=0, \quad \text{for any integer } p;$$

i. e., $\zeta_1, \zeta_2,\cdots,\zeta_n,\cdots$ converges to a number. In the same way ζ_{-1}, $\zeta_{-2},\cdots, \zeta_{-n},\cdots$ converges to a number. Hence *the order of the cycle* (of the group of sequence) *is at most* 2, since $\lim_{n\to\infty}(\zeta_n-\zeta_{n+p})=0$ is the necessary and sufficient condition that the limited sequence $\{\zeta_n\}$ converges to a finite determinate limit, and the same for $\{\zeta_{-n}\}$.

Next let

$$\lim_{n\to\infty} \zeta_n=\zeta', \ \lim_{n\to\infty} \zeta_{-n}=\zeta''.$$

If the series $\sum_{n=1}^{\infty}(\zeta_n-\zeta')$ and $\sum_{n=1}^{\infty}(\zeta_{-n}-\zeta'')$ be absolutely convergent, we say that the cycle C is *simple*. As a special case, a cycle of the finite degree is simple. But a cycle of the infinite degree is not in general simple.

We remark that any simple cycle in the abstract sense is admitted by our sequence. For, using the same notations, let C be simple, then since $\sum_{n=1}^{\infty}(\zeta_n-\zeta')$ and $\sum_{n=1}^{\infty}(\zeta_{-n}-\zeta'')$ are absolutely convergent, the series $\sum_{\nu=1}^{\infty}(z_\nu-z_{c_\nu})$ where $\sum z_{c_\nu}$ is the transformed series of $\sum z_\nu$ by C, is nothing but

$$\sum_{n=1}^{\infty}(\zeta_n-\zeta_{n+1})+(\zeta_{-1}-\zeta_1)+\sum_{n=1}^{\infty}(\zeta_{-n-1}-\zeta_{-n})$$
$$=\sum_{n=1}^{\infty}\{(\zeta_n-\zeta')-(\zeta_{n+1}-\zeta')\}+(\zeta_{-1}-\zeta_1)+\sum_{n=1}^{\infty}\{(\zeta_{-n-1}-\zeta'') - (\zeta_{-n}-\zeta'')\}.$$

This series is absolutely convergent.

7. Using the same notations as in the last section, we shall decompose first the cycle of the infinite degree

$$C=(\cdots\cdots\cdots -2 \quad -1 \quad 1 \quad 2 \quad 3\cdots\cdots\cdots)$$

into a product of transpositions. A transposition is a cycle consisting of two elements. For example, by

$$(1\ \ 2)(-1\ \ 1)(-1\ \ -2)\Big|(-2\ \ 3)(-2\ \ -3)\Big|$$
$$\times(-3\ \ 4)(-3\ \ -4)\Big|(-4\ \ 5)$$

we have the interchange as follows:

$$\left.\begin{matrix} \cdots\cdots\cdots -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5\cdots\cdots \\ \cdots\cdots\cdots -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & -4\ \cdots \end{matrix}\right\};$$

while by
$$(1\ \ 2)(-1\ \ 1)(-1\ \ -2)\Big|(-2\ \ 3)(-2\ \ -3)\Big|$$
$$\times(-3\ \ 4)(-3\ \ -4)\Big|(-4\ \ 5)(-4\ \ -5),$$

we have the interchange

$$\left.\begin{matrix} \cdots\cdots -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5\cdots\cdots \\ \cdots\cdots -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & -5\cdots \end{matrix}\right\}$$

where the dotted parts are unchanged and the sections are marked to make clear the manner of the composition of the transpositions. We have therefore

$$C=(1\ \ 2)(-1\ \ 1)(-1\ \ -2)(-2\ \ 3)(-2\ \ -3)\cdots$$
$$\cdots\cdots(-n\ \ \overline{n+1})(-n\ \ -\overline{n+1})\cdots\cdots\cdots.$$

