On a Unified Theory of Gravitational and Electromagnetic Fields

By

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Abstract

The object of this paper is to unify the gravitational and electromagnetic fields in a simple way and to deduce equations of motion of charged particles as well as those of light-rays.

To do this, the writer makes the following assumptions:

(i) Our space-time world is a four-dimensional continuum with a semi-symmetric and metric connection, whose fundamental tensor $g_{\mu\nu}$ and vector φ_ν are interpreted as the gravitational and electromagnetic potentials respectively.

(ii) The field-laws are derived from a Hamiltonian function $R-L$, where R is the scalar curvature of the world and L is an invariant formed by $g_{\mu\nu}$ and φ_ν.

Starting from these assumptions and applying several mathematical theorems, the writer shows that all the results given in the theory of general relativity together with Maxwell-Lorentz's theory are obtained in a compact form without any modification.

An attractive theory based on parallelism at a distance has been published by Einstein[1] a few years ago. To develope his theory he used the conception of an orthogonal ennuple, which was also used by Levi-Civita,[2] and he succeeded in finding the field-equations in the first approximation, but he could not find the equations of the path traced by a particle carrying an electric charge. Other theories starting from the notion of infinitesimal parallel displacement have been developed by Weyl,[3] Eddington[4] and also by Einstein.[5]

But by these theories it is not possible to compare lengths (except

1. A. Einstein: Berlin. Berichte, 1 (1929)
2. T. Levi-Civita: Berlin. Berichte, 138 (1929)
3. H. Weyl: Berlin. Berichte, 465 (1918)
4. A. S. Eddington: Proc. Roy. Soc. A. **99**, 104 (1921)
5. A. Einstein: Berl. Berichte, 32, 76, 137 (1923)

zero length) at different points, because the length of any vector is changed by a parallel displacement. Though Einstein's new theory is free from this difficulty, it seems unnatural and complicated in its geometrical structure.

In this paper we try to show that Einstein's celebrated equations of the gravitational field and Maxwell-Lorentz's equations of the electromagnetic field in the relativistic form can be admitted without modification in a four dimensional continuum having the characteristic properties such that the fundamental tensor- and vector fields may exist and the comparison of lengths at different points is possible, and also to show that the equations of the path traced by a particle carrying a charge can be deduced, in an ordinary form, by means of energy-momentum-equations. Our unification of the gravitational and electromagnetic fields is accomplished by using a differential geometry which is a special case of a very general one based on the principle of linear connection given by Schouten.[1]

In 1928 Leopordo Infeld[2] had already attacked the problem by the aid of a differential geometry which is contained in Schouten's as another special case and obtained the field-equations in a slightly different form. But the writer thinks that the following theory is more simple and complete than the line of approach adopted by Infeld.

§ 1. Geometrical preliminaries.

In this section we describe briefly our geometrical theory in the manner in which it is extended by Schouten.[3] Suppose an n-dimensional continuum defined by any ordered set of n independent real variables $x^\nu (\nu=1, 2, \ldots\ldots, n)$, which are called the coordinates of points in it. Moreover, consider that this continuum obeys the principle of linear connection; a connection which satisfies the following five conditions is called a linear one :—

(i) Any tensor and its absolute differential are tensors of the same kind.

We denote the absolute differential of any tensor Φ by the symbol $\delta\Phi$, and its ordinary differential by the symbol $d\Phi$.

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1. J. A. Schouten: Der Ricci-Kalcül.
2. L. Infeld: Zs. f. Phys., **50**, 137 (1928)
3. loc. cit. Chap. II.
Schouten established his theory discriminating the "eigentlich" tensor from the ordinary one. But in our case we need not consider this difference, so as to develope the concept of the "Überschiebungsinvariant" connection.

(ii) The absolute differential of any tensor is a linear function of coordinate differentials.

(iii) The absolute differential of a sum is the sum of the absolute differentials of its components, thus

$$\delta(\Phi+\Psi)=\delta\Phi+\delta\Psi.$$

(iv) The absolute differentiation of an outer product obeys the same rule as the ordinary differentiation, namely

$$\delta(\Phi\Psi)=\Psi\delta\Phi+\Phi\delta\Psi.$$

(v) The absolute differential of any invariant is equal to its ordinary differential: viz.

$$\delta A=dA,$$

where A is an invariant.

When the absolute differential of any tensor Φ is expressed by the form

$$\delta\Phi=dx^{\mu}\nabla_{\mu}\Phi,$$

we may call $\nabla_{\mu}\Phi$ the covariant derivative of Φ. In the foregoing connection Schouten showed that the absolute differentials of any contravariant vector A^{ν} and any covariant vector B_{ν} are respectively given by the formulae[1]

$$\delta A^{\nu}=dA^{\nu}+\Gamma^{\nu}_{\lambda\mu}A^{\lambda}dx^{\mu}$$

$$\delta B_{\nu}=dB_{\nu}-\Gamma'^{\lambda}_{\nu\mu}B_{\lambda}dx^{\mu},$$

where $\Gamma^{\nu}_{\lambda\mu}$ and $\Gamma'^{\nu}_{\lambda\mu}$ are $2n^3$ arbitrarily chosen functions of x's.

