On the Definition of Functions by the Recurrence Formula so as to satisfy the Linear Homogeneous Differential Equations of the Second Order

By

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1. Legendre polynomials, Bessel functions and many other polynomials commonly known satisfy certain recurrence formulas. These functions are defined as the integrals of certain linear homogeneous differential equations of the second order. Sometimes they are defined as the coefficients of the expansions of certain functions. In general as a consequence of these definitions, those functions satisfy certain recurrence formulas. Now conversely let us define a sequence of functions $\{F_n(x)\}$ by the recurrence formula $F_{n+1}(x) = A_n(x)F_n(x) + B_{n-1}(x)F_{n-1}(x)$ where the sequences of functions $A_1(x)$, $A_2(x), \ldots\ldots$, $A_n(x), \ldots\ldots$ and $B_0(x)$, $B_1(x), \ldots\ldots$, $B_{n-1}(x), \ldots\ldots$ are given, and find the differential equations

$$\frac{d^2F}{dx^2} + \alpha_n(x)\frac{dF}{dx} + \beta_n(x)F = 0$$

satisfied by these functions $F_n(x)$.

2. If we give at random two sequences of functions $A_1(x)$, $A_2(x), \ldots\ldots$, $A_n(x), \ldots\ldots$ and $B_0(x)$, $B_1(x), \ldots\ldots$, $B_{n-1}(x), \ldots\ldots$ and put

$$F_{n+1}(x) = A_n(x)F_n(x) + B_{n-1}(x)F_{n-1}(x), \quad n = 1, 2, \ldots\ldots,$$

then a sequence of functions $\{F_n(x)\}$ is surely defined, provided we give two initial functions $F_0(x)$, $F_1(x)$. This equality is quite a general recurrence formula. But since we give at random the sequence $\{A_n(x)\}$, when only the n first functions $A_1(x)$, $A_2(x), \ldots\ldots$, $A_n(x)$ are given, we can not know the next function $A_{n+1}(x)$. That is $A_n(x)$ is not the analytical expression of n. It is the same with $\{B_{n-1}(x)\}$. The recurrence formulas commonly used are different at this point. For example Legendre polynomials $P_n(x)$ satisfy the recurrence formula

$$P_{n+1}(x)=\frac{2n+1}{n+1}xP_n(x)-\frac{n}{n+1}P_{n-1}(x),\ n=1,2,\ldots\ldots,$$

so that $$A_n(x)=\frac{2n+1}{n+1}x,\quad B_{n-1}(x)=-\frac{n}{n+1},$$

and they are analytical expressions of n. Therefore in the following we suppose that $A_n(x)$, $B_n(x)$ are given by some analytical expressions of x and n, say

$$A_n(x)\equiv A(x, n),$$
$$B_n(x)\equiv B(x, n).$$

Then $F_n(x)$ are the solutions of the difference equation

$$F_{\nu+2}=A_{\nu+1}F_{\nu+1}+B_\nu F_\nu.$$

containing the original variable x as a parameter. If $A(x, n)$, $B(x,n)$ and the initial functions $F_0(x)$, $F_1(x)$ are polynomials of x, then $F_n(x)$ are also polynomials of x.

3. To construct the linear homogeneous differential equations of the second order satisfied by the functions $F_n(x)$, $F_{n+1}(x),\ldots\ldots$, we put first

$$\left.\begin{aligned}\frac{dF_n}{dx}&=H_nF_n+K_{n-1}F_{n-1}\\ \frac{dF_{n-1}}{dx}&=L_nF_n+M_{n-1}F_{n-1}\end{aligned}\right\}\qquad(1)$$

where H_n, K_{n-1}, L_n, M_{n-1} are certain functions of x. Since the sequence $\{F_n(x)\}$ is already defined, only one of the functions $H_n(x)$ and $K_{n-1}(x)$ is arbitrary. It is the same with the functions $L_n(x)$, $M_{n-1}(x)$. By these simultaneous relations we easily get the differential equation satisfied by $F_n(x)$, namely

$$\left.\begin{aligned}&\frac{d^2F}{dx^2}+\alpha_n\frac{dF}{dx}+\beta_nF=0,\\ \text{where}\quad &\alpha_n\equiv-\left(H_n+M_{n-1}+\frac{K'_{n-1}}{K_{n-1}}\right),\\ &\beta_n\equiv-(H'_n+K_{n-1}L_n)+H_n\left(M_{n-1}+\frac{K'_{n-1}}{K_{n-1}}\right),\end{aligned}\right\}\qquad(2)$$

K'_{n-1}, H'_n being the differential quotients with respect to x of K_{n-1} and H_n respectively.