From the remark at the end of the last section, it is easy to prove that the equality conforms to the condition for the product of substitutions. Thus *any cycle may be decomposed into a product of transpositions with the same elements.*

Now it is easy to decompose the cycle C into a product of simple cycles of the infinite degree. For example, we have

$$(-2\ \ -3)=(\cdots\cdots\beta\ \ \alpha\ \ -2\ \ -3\ \ a\ \ \ b\ \ \ c\cdots\cdots)$$
$$\times(\cdots\cdots\cdots\ c\ \ \ b\ \ \ a\ \ -2\ \ \alpha\ \ \beta\cdots\cdots)$$

where a, b, c,$\cdots\cdots$ and α, β,$\cdots\cdots$ are some integers in C and greater than 3 in absolute value. To make the component cycles simple, we have only to take for a, b, c,$\cdots\cdots$ the suffixes of a partial sequence ζ_a, ζ_b, ζ_c,$\cdots\cdots\cdots$ of the sequence ζ_4, ζ_5, ζ_6,$\cdots\cdots\cdots$ such that the series

$(\zeta_a-\zeta')+(\zeta_b-\zeta')+(\zeta_c-\zeta')+\cdots\cdots\cdots$ becomes absolutely convergent where $\lim_{n\to\infty}\zeta_n=\zeta'$. We take α, $\beta,\cdots\cdots\cdots$ in the same way. Instead of the sequence ζ_4, ζ_5, $\zeta_6,\cdots\cdots$ we may also apply the sequence ζ_{-4}, ζ_{-5}, $\zeta_{-6},\cdots\cdots$.

These considerations are quite general and we have only to notice that we should not use the elements of the transpositions in the product of C lying on the left of the transposition now taken into consideration. Thus *any cycle may be decomposed into a product of simple cycles of the infinite degree with the same elements.*

We have decomposed any substitution S into a product of cycles with no common elements:

$$S=C_1C_2\cdots\cdots\cdots C_n\cdots\cdots\cdots.$$

Now we can decompose each cycle C_n into a product of simple cycles such that

$$C_n=C_{n\,1}C_{n\,2}\cdots\cdots\cdots C_{n\,m}\cdots\cdots\cdots,$$

then since C_1, $C_2,\cdots$, $C_n,\cdots$ have no common elements, the product

$$C_{1\,1}C_{1\,2}C_{2\,1}\cdots\cdots C_{1\,n}C_{2\,n-1}\cdots\cdots C_{n\,1}\cdots\cdots\cdots$$

is equal in its effect to the product $C_1C_2\cdots\cdots C_n\cdots\cdots$.
Therefore *any substitution may be decomposed into a product of simple cycles of the infinite degree.*

8. It is not difficult to prove that our new product

$$C_{1\,1}C_{1\,2}C_{2\,1}\cdots\cdots C_{1\,n}C_{2\,n-1}\cdots\cdots C_{n\,1}\cdots\cdots\cdots\cdots$$

conforms to the condition for the product given in § 4. Using the original notations, consider an element ν'; then since

$$S=C_1C_2\cdots\cdots\cdots C_n\cdots\cdots\cdots,$$

we can determine two integers $\bar{\nu}$ and $\bar{n}$ such that, writing

$$C_1C_2\cdots\cdots C_n\equiv\binom{\nu}{p_\nu^n},$$

we have $\quad p_j^n=p_j^{\bar{n}},\ j=1,\ 2,\cdots\cdots,\ \bar{\nu}$, for any $n>\bar{n}$,