Let us confine ourselves to the consideration of the case which Schouten called the "Überschiebungsinvariant" connection i. e. $\Gamma^{\nu}_{\lambda\mu}=\Gamma'^{\nu}_{\lambda\mu}$, and call it simply "the connection". Then, in our connection, we have

$$\delta A^{\nu}=dA^{\nu}+\Gamma^{\nu}_{\lambda\mu}A^{\lambda}dx^{\mu}, \qquad (1\cdot 1)$$

and

$$\delta B_{\nu}=dB_{\nu}-\Gamma^{\lambda}_{\nu\mu}B_{\lambda}dx^{\mu}, \qquad (1\cdot 2)$$

from which it can easily be shown that, if $A^{\beta_1\ldots\ldots\beta_q}_{\alpha_1\ldots\ldots\alpha_p}$ are the components of a tensor, its absolute differential is given by

$$\delta A^{\beta_1\ldots\ldots\beta_q}_{\alpha_1\ldots\ldots\alpha_p}=dA^{\beta_1\ldots\ldots\beta_q}_{\alpha_1\ldots\ldots\alpha_p}+\sum_{i=1}^{q}\Gamma^{\beta_i}_{\lambda\mu}A^{\beta_1\ldots\beta_{i-1}\lambda\beta_{i+1}\ldots\beta_q}_{\alpha_1\ldots\ldots\ldots\ldots\ldots\ldots\alpha_p}dx^{\mu}$$
$$-\sum_{j=1}^{p}\Gamma^{\lambda}_{\alpha_j\mu}A^{\beta_1\ldots\ldots\ldots\ldots\ldots\ldots\beta_q}_{\alpha_1\ldots\alpha_{j-1}\lambda\alpha_{j+1}\ldots\alpha_p}dx^{\mu}. \qquad (1\cdot 3)$$

1. loc. cit. p. 65.

If we express the preceding formulae (1·1), (1·2) and (1·3) in forms of covariant derivatives, we have

$$\nabla_\mu A^\nu = \frac{\partial A^\nu}{\partial x^\mu} + \Gamma^\nu_{\lambda\mu} A^\lambda, \qquad (1\cdot 4)$$

$$\nabla_\mu B_\nu = \frac{\partial B_\nu}{\partial x^\mu} - \Gamma^\lambda_{\nu\mu} B_\lambda, \qquad (1\cdot 5)$$

and

$$\nabla_\mu A^{\beta_1\cdots\cdots\beta_q}_{\alpha_1\cdots\cdots\alpha_p} = \frac{\partial A^{\beta_1\cdots\cdots\beta_q}_{\alpha_1\cdots\cdots\alpha_p}}{\partial x^\mu} + \sum_{i=1}^{q} \Gamma^{\beta_i}_{\lambda\mu} A^{\beta_1\cdots\beta_{i-1}\lambda\beta_{i+1}\cdots\beta_q}_{\alpha_1\cdots\cdots\cdots\cdots\cdots\alpha_p} - \sum_{j=1}^{p} \Gamma^\lambda_{\alpha_j\mu} A^{\beta_1\cdots\cdots\cdots\cdots\cdots\beta_q}_{\alpha_1\cdots\alpha_{j-1}\lambda\alpha_{j+1}\cdots\alpha_p} \qquad (1\cdot 6)$$

Applying (1·6) to the tensor $\delta^\nu_\lambda (=1$ for $\lambda=\nu$; $=0$ for $\lambda \neq \nu)$, we get

$$\nabla_\mu \delta^\nu_\lambda = \Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\lambda\mu} = 0, \qquad (1\cdot 7)$$

which is the characteristic property of our connection.

Our connection is then determinate, when the coefficients of connection $\Gamma^\nu_{\lambda\mu}$ are defined as functions of the coordinates x's. In order that the connection may itself be independent of the choice of coordinates, the set of n^3 functions $\Gamma^\nu_{\lambda\mu}$ must be transformed, in a definite way, into another set of n^3 functions $\bar{\Gamma}^\nu_{\lambda\mu}$ if we take another system of coordinates $\bar{x}$'s.

It is evident that the transformation-equations of the Γ's are given by

$$\bar{\Gamma}^\nu_{\lambda\mu} = \Gamma^\alpha_{\beta\gamma} \frac{\partial x^\beta}{\partial \bar{x}^\lambda} \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \frac{\partial \bar{x}^\nu}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\mu} \frac{\partial \bar{x}^\nu}{\partial x^\alpha}, \qquad (1\cdot 8)$$

when we perform a transformation of coordinates

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, x^2, \ldots\ldots, x^n)$$

whose Jacobian is not identically zero.

According to (1·8) it follows that the difference

$$2P^{\ \nu}_{\lambda\mu} \equiv \Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\mu\lambda} \qquad (1\cdot 9)$$

is also a tensor of the third rank which is anti-symmetric with respect to λ and μ. In particular, we say that the connection is semi-symmetric, when $P^{\ \nu}_{\lambda\mu}$ has the following form

$$P^{\ \nu}_{\lambda\mu} = \frac{1}{2}(\varphi_\lambda \delta^\nu_\mu - \varphi_\mu \delta^\nu_\lambda), \qquad (1\cdot 10)$$

where φ_λ are the components of a vector.

As in Riemannian geometry, if we define the curl of a vector A_ν by the tensor of the second rank $\Gamma_\mu A_\nu - \Gamma_\nu A_\mu$, we obtain, in a semi-symmetric connection,

$$\Gamma_\mu A_\nu - \Gamma_\nu A_\mu = \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) - \varphi_\nu A_\mu + \varphi_\mu A_\nu. \tag{1·11}$$

From this it follows that the curl of the fundamental vector φ_ν is

$$\Gamma_\mu \varphi_\nu - \Gamma_\nu \varphi_\mu = \frac{\partial \varphi_\nu}{\partial x^\mu} - \frac{\partial \varphi_\mu}{\partial x^\nu}. \tag{1.12}$$

It is a remarkable fact that in our semi-symmetric connection only the curl of the fundamental vector φ_ν has the same form as in Riemannian geometry, as we have just seen.