Next to find the differential equation satisfied by F_{n+1}, first from the equalities

$$\frac{dF_n}{dx}=H_nF_n+K_{n-1}F_{n-1},$$

$$\frac{dF_{n-1}}{dx}=L_nF_n+M_{n-1}F_{n-1},$$

$$F_{n+1}=A_nF_n+B_{n-1}F_{n-1},$$

we have the relations

$$\frac{dF_{n+1}}{dx}=H_{n+1}F_{n+1}+K_nF_n,$$

$$\frac{dF_n}{dx}=L_{n+1}F_{n+1}+M_nF_n,$$

where

$$\left.\begin{aligned} &H_{n+1}\equiv A_nL_{n+1}+M_{n-1}+\frac{B'_{n-1}}{B_{n-1}}\\ &K_n\equiv A_nH_n+B_{n-1}L_n+A'_n-A_nH_{n+1}\\ &L_{n+1}\equiv\frac{K_{n-1}}{B_{n-1}}\\ &M_n\equiv H_n-A_nL_{n+1}. \end{aligned}\right\} \qquad (3)$$

Therefore the required differential equation is

$$\left.\begin{aligned} &\frac{d^2F_{n+1}}{dx^2}+\alpha_{n+1}\frac{dF_{n+1}}{dx}+\beta_{n+1}F_{n+1}=0,\\ \text{where}\quad &\alpha_{n+1}\equiv-\left(H_{n+1}+M_n+\frac{K'_n}{K_n}\right),\\ &\beta_{n+1}\equiv-(H'_{n+1}+K_nL_{n+1})+H_{n+1}\left(M_n+\frac{K'_n}{K_n}\right). \end{aligned}\right\} \qquad (4)$$

4. If we give e. g., K_{n-1}, M_{n-1} as analytical expressions of x and n, then by (1) and (2), α_n is an analytical expression of x and n, say $\alpha_n\equiv\alpha(x, n)$. Yet it is not certain that $\alpha_{n+1}=\alpha(x, n+1)$. The same with β_n. Instead of following up that line of study we shall in the following find the conditions that $\alpha_n=\alpha_{n+1}$.

For $\alpha_n=\alpha_{n+1}$, we have by (2) and (4),

$$H_{n+1}+M_n+\frac{K'_n}{K_n}=H_n+M_{n-1}+\frac{K'_{n-1}}{K_{n-1}}.$$

Putting the values of H_{n+1}, M_n in the left of the equality, by formula (3), we easily get the condition

$$\frac{B'_{n-1}}{B_{n-1}}+\frac{K'_n}{K_n}=\frac{K'_{n-1}}{K_{n-1}},$$

or, integrating, we have

$$B_{n-1}K_n=c_nK_{n-1}, \tag{5}$$

where c_n is constant with respect to x. If the condition be satisfied, we have by the third formula of (3),

$$\frac{K'_n}{K_n}=\frac{L'_{n+1}}{L_{n+1}},$$

or, integrating, we have

$$L_{n+1}=C_nK_n, \tag{6}$$

where C_n is constant with respect to x. c_n and C_n are not independent. For since by (3),

$$L_{n+1}=\frac{K_{n-1}}{B_{n-1}},$$

we have by (6) and (5),

$$C_nK_n=\frac{K_n}{c_n}.$$

Therefore $c_nC_n=1$.

In the same way the condition for $a_{n+1}=a_{n+2}$ may be given by

$$L_{n+2}=C_{n+1}K_{n+1},$$

where C_{n+1} is constant with respect to x. Therefore if the condition (6) be satisfied for n, $n+1$, $n+2$, ..., then $a_n=a_{n+1}=a_{n+2}=\ldots\ldots$, i. e., they are independent of the suffixes.