and the integers $1,\ 2,\cdots,\ \nu'$ are a part of the integers $p_j^{\bar{n}},\ j=1,\ 2,\cdots,\ \bar{\nu}$. Since the cycles C_1, $C_2,\cdots$, $C_n,\cdots$ have no common elements, there is a cycle with the greatest suffix, say C_m, which contains some of the elements $p_j^{\bar n}$, $j=1, 2, \cdots\cdots, \bar\nu$, while C_{m+1}, $C_{m+2}, \cdots\cdots$ do not contain any of them. It is evident that $\bar n \geqq m$. Consider in general a cycle C_k, $(k \leqq m)$ which contains some of the elements $p_j^{\bar n}, j=1, 2, \cdots\cdots \bar\nu$. Then we have the decomposition

$$C_k = C_{k\,1}\ C_{k\,2} \cdots\cdots\cdots C_{k\,n} \cdots\cdots\cdots,$$

where $C_{k\,1}, C_{k\,2}, \cdots\cdots$, $C_{k\,n} \cdots\cdots$ are simple cycles as considered in the preceding section. Write

$$C_{k\,1} C_{k\,2} \cdots\cdots C_{k\,n} \equiv \begin{pmatrix} \mu \\ q_\mu^n \end{pmatrix},$$

and let the greatest integer among $p_j^{\bar n}$, $j=1, 2, \cdots\cdots \bar\nu$ be ν_0 such that they are a part of $1, 2, \cdots\cdots \nu_0$. Then for $\mu = \nu_0$, we can determine two integers μ_k, n_k such that

$$q_j^n = q_j^{n_k}\ , \ j=1, 2, \cdots\cdots, \mu_k, \text{ for any } n > n_k.$$

where the elements $1, 2, \cdots, \nu_0$ are a part of $q_j^{n_k}$, $j=1, 2, \cdots, \mu_k$. $(\nu_0 \leqq \mu_k)$ Therefore the effect of the cycle C_k on the elements $1, 2, \cdots\cdots, \mu_k$ is equal to that of $C_{k\,1} C_{k\,2} \cdots\cdots C_{k\,n_k}$, where $\nu_0 \leqq \mu_k$, the greatest of $p_j^{\bar n}$, $j=1, 2, \cdots, \bar\nu$ which contain $1, 2, \cdots\cdots, \nu'$ as a part, i. e., $\mu_k \geqq \nu'$. Thus some of the elements $p_j^{\bar n}$, $j=1, 2, \cdots, \nu'$ and contained in C_k are settled by the product $C_{k\,1} C_{k\,2} \cdots\cdots C_{k\,n}$ in their proper positions due to the substitution S, provided $n \geqq n_k$. Also if C_k, $(k \geqq m)$ does not contain any of the elements $p_j^{\bar n}$, $j=1, 2, \cdots\cdots, \bar\nu$, then these elements are not affected by any product $C_{k\,1} C_{k\,2} \cdots\cdots C_{k\,n}$, since the elements of these cycles consist of those of C_k. Thus for this cycle C_k, $n_k = 0$, so that $n_{m+1} = \cdots\cdots = 0$. Therefore by any product

$$C_{1\,1} C_{1\,2} \cdots\cdots C_{1\,n} \times C_{2\,1} C_{2\,2} \cdots\cdots C_{2\,n-1}$$
$$\times\ \cdots\cdots\cdots\cdots \times C_{n-1\,1}\ C_{n-1\,2} \times C_{n\,1}$$

where n is taken so great that

$$n \geqq n_1,\ n-1 \geqq n_2, \cdots\cdots,\ n+1-m \geqq n_m,$$

we have the rearrangements for the first ν elements :

$$\left.\begin{matrix} 1 & 2 & \cdots\cdots\cdots\cdots \bar\nu \\ p_1^{\bar n} & p_2^{\bar n} & \cdots\cdots\cdots\cdots p_\nu^{\bar n} \end{matrix}\right\}.$$