In the preceding arguments we have obtained the tensor-field $P_{\lambda\mu}^{\cdot\cdot\nu}$ formed by the fundamental vector-field φ_ν as in (1·10), but it is not sufficient to determine the $\Gamma_{\lambda\mu}^{\nu}$-field by means of a single vector-field φ_ν. After Schouten, this determination of the $\Gamma_{\lambda\mu}^{\nu}$-field can be accomplished by the aid of the vector-field φ_ν and a certain tensor-field.

If we take any symmetric contravariant tensor $g^{\lambda\nu}$ whose determinant g has rank n, there exists a unique covariant tensor $g_{\lambda\nu}$ which is defined as the cofactor of $g^{\lambda\nu}$ in this determinant, divided by g. Hence we have

$$g_{\lambda\mu} g^{\mu\nu} = \delta_\lambda^\nu. \tag{1·13}$$

For the covariant derivatives of $g^{\lambda\nu}$ and $g_{\lambda\nu}$, we use the symbols $Q_\mu^{\cdot\lambda\nu}$ and $S_{\mu\lambda\nu}$ respectively: viz

$$\left.\begin{aligned} \Gamma_\mu g^{\lambda\nu} &= Q_\mu^{\cdot\lambda\nu}, \\ \Gamma_\mu g_{\lambda\nu} &= S_{\mu\lambda\nu}. \end{aligned}\right\} \tag{1·14}$$

Differentiating (1·13) covariantly and substituting (1·14) in it, we get

$$S_{\mu\lambda\nu} = -g_{\lambda\alpha} g_{\nu\beta} Q_\mu^{\cdot\alpha\beta}.$$

Accordingly the two sets of equations

$$Q_\mu^{\cdot\lambda\nu} = \frac{\partial g^{\lambda\nu}}{\partial x^\mu} + \Gamma_{\alpha\mu}^{\lambda} g^{\alpha\nu} + \Gamma_{\alpha\mu}^{\nu} g^{\lambda\alpha} = 0 \tag{1·15}$$

and

$$S_{\mu\lambda\nu}=\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}-\Gamma^{\alpha}_{\lambda\mu}g_{\alpha\nu}-\Gamma^{\alpha}_{\nu\mu}g_{\lambda\alpha}=0 \tag{1·16}$$

are equivalent to one another.

If $\Gamma^{\nu}_{\lambda\mu}$ satisfy (1·16) or (1·15), we call this connection a metric one. Although we can determine $\Gamma^{\nu}_{\lambda\mu}$, as Schouten showed[1], by the two general tensor-fields $P_{\lambda\mu}^{\ \ \nu}$ and $Q_{\mu}^{\lambda\nu}$, we may express $\Gamma^{\nu}_{\lambda\mu}$ with these only when the connection is semi-symmetric and metric.

In this case $\Gamma^{\nu}_{\lambda\mu}$ satisfies (1·16) i. e.

$$\Gamma^{\alpha}_{\mu\nu}g_{\lambda\alpha}+\Gamma^{\alpha}_{\lambda\nu}g_{\mu\alpha}=\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}}. \tag{1·17a}$$

Similarly

$$\Gamma^{\alpha}_{\nu\lambda}g_{\mu\alpha}+\Gamma^{\alpha}_{\mu\lambda}g_{\nu\alpha}=\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}, \tag{1·17b}$$

$$\Gamma^{\alpha}_{\lambda\mu}g_{\nu\alpha}+\Gamma^{\alpha}_{\nu\mu}g_{\lambda\alpha}=\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}. \tag{1·17c}$$

Adding (1·17)b and (1·17)c and subtracting (1·17)a, we obtain by (1.9)

$$2\Gamma^{\alpha}_{\lambda\mu}g_{\nu\alpha}=\left(\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}+\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}-\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}}\right)$$
$$+2(P_{\lambda\mu}^{\ \ \alpha}g_{\nu\alpha}+P_{\mu\nu}^{\ \ \alpha}g_{\lambda\alpha}+P_{\lambda\nu}^{\ \ \alpha}g_{\mu\alpha}),$$

where $P_{\lambda\mu}^{\ \ \nu}$ is, of course, given by (1·10).

If we solve $\Gamma^{\nu}_{\lambda\mu}$ from the above equations and use (1·10), we can easily show that

$$\Gamma^{\nu}_{\lambda\mu}=\left\{\begin{matrix}\nu\\ \lambda\mu\end{matrix}\right\}+\varphi_{\lambda}\delta^{\nu}_{\mu}-g_{\lambda\mu}\varphi^{\nu}, \tag{1·18}$$

where $\left\{\begin{matrix}\nu\\ \lambda\mu\end{matrix}\right\}$ is the Christoffel 3-index-symbol formed with $g_{\mu\nu}$, and φ^{ν} is the associated vector of φ_{λ} with respect to $g_{\mu\nu}$. (Hereafter we will use these notations as in Riemannian geometry.)

From (1·18) it will be seen that the coefficients of connection are completely determined by the vector-field φ_{ν} and the tensor-field $g_{\mu\nu}$.

If we put

$$\varphi_{\lambda}\delta^{\nu}_{\mu}-g_{\lambda\mu}\varphi^{\nu}=T_{\lambda\mu}^{\ \ \nu}, \tag{1·19}$$

we have, from (1·18),

$$\Gamma^{\nu}_{\lambda\mu}=\left\{\begin{matrix}\nu\\ \lambda\mu\end{matrix}\right\}+T_{\lambda\mu}^{\ \ \nu}. \tag{1·20}$$