Let n_0 be the least integer satisfying the above inequalities. Since the elements of cycles $(C_{1\,1}, C_{1\,2},\cdots, C_{1\,n})$, $(C_{2\,1}, C_{2\,2},\cdots, C_{2\,n-1}),\cdots$ and $(C_{n\,1})$ are not common, their order in the product may be changed provided that the order of the cycles $C_{1\,1}, C_{1\,2},\cdots C_{1\,n}$ is conserved and is the same for the other cycles in the brackets. Thus the effect of the product written above is identical to that of the product $C_{1\,1}C_{1\,2}C_{2\,1}\cdots C_{1\,n}\,C_{2\,n-1}\cdots C_{n\,1}$, where $n\geq n_0$. The number of the factors is equal to $\dfrac{n(n+1)}{2}$. Put

$$\bar{l}\equiv\frac{n_0(n_0+1)}{2}, \text{ and}$$

$$C_{1\,1}C_{1\,2}C_{2\,1}\cdots\cdots C_{1\,n}C_{2\,n-1}\cdots C_{n\,1}\cdot C_{1\,n+1}\cdots C_{i\,n+1-i}\equiv\begin{pmatrix}\lambda\\ r^l_\lambda\end{pmatrix},$$

where $l(>l_0)$ denotes the number of the cycles. Then by our choice of n_0, the product

$$C_{1\,n_0+1}\,C_{2\,n_0}\cdots\cdots\cdots C_{i\,n_0+1-i}$$

does not affect the elements $p^{\bar{n}}_j$, $j=1, 2,\cdots, \bar{\nu}$. Therefore for any given $\lambda=\nu'$, we have determined two integers $\bar{\nu}$, $\bar{l}$ such that

$$r^l_j=p^{\bar{n}}_j,\ j=1,\ 2,\cdots\cdots,\ \bar{\nu},\ \text{for any } l\geq\bar{l},$$

or writing $\bar{\lambda}\equiv\bar{\nu}$, we have

$$r^l_j=r^{\bar{l}}_j,\ j=1,\ 2,\cdots\cdots,\ \bar{\lambda},\quad \text{for any } l>\bar{l}$$

Thus the product $C_{1\,1}C_{1\,2}C_{21}\cdots\cdots$ conforms to the condition for the product and it is equal to S. Q. E. D.

9. We have remarked that in the decomposition of a substitution into cycles, the sum of the nuclei of the cycles may not be equal to the nucleus of the substitution in so far as the number of the cycles is infinite. But we may decompose any substitution S into a product of simple cycles such that the sum of their nuclei is equal to that of S provided the order of S is not greater than $\mathfrak{a}$. For this, consider as usual the decomposition

$$S=C_1C_2\cdots\cdots C_n\cdots\cdots,$$

where the cycles C_1, $C_2,\cdots\cdots$, $C_n,\cdots\cdots$ have no common elements and we have the decomposition

$$C_n=T_{n\,1}T_{n\,2}\cdots\cdots\cdots,$$

where $T_{n\,1}$, $T_{n\,2},\cdots\cdots$ are transpositions formed in § 7 and their elements are those of C_n. Now we may prove as in the preceding section that

$$S = T_{1\,1} T_{1\,2} T_{2\,1} \cdots\cdots T_{1\,n} T_{2\,n-1} \cdots\cdots T_{n\,1} \cdots\cdots.$$

Thus *any substitution may be decomposed into a product of transpositions.* Let the nucleus of S be

$$N_S = (z^{(1)}, z^{(2)}, \cdots\cdots\cdots, z^{(m)}, \cdots\cdots\cdots),$$

and establish one to one correspondence between this set and the set of component transpositions of S such that $z^{(1)}$ corresponds to $T_{1\,1}$, $z^{(2)}$ to $T_{1\,2}, \cdots\cdots\cdots$. Now suppose $z^{(m)}$ corresponds to $T_{h\,k}$ and let