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1. loc. cit. pp. 72–73

According to the tensorial property of $T_{\lambda\mu}^{\ \nu}$ defined by (1·19), we infer that the quantities $\mathring{\nabla}_\mu A^\nu$ and $\mathring{\nabla}_\mu B_\nu$ defined by

$$\left.\begin{aligned}\mathring{\nabla}_\mu A^\nu &= \frac{\partial A^\nu}{\partial x^\mu} + \left\{ {\nu \atop \alpha\mu} \right\} A^\alpha, \\ \mathring{\nabla}_\mu B_\nu &= \frac{\partial B_\nu}{\partial x^\mu} - \left\{ {\alpha \atop \nu\mu} \right\} B_\alpha\end{aligned}\right\} \tag{1·21}$$

are also tensors of the second rank, since the covariant derivatives of these vectors A^ν and B_ν can be written in the form

$$\left.\begin{aligned}\nabla_\mu A^\nu &= \mathring{\nabla}_\mu A^\nu + T_{\alpha\mu}^{\ \nu} A^\alpha, \\ \nabla_\mu B_\nu &= \mathring{\nabla}_\mu B_\nu - T_{\nu\mu}^{\ \alpha} B_\alpha.\end{aligned}\right\} \tag{1·22}$$

In particular, if we contract (1·22) with respect to μ and ν, we have

$$\nabla_\nu A^\nu = \mathring{\nabla}_\nu A^\nu + (n-1)\varphi_\nu A^\nu. \tag{1·23}$$

Evidently these formulae can be extended, without difficulty, to the covariant derivatives of a tensor of any rank.

§ 2. Theory of continuum with a semi-symmetric and metric connection.

In our n-dimensional continuum let C be any curve defined by the equations

$$x^\nu = x^\nu(t),$$

where t is a parameter. As in Riemannian Geometry, a unique solution A^ν of the system of differential equations

$$\frac{dA^\nu}{dt} + \Gamma_{\lambda\mu}^{\nu} A^\lambda \frac{dx^\mu}{dt} = 0, \tag{2·1}$$

satisfying the initial conditions

$$A^\nu(t_0) = \underset{0}{A}{}^\nu,$$

where $\underset{0}{A}{}^\nu$ is any given vector at a point t_0 of C is said to be a family of vectors parallel along C to the given vector $\underset{0}{A}{}^\nu$ at the point t_0 of C.

By (1·1) the system of differential equations (2·1) is obviously equivalent to the system

$$\delta A^\nu = 0. \tag{2·2}$$

Hence the absolute change of the squared length A^2 of the vector A^ν defined by

$$A^2 = g_{\mu\nu} A^\mu A^\nu \tag{2·3}$$

is $\delta g_{\mu\nu}A^\mu A^\nu$, when A^μ is carried by an infinitesimal parallel displacement. But this value is evidently zero, since our connection satisfies the metric conditions (1·16) i. e. $\delta g_{\mu\nu}=0$.

Similarly, if we define the angle θ between any two vectors A^μ and B^ν by the equation

$$\cos\theta = g_{\mu\nu}A^\mu B^\nu \tag{2·4}$$

we can prove that its absolute change is also zero. Therefore we get the theorem: The length of a vector and the angle between two vectors do not change their values respectively through the parallel displacement of the vectors along any given curve. This makes it possible to compare lengths and angles at different points without any ambiguity.

Accordingly, if we define the length of C by the integral $\int_c ds$, it has a definite meaning and when we choose it as a parameter of C, $\dfrac{dx^\nu}{ds}$ gives the tangent-vector of C whose length is always unity.

If we introduce the contravariant vector

$$N^\nu = \frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\lambda\mu}\frac{dx^\lambda}{ds}\frac{dx^\mu}{ds}, \tag{2·5}$$

it is obvious that $N^\nu ds$ represents the absolute differential of the tangent vector of C. Accordingly we call N^ν the principal normal of C.

When $N^\nu=0$ along C, the curve is said to be a geodesic. Strictly speaking, geodesics are curves such that the absolute changes of their tangent-vectors along them are identically zero.

Hence we have

$$\frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\lambda\mu}\frac{dx^\lambda}{ds}\frac{dx^\mu}{ds} = 0, \tag{2·6}$$

as the differential equations of geodesics.

But by means of the relation

$$\varphi_\lambda\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds} = \varphi^\nu$$

and (1·18), equations (2·5) and (2·6) can be written as

$$N^\nu = \frac{d^2x^\nu}{ds^2} + \left\{ {\nu \atop \lambda\mu} \right\}\frac{dx^\lambda}{ds}\frac{dx^\mu}{ds} \tag{2·7}$$

and

$$\frac{d^2x^\nu}{ds^2}+\left\{ {\nu \atop \lambda\mu} \right\}\frac{dx^\lambda}{ds}\frac{dx^\mu}{ds}=0 \qquad (2\cdot 8)$$

respectively.

In particular, when a curve of zero-length satisfies the differential equations of the form (2·8), in which s is simply used as a parameter of the curve, we will call it a minimal geodesic.

From what has been said, it will be seen that the expressions of the principal normal of a curve and the differential equations of a geodesic in our geometry are of the same form as in Riemannian Geometry.

We are now going to construct the curvature tensor in our connection.

In order to do this, take a surface defined by

$$x^\nu=f^\nu(u, v)$$

where the functions f^ν and their derivatives up to the third order exist and are continuous at a point P, and consider a circuit passing through this point $P(u, v)$, $Q(u+\Delta u, v)$, $R(u+\Delta u, v+\Delta v)$, $S(u, v+\Delta v)$ and P. If a vector A^ν is transported parallel to itself around this circuit, it is well known that the difference $\Delta(A^\nu)_P$ of the final and initial values at P of the components of the vector A^ν is given by the equations

$$\Delta(A^\nu)_P=\left(A^\lambda R^\nu{}_{\lambda\mu\sigma}\frac{\partial x^\sigma}{\partial u}\frac{\partial x^\mu}{\partial v}\right)_P\Delta u\Delta v, \qquad (2\cdot 9)$$

where $R^\nu{}_{\lambda\mu\sigma}$ is called the curvature tensor of the continuum and is given by

$$R^\nu{}_{\lambda\mu\sigma}=\frac{\partial\Gamma^\nu_{\lambda\sigma}}{\partial x^\mu}-\frac{\partial\Gamma^\nu_{\lambda\mu}}{\partial x^\sigma}+\Gamma^\kappa_{\lambda\sigma}\Gamma^\nu_{\kappa\mu}-\Gamma^\kappa_{\lambda\mu}\Gamma^\nu_{\kappa\sigma}. \qquad (2\cdot 10)$$

Since our continuum is subjected to a semi-symmetric and metric connection, we must substitute (1·18) for the $\Gamma^\nu_{\lambda\mu}$ in the right hand side of (2·10). And, after a tedious calculation, we can show that

$$R^\nu{}_{\lambda\mu\sigma}=B^\nu{}_{\lambda\mu\sigma}+\delta^\nu_\sigma V_\mu\varphi_\lambda-\delta^\nu_\mu V_\sigma\varphi_\lambda-g_{\lambda\sigma}V_\mu\varphi^\nu$$
$$+g_{\lambda\mu}V_\sigma\varphi^\nu+(\delta^\nu_\mu g_{\lambda\sigma}-\delta^\nu_\sigma g_{\lambda\mu})\Phi, \qquad (2\cdot 11)$$

where $B^\nu{}_{\lambda\mu\sigma}$ is the Riemann-Christoffel tensor formed by $g_{\lambda\mu}$, i. e.

$$B^\nu{}_{\lambda\mu\sigma}=\frac{\partial}{\partial x^\mu}\left\{ {\nu \atop \lambda\sigma} \right\}-\frac{\partial}{\partial x^\sigma}\left\{ {\nu \atop \lambda\mu} \right\}+\left\{ {\kappa \atop \lambda\sigma} \right\}\left\{ {\nu \atop \kappa\mu} \right\}-\left\{ {\kappa \atop \lambda\mu} \right\}\left\{ {\nu \atop \kappa\sigma} \right\} \qquad (2\cdot 12)$$

and Φ is the squared length of φ_ν, i. e.

$$\Phi=\varphi_\nu\varphi^\nu. \tag{2·13}$$

Writing

$$R^\sigma{}_{\lambda\mu\sigma}=R_{\lambda\mu} \tag{2·14}$$

and

$$B^\sigma{}_{\lambda\mu\sigma}=G_{\lambda\mu}, \tag{2·15}$$

we get the contracted tensor

$$R_{\lambda\mu}=G_{\lambda\mu}+(n-2)\Gamma_\mu\varphi_\lambda+g_{\lambda\mu}\Gamma_\nu\varphi^\nu-(n-1)g_{\lambda\mu}\Phi, \tag{2·16}$$

whose symmetrical part $K_{\lambda\mu}$ and anti-symmetrical part $F_{\lambda\mu}$ are respectively given by

$$K_{\lambda\mu}=G_{\lambda\mu}+\frac{n-2}{2}(\Gamma_\mu\varphi_\lambda+\Gamma_\lambda\varphi_\mu)+g_{\lambda\mu}\Gamma_\nu\varphi^\nu-(n-1)g_{\lambda\mu}\Phi \tag{2·17}$$

and

$$F_{\lambda\mu}=\frac{n-2}{2}(\Gamma_\mu\varphi_\lambda-\Gamma_\lambda\varphi_\mu). \tag{2·18}$$

Multiplying (2·16) by $g^{\lambda\mu}$, we get

$$R=G+2(n-1)\Gamma_\nu\varphi^\nu-n(n-1)\Phi, \tag{2·19}$$

where we put

$$R=g^{\lambda\mu}R_{\lambda\mu}, \tag{2·20}$$

and

$$G=g^{\lambda\mu}G_{\lambda\mu}. \tag{2·21}$$

In particular, for $n=4$, we can reduce these formulae to a more simple form, viz.

$$R_{\lambda\mu}=G_{\lambda\mu}+2\Gamma_\mu\varphi_\lambda+g_{\lambda\mu}\Gamma_\nu\varphi^\nu-3g_{\lambda\mu}\Phi, \tag{2·22}$$

$$K_{\lambda\mu}=G_{\lambda\mu}+(\Gamma_\mu\varphi_\lambda+\Gamma_\lambda\varphi_\mu)+g_{\lambda\mu}\Gamma_\nu\varphi^\nu-3g_{\lambda\mu}\Phi, \tag{2·23}$$

$$F_{\lambda\mu}=\Gamma_\mu\varphi_\lambda-\Gamma_\lambda\varphi_\mu, \tag{2·24}$$

$$R=G+6\Gamma_\nu\varphi^\nu-12\Phi. \tag{2·25}$$

From (2·24) it will be seen that the anti-symmetrical part $F_{\lambda\mu}$ of the contracted curvature tensor $R_{\lambda\mu}$ is the curl of the fundamental vector φ_ν. Accordingly, in consequence of (1·12), we can write it

$$F_{\lambda\mu}=\frac{\partial\varphi_\lambda}{\partial x^\mu}-\frac{\partial\varphi_\mu}{\partial x^\lambda}. \tag{2·26}$$

§ 3. Field-equations.

The construction of our physical theory depends on the following assumptions.

(i) Our space-time world is a four-dimensional continuum with a semi-symmetric and metric connection.

(ii) We interpret the fundamental tensor $g_{\lambda\mu}$ and vector φ_ν as the gravitational potentials and the electromagnetic potential-vector respectively.

(iii) The field-laws in our space-time world are describable by two systems of such differential equations that the Lagrangian derivatives of a scalar-density $\mathfrak{H}$, called the Hamiltonian function, with respect to $g^{\lambda\mu}$ and φ_ν are proportional to the energy-tensor-density $\mathfrak{T}_{\lambda\mu}$ and charge-current vector-density $\mathfrak{S}^\nu$ respectively.