$$T_{k\,k} = (\xi\ \eta),$$

where ξ, η are certain elements of C_h, Since $z^{(m)}$ is a point of N_S, there is a partial sequence say z_{s_j}, $j=1, 2, \cdots\cdots$ tending to $z^{(m)}$ where $\varepsilon_j^s = 1$, $j=1, 2, \cdots\cdots$. Some integers of the sequence $(s_1, s_2, \cdots\cdots)$ may be contained in some transpositions standing to the left of $T_{h\,k}$ in the product expression of S. But their number is finite. Excepting these integers, we choose two partial sequences $(a, b, c, \cdots\cdots)$ and $(\alpha, \beta, \cdots)$ out of $(s_1, s_2, \cdots\cdots)$ such that the series

$$(z_a - z^{(m)}) + (z_b - z^{(m)}) + (z_c - z^{(m)}) + \cdots\cdots\cdots$$

and

$$(z_\alpha - z^{(m)}) + (z_\beta - z^{(m)}) + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

shall be absolutely convergent. On the other hand we have

$$\begin{aligned} T_{h\,k} &= (\xi\ \eta) \\ &= (\cdots\cdots\beta\ \ \alpha\ \ \xi\ \ \eta\ \ a\ \ b\ \ c\cdots\cdots) \\ &\times (\cdots\cdots\cdots\ c\ \ b\ \ a\ \ \xi\ \ \alpha\ \ \beta\cdots\cdots) \end{aligned}$$

where the two component cycles are simple and have $z^{(m)}$ as their nucleus. Such a decomposition is possible for any transposition corresponding to an element of N_S. Denoting these cycles by Γ's, we have

$$S = \Gamma_1 \Gamma_2 \cdots\cdots\cdots.$$

We remark that if Γ_l is a component cycle of $T_{h\,k}$ then since Γ_l does not contain any elements of cycles standing on the left of $T_{h\,k}$ in the product expression of S, this product satisfies the condition for the product of substitutions. Thus *any substitution* S *of order not greater than* $\mathfrak{a}$ *may be decomposed into a product of simple cycles* $\Gamma_1, \Gamma_2, \cdots$:

$$S = \Gamma_1 \Gamma_2 \cdots\cdots\cdots\cdots\cdots\cdots.$$

such that $N_S = N_{\Gamma_1} \dot{+} N_{\Gamma_2} \dot{+} \cdots\cdots.$

10. Any sequence admits of any transposition. Therefore the transpositions are not concerned with the condition for the equality of the groups of two different sequences. On the contrary, a simple cycle (of the infinite degree) of a sequence is not necessarily admitted by another sequence. But *if two sequences have the same group, they must have all the same simple cycles.* This is a slightly closer condition for the equality of the groups of two sequences than that which was proved in a former paper. For let two (limited) sequences be $\{z_\nu\}$ and $\{\zeta_\nu\}$ having the same group and let

$$C \equiv (\cdots\cdots\cdots \nu_\beta \quad \nu_\alpha \quad \nu_a \quad \nu_b \quad \nu_c \cdots\cdots\cdots)$$

be a simple cycle of the sequence $\{z_\nu\}$, then by definition, in the limit,

$$z_{\nu_a},\ z_{\nu_b},\ z_{\nu_c},\cdots\cdots\cdots \to z',$$

$$z_{\nu_\alpha},\ z_{\nu_\beta},\ \cdots\cdots\cdots\cdots \to z'',$$

where z', z'' are the limit-points, such that the series

$$|z_{\nu_a} - z'| + |z_{\nu_b} - z'| + |z_{\nu_c} - z'| + \cdots\cdots\cdots$$

and
$$|z_{\nu_\alpha} - z''| + |z_{\nu_\beta} - z''| + \cdots\cdots\cdots\cdots$$

are both convergent and, as we have proved (§ 6), the cycle C is admitted by the sequence $\{z_\nu\}$ i. e., C is a substitution of the group. Since by hypothesis the two sequences have the same group, the cycle C must be admitted by the sequence $\{\zeta_\nu\}$. Consequently, as we have proved (§ 6), in the limit