First let us adopt as the Hamiltonian function $\mathfrak{H}$

$$\mathfrak{H} \equiv H\sqrt{-g} = (R-L)\sqrt{-g}, \tag{3·1}$$

where R is the scalar curvature of our space-time world given by (2·25) and

$$L = \frac{\varkappa}{2} F_{\alpha\beta}F^{\alpha\beta} + 6\Phi, \tag{3·2}$$

$F_{\alpha\beta}$ and Φ being given by (2·24) and (2·13) respectively and $\varkappa$ a universal constant.

Then let us consider the variation of the Hamiltonian integral

$$I = \int \mathfrak{H}\, dx, \tag{3·3}$$

where $dx \equiv dx^1 dx^2 dx^3 dx^4$, in a region for arbitrary small variations $\delta g^{\lambda\mu}$ and $\delta\varphi_\nu$ under the conditions that they will vanish with their first derivatives at the boundary of the region.

In consequence of (2·25), (3·1) and (3·3), the variation of I is given by

$$\begin{aligned} \delta I &= \delta \int (R-L)\sqrt{-g}\, dx \\ &= \delta \int \left(G + 6\Gamma_\nu \varphi^\nu - 18\Phi - \frac{\varkappa}{2} F_{\alpha\beta}F^{\alpha\beta} \right) \sqrt{-g}\, dx. \end{aligned} \tag{3·4}$$

Applying (1·23) to the fundamental vector φ^ν, we have

$$\Gamma_\nu \varphi^\nu = \mathring{\Gamma}_\nu \varphi^\nu + 3\Phi.$$

Hence we get

$$(\Gamma_\nu\varphi^\nu)\sqrt{-g}=\frac{\partial}{\partial x^\nu}(\sqrt{-g}\varphi^\nu)+3\Phi\sqrt{-g}.$$

Substituting this in (3·4), we obtain

$$\delta I=\delta\int\left(G-\frac{\varkappa}{2}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,dx+6\delta\int\frac{\partial}{\partial x^\nu}(\sqrt{-g}\varphi^\nu)dx.$$

But by the boundary conditions, the second term in the right hand side of the above equation vanishes.

Hence we get

$$\delta I=\delta\int\left(G-\frac{\varkappa}{2}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,dx. \qquad (3\cdot 5)$$

If we denote the Lagrangian derivatives of $\mathfrak{H}$ with respect to $g^{\lambda\mu}$ and φ_ν by the symbols $[\mathfrak{H}]_{\lambda\mu}$ and $[\mathfrak{H}]^\nu$ respectively, δI can be written

$$\delta I=\int\{[\mathfrak{H}]_{\lambda\mu}\delta g^{\lambda\mu}+[\mathfrak{H}]^\nu\delta\varphi_\nu\}\,dx.$$

From this it follows that the Lagrangian derivatives of $\mathfrak{H}$ with respect to $g^{\lambda\mu}$ and φ_ν are equal to those of $\left(G-\frac{\varkappa}{2}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}$ respectively.

Since

$$\frac{1}{\sqrt{-g}}\left[\left(G-\frac{\varkappa}{2}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\right]_{\lambda\mu}=G_{\lambda\mu}-\frac{1}{2}g_{\lambda\mu}G-\varkappa\left(g^{\alpha\beta}F_{\lambda\alpha}F_{\mu\beta}-\frac{1}{4}g_{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

and

$$\frac{1}{\sqrt{-g}}\left[\left(G-\frac{\varkappa}{2}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\right]^\nu=2\varkappa\mathring{\nabla}_\sigma F^{\nu\sigma}$$

are well known formulae[1] in the theory of general relativity, we have

$$\frac{1}{\sqrt{-g}}[\mathfrak{H}]_{\lambda\mu}=G_{\lambda\mu}-\frac{1}{2}g_{\lambda\mu}G-\varkappa\left(g^{\alpha\beta}F_{\lambda\alpha}F_{\mu\beta}-\frac{1}{4}g_{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}\right) \qquad (3\cdot 6)$$

and

$$\frac{1}{\sqrt{-g}}[\mathfrak{H}]^\nu=2\varkappa\mathring{\nabla}_\sigma F^{\nu\sigma}, \qquad (3\cdot 7)$$

where $\mathfrak{H}\equiv\left(R-\frac{\varkappa}{2}F_{\alpha\beta}F^{\alpha\beta}-6\Phi\right)\sqrt{-g}.$

1. A. S. Eddington: The mathematical theory of Relativity, 2nd. ed., §60, §79.

Now, according to our assumption (iii), we propose

$$\left.\begin{aligned} \frac{1}{\sqrt{-g}}[\mathfrak{H}]_{\lambda\mu} &= -\varkappa T_{\lambda\mu} \\ \frac{1}{\sqrt{-g}}[\mathfrak{H}]^{\nu} &= 2\varkappa S^{\nu} \end{aligned}\right\} \tag{3·8}$$

as the field-equations.

By (3·6) and (3·7), the above equations (3·8) can be put in the following form

$$G_{\lambda\mu} - \frac{1}{2} g_{\lambda\mu} G = -\varkappa\left(T_{\lambda\mu} - g^{\alpha\beta} F_{\lambda\alpha} F_{\mu\beta} + \frac{1}{4} g_{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta}\right), \tag{3·9}$$

$$\mathring{\nabla}_{\sigma} F^{\nu\sigma} = S^{\nu}, \tag{3·10}$$

which are identical with those given by the general theory of relativity. In order to express (3·10) with our covariant derivatives instead of those of Riemannian Geometry, we use the identities

$$\nabla_{\sigma} F^{\nu\sigma} = \mathring{\nabla}_{\sigma} F^{\nu\sigma} + 2\varphi_{\sigma} F^{\nu\sigma},$$

which can easily be shown by (1·6) and (1·18).