$$\zeta_{\nu_a},\ \zeta_{\nu_b},\ \zeta_{\nu_c},\cdots\cdots \cdot \to \zeta',$$

$$\zeta_{\nu_\alpha},\ \zeta_{\nu_\beta},\cdots\cdots\cdots\cdots\cdots\cdots \to \zeta''.$$

where ζ', ζ'' are the limit-points. Now if the cycle C be not simple with respect to the sequence $\{\zeta_\nu\}$, then at least one of the series

$$|\zeta_{\nu_a} - \zeta'| + |\zeta_{\nu_b} - \zeta'| + |\zeta_{\nu_c} - \zeta'| + \cdots\cdots\cdots$$

and
$$|\zeta_{\nu_\alpha} - \zeta''| + |\zeta_{\nu_\beta} - \zeta''| + \cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

is divergent, though the terms of both series converge to zero in the limit. Assume the first to be divergent, then we may divide the sequence $(\nu_a, \nu_b, \nu_c, \cdots\cdots)$ into two parts say $(\lambda_1, \lambda_2, \cdots\cdots)$ and $(\mu_1, \mu_2, \cdots)$ such that the series

$$|\zeta_{\lambda_1}-\zeta'|+|\zeta_{\lambda_2}-\zeta'|+\cdots\cdots\cdots$$

is divergent while the series

$$|\zeta_{\mu_1}-\zeta'|+|\zeta_{\mu_2}-\zeta'|+\cdots\cdots\cdots$$

is convergent. Consider the cycle

$$C'\equiv(\cdots\cdots\nu_3\quad\nu_a\quad\lambda_1\quad\mu_1\quad\lambda_2\quad\mu_2\cdots\cdots).$$

Then, since

$$\begin{aligned}&|\zeta_{\lambda_1}-\zeta_{\mu_1}|+|\zeta_{\mu_1}-\zeta_{\lambda_2}|+|\zeta_{\lambda_2}-\zeta_{\mu_2}|+\cdots\cdots\cdots\\ =&|\zeta_{\lambda_1}-\zeta')-(\zeta_{\mu_1}-\zeta')|+|(\zeta_{\mu_1}-\zeta')-(\zeta_{\lambda_2}-\zeta')|\\ &\qquad+|(\zeta_{\lambda_2}-\zeta')-(\zeta_{\mu_2}-\zeta')|+\cdots\cdots\\ \geqq&|\zeta_{\lambda_1}-\zeta'|+2|\zeta_{\lambda_2}-\zeta'|+\cdots\cdots\cdots\\ &-2\{|\zeta_{\mu_1}-\zeta'|+|\zeta_{\mu_2}-\zeta'|+\cdots\cdots\}=\infty,\end{aligned}$$

the cycle C' is not admitted by the sequence $\{\zeta_\nu\}$. But the series

$$\begin{aligned}&|z_{\lambda_1}-z'|+|z_{\lambda_2}-z'|+\cdots\cdots\cdots\\ +&|z_{\mu_1}-z'|+|z_{\mu_2}-z'|+\cdots\cdots\cdots\end{aligned}$$

is nothing but the convergent series $|z_{\nu_a}-z'|+|z_{\nu_b}-z'|+|z_{\nu_c}-z'|+\cdots$. Therefore C' is a simple cycle of the sequence $\{z_\nu\}$, a substitution of its group. This is contradictory to the hypothesis that our two sequences have the same group. Q. E. D.

We remark that the converse of the theorem is not true i. e., two sequences with all the same simple cycles may not have the same group. For example the sequences $\left\{\frac{1}{\nu}\right\}$ and $\left\{(-1)^\nu\frac{1}{\nu}\right\}$ have all the same simple cycles, though the latter does not admit the subsitution $(1\ 2)(3\ 4)(5\ 6)\cdots\cdots$, which is admitted by the former. This shows that the infinite product of certain simple cycles is not necessarily a substitution of the group of a sequence.