Hence the Maxwell-Lorentz equation (3·10) can also be written

$$\nabla_{\sigma} F^{\nu\sigma} - 2\varphi_{\sigma} F^{\nu\sigma} = S^{\nu}. \tag{3·11}$$

§ 4. Laws of conservation.

By the so-called "Kastenexperiment" Einstein proposed the equivalence hypothesis, which says that with an appropriately chosen local reference frame special relativity holds for every infinitesimal four-dimensional domain of the world. Evidently this assumption of Einstein's corresponds to the mathematical theorem: In a Riemannian continuum, a system of coordinates can be chosen for which the Christoffel 3-index-symbols i. e. the coefficients of the Riemannian connection, vanish at any given point.

Taking a step further, it is an interesting problem for us to ask whether a reference frame on which the gravitational and electromagnetic phenomena have no influence can exist or not. This is the same as asking whether a system of coordinates for which the coefficients of our semi-symmetric and metric connection vanish at any given point can exist or not.

If such a system of coordinates could be chosen in our connection, the transformed $\overline{\Gamma}^{\nu}_{\lambda\mu}$ would be symmetric with respect to λ and

μ, as can easily be seen by (1·8). Hence at the point considered, $\bar{P}_{\lambda\mu}^{\nu}=0$. From the tensorial property of $P_{\lambda\mu}^{\nu}$, it follows that $P_{\lambda\mu}^{\nu}$ would be zero for any coordinates i. e. our connection would be a symmetric one. This contradicts our assumption. Hence it is not possible to choose a system of coordinates for which the coefficients of our connection vanish at any given point. Of course, the non-existence of such a coordinate system will be also assured by the aid of a general theorem[1]: When, and only when, at a point the coefficients of a connection are symmetric in the subscripts, coordinate systems can be chosen, with the point as origin, such that the coefficients are zero at the point.

From this very important fact we infer that, when the law of conservation is satisfied by a tensor $M_{\lambda\mu}$, its mathematical form is not

$$\nabla_{\nu} M_{\lambda}^{\nu}=0,$$

but

$$\mathring{\nabla}_{\nu} M_{\lambda}^{\nu}=0.$$

For, if we select a sufficiently small region of our space-time world, the law of conservation must be written at least in the form of vanishing of elementary divergence

$$\frac{\partial M_{\lambda}^{1}}{\partial x^{1}}+\frac{\partial M_{\lambda}^{2}}{\partial x^{2}}+\frac{\partial M_{\lambda}^{3}}{\partial x^{3}}+\frac{\partial M_{\lambda}^{4}}{\partial x^{4}}=0.$$

If we interpret

$$M_{\lambda\mu}=T_{\lambda\mu}-g^{\alpha\beta}F_{\lambda\alpha}F_{\mu\beta}+\frac{1}{4}g_{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta} \qquad (4\cdot 1)$$

as the whole energy-momentum-tensor, it will consist of two parts $T_{\lambda\mu}$ due to matter and $E_{\lambda\mu}$ due to electromagnetic field, where

$$E_{\lambda\mu}=-g^{\alpha\beta}F_{\lambda\alpha}F_{\nu\beta}+\frac{1}{4}g_{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}. \qquad (4\cdot 2)$$

Since, by means of (4·1), (3·9) can be written in the form

$$G_{\lambda\mu}-\frac{1}{2}g_{\lambda\mu}G=-\varkappa M_{\lambda\mu},$$

and the Riemannian divergence of $\left(G_{\lambda\mu}-\frac{1}{2}g_{\lambda\mu}G\right)$ is identically zero, we obtain

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1. L. P. Eisenhart: Non-Riemannian geometry (1927), §20.

$$\left.\begin{aligned} &\hat{\nabla}_\nu M_\lambda{}^\nu = 0, \\ \text{or} \qquad &\frac{\partial \mathfrak{M}_\lambda{}^\nu}{\partial x^\nu} + \frac{1}{2}\mathfrak{M}_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^\lambda} = 0, \end{aligned}\right\} \tag{4·3}$$

as representing the law of conservation of the whole energy-momentum-tensor.

Similarly from the identity

$$\hat{\nabla}_\nu \hat{\nabla}_\sigma F^{\nu\sigma} = 0,$$

it follows that the charge-current-vector S^ν satisfies the law of conservation

$$\left.\begin{aligned} &\hat{\nabla}_\nu S^\nu = 0, \\ \text{or} \qquad &\frac{\partial \mathfrak{S}^\nu}{\partial x^\nu} = 0. \end{aligned}\right\} \tag{4·4}$$

According to the theory of integral invariants, if I is an (absolute) integral invariant such that

$$I = \int W\sqrt{-g}\, dx,$$

where W is an invariant function involving only $g^{\lambda\mu}$ and φ_ν and their derivatives as its arguments, its Lagrangian derivatives with respect to $g^{\lambda\mu}$ and φ_ν satisfy identically the Lie differential equations[1]

$$[\mathfrak{W}]_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^\lambda} + [\mathfrak{W}]^\alpha \frac{\partial \varphi_\alpha}{\partial x^\lambda} + \frac{\partial}{\partial x^\alpha}\left\{2[\mathfrak{W}]_{\lambda\beta}g^{\alpha\beta} - [\mathfrak{W}]^\alpha \varphi_\lambda\right\} = 0. \tag{4·5}$$

If we adopt H given by (3·1) as W in (4·5), we get, by (3·8),

$$\frac{\partial \mathfrak{T}_\lambda{}^\alpha}{\partial x^\alpha} + \frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\lambda}\mathfrak{T}_{\alpha\beta} + \mathfrak{S}^\alpha\left(\frac{\partial \varphi_\lambda}{\partial x^\alpha} - \frac{\partial \varphi_\alpha}{\partial x^\lambda}\right) + \varphi_\lambda \frac{\partial \mathfrak{S}^\alpha}{\partial x^\alpha} = 0.$$

Applying (2·26) and (4·4) to this, we get

$$\frac{\partial \mathfrak{T}_\lambda{}^\alpha}{\partial x^\alpha} + \frac{1}{2}\mathfrak{T}_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^\lambda} + F_{\lambda\alpha}\mathfrak{S}^\alpha = 0, \tag{4·6}$$

and, subtracting this from (4·3), we obtain

$$\frac{\partial \mathfrak{E}_\lambda{}^\alpha}{\partial x^\alpha} + \frac{1}{2}\mathfrak{E}_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^\lambda} - F_{\lambda\alpha}\mathfrak{S}^\alpha = 0. \tag{4·7}$$

These are exactly the same equations as we have already obtained and are known as representing the fact that the rate of change of

1. R. Weitzenböck: Invariantentheorie (1923), Chap. XIV, § 5.

momentum of matter per unit volume is equal to the total ponderable force exerted upon it by the field (for $\mu=1, 2, 3$) and the rate of change of energy of matter per unit volume is equal to the rate at which work is done by this force. (for $\mu=4$)

§ 5. Equations of motion.

In order to find the equations of the path traced by a particle carrying an electric charge, it is enough to pursue the same course as that of general relativity.

Let us consider the case of incoherent matter

$$\mathfrak{T}_\lambda^\alpha=\mu_0 u_\lambda u^\alpha \sqrt{-g}, \tag{5·1}$$

where μ_0 is the proper density of mass and u^ν is the tangent-vector of the track

$$u^\nu=\frac{dx^\nu}{ds}, \tag{5·2}$$

and assume

$$\mathfrak{S}^\nu=\rho_0 u^\nu \sqrt{-g} \tag{5·3}$$

for the vector-density of charge and current, where ρ_0 is the proper-density of electric charge.

As is well known, if we substitute (5·1), (5·2) and (5·3) in (4·6), we get

$$\mu_0 \sqrt{-g}\frac{du_\lambda}{ds}+u_\lambda\frac{\partial}{\partial x^\nu}(\mu_0 \sqrt{-g}u^\nu)$$
$$=\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\mu_0 \sqrt{-g}\, u^\alpha u^\beta-F_{\lambda\alpha}\rho_0 \sqrt{-g}\, u^\alpha, \tag{5·4}$$

and, multiplying u^λ to (5·4), the law of conservation of proper-mass

$$\frac{\partial}{\partial x^\nu}(\mu_0 \sqrt{-g}\, u^\nu)=0 \tag{5·5}$$

can be obtained.

From (5·4) and (5·5), we finally get

$$\frac{d^2x^\nu}{ds^2}+\begin{Bmatrix}\nu\\ \alpha\beta\end{Bmatrix}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=-\frac{1}{\mu_0}F^\nu{}_\alpha S^\alpha, \tag{5·6}$$

as the differential equations of a world-line taken by a particle carrying a charge. These equations (5·6) are formally identical with those of general relativity.

By (2·5) and (2·7), (5·6) can be written in the form

$$N^{\nu}=-\frac{1}{\mu_0}F^{\nu}{}_{\alpha}S^{\alpha}, \tag{5·7}$$

which shows that the principal normal of a world-line taken by a particle carrying an electric charge is proportional to the Lorentz force exerted upon it.

From the physical and geometrical point of view, it is an interesting fact that this interpretation of the equations of path is identical with that of general relativity.

Now we will proceed further to consider the track of an electromagnetic wave.

Since the field-equations (3·9) and (3·10) involve only the gravitational potentials $g_{\lambda\mu}$ and the electromagnetic field-tensor $F_{\lambda\mu}$, with their derivatives, but not the electromagnetic potential-vector φ_{ν} explicitly, there exists an arbitrariness of φ_{ν}.

It is usual to avoid this arbitrariness of φ_{ν} by imposing the condition

$$\overset{*}{\nabla}_{\nu}\varphi^{\nu}=0. \tag{5·8}$$

By means of equation (5·8), (3·10) is reducible to the form

$$g_{\alpha\beta}\frac{\partial^2\varphi_{\nu}}{\partial x^{\alpha}\partial x^{\beta}}+(\text{terms not involving second derivatives of the } \varphi\text{'s})=0. \tag{5·9}$$

On the other hand, following Hadamard[1], Vessiot[2], de Donder[3] and Whittaker[4], we know that a ray for a wave is identical with the bicharacteristic of the partial differential equations of the second order which represent the wave. Hence, from (5·9), we have

$$\left.\begin{aligned}&\frac{d^2x^{\nu}}{dt^2}+\left\{{\nu \atop \alpha\beta}\right\}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}=0,\\ &ds=0,\end{aligned}\right\} \tag{5·10}$$

for the differential equations of the bicharacteristics of (5·9). Consequently we see, from the description in § 2, that this track of a light ray is nothing less than a minimal geodesic in our connection.

Thus we have seen that all the results obtained by the theory of general relativity (with the Maxwell-Lorentz theory) are unified

1. J. Hadamard: Leçon sur la propagation des Ondes (1903), Chap. VII.
2. E. Vessiot: Com. Rend., **166**, 349 (1918)
3. de Donder: La gravifique einsteinnienne (1921), § 29, § 45
4. E. T. Whittaker: Proc. Camb. Phil. Soc., **24**, 32 (1928)

geometrically and physically in a simple form by using a semi-symmetric and metric connection.

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