5. For the functions commonly known, the above conditions are always satisfied as is shown in the following list.

Legendre polynomial $P_n(x)$.

$$P_{n+1}=\frac{2n+1}{n+1}xP_n-\frac{n}{n+1}P_{n-1},$$

$$P'_{n+1}=\frac{(n+1)x}{x^2-1}P_{n+1}-\frac{n+1}{x^2-1}P_n,$$

$$P'_n=\frac{n+1}{x^2-1}P_{n+1}-\frac{(n+1)x}{x^2-1}P_n,$$

$$\frac{d^2P_{n+1}}{dx^2}+\frac{2x}{x^2-1}\frac{dP_{n+1}}{dx}-\frac{(n+1)(n+2)}{x^2-1}P_{n+1}=0.$$

The middle two equalities are obtained from the definition of the polynomials,

$$\frac{1}{\sqrt{1-2xz+z^2}}=P_0(x)+P_1(x)z+\ldots\ldots+P_n(x)z^n+\ldots\ldots,$$
$$(|z|<1)$$

by aid of the recurrence formula.

Bessel functions $J_n(x)$.

$$J_{n+1}=\frac{2n}{x}J_n-J_{n-1},$$

$$J'_{n+1}=-\frac{n+1}{x}J_{n+1}+J_n,$$

$$J'_n=-J_{n+1}+\frac{n}{x}J_n,$$

$$\frac{d^2J_{n+1}}{dx^2}+\frac{1}{x}\frac{dJ_{n+1}}{dx}+\left(1-\frac{(n+1)^2}{x^2}\right)J_{n+1}=0.$$

The middle two equalities for integral values of n may also be obtained from the definition of the functions

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}=\sum_{n=-\infty}^{\infty}J_n(x)z^n$$

by aid of the recurrence formula.

This is quite different from *Tchebyscheff's polynomials* $T_n(x)$. They may be defined as the coefficients of the expansion

$$\frac{1-z^2}{1-2xz+z^2}=\sum_{n=0}^{\infty}T_n(x)(2z)^n,$$

and they satisfy the recurrence formula

$$T_{n+1}=xT_n-\frac{1}{4}T_{n-1},\quad n\geqq 2,$$

and the differential equation

$$\frac{d^2T_{n+1}}{dx^2}-\frac{x}{1-x^2}\frac{dT_{n+1}}{dx}+\frac{(n+1)^2}{1-x^2}T_{n+1}=0.$$

To determine four functions such as

$$\left.\begin{aligned}\frac{dT_n}{dx}&=H_nT_n+K_{n-1}T_{n-1}\\ \frac{dT_{n-1}}{dx}&=L_nT_n+M_{n-1}T_{n-1}\end{aligned}\right\},$$

if we treat as in the cases of $P_n(x)$ and $J_n(x)$, we differentiate the definitional equation with respect to x. Then

$$2z\sum_{n=0}^{\infty}T_n\cdot(2z)^n=(1-2xz+z^2)\sum_{n=0}^{\infty}T_n'\cdot(2z)^n.$$

Equating the coefficients of n^{th} degree in $2z$, we have

$$T_{n-1}=T_n'-xT_{n-1}'+\frac{1}{4}T_{n-2}',\quad n\geqq 2.$$

By the recurrence formula we have

$$T_n'=T_{n-1}+xT_{n-1}'-\frac{1}{4}T_{n-2}',$$

which is identical with the preceding equality. Hence we can not determine the functions H_n, K_{n-1}, L_n, M_{n-1}. In this case, since

$$T_n(x)=\left(\frac{x+\sqrt{x^2-1}}{2}\right)^n+\left(\frac{x-\sqrt{x^2-1}}{2}\right)^n,\quad n\geqq 1,$$

we have

$$\begin{aligned}\frac{dT_n}{dx}&=n\frac{1+\dfrac{x}{\sqrt{x^2-1}}}{2}\left(\frac{x+\sqrt{x^2-1}}{2}\right)^{n-1}\\ &\quad+n\frac{1-\dfrac{x}{\sqrt{x^2-1}}}{2}\left(\frac{x-\sqrt{x^2-1}}{2}\right)^{n-1}\\ &=\frac{n}{\sqrt{x^2-1}}\left\{(x+\sqrt{x^2-1})^n-(x-\sqrt{x^2-1})^n\right\},\end{aligned}$$

putting

$$\equiv H_nT_n+K_{n-1}T_{n-1},$$

therefore

$$\begin{aligned}&=\left(H_n+\frac{2K_{n-1}}{x+\sqrt{x^2-1}}\right)\left(\frac{x+\sqrt{x^2-1}}{2}\right)^n\\ &\quad+\left(H_n+\frac{2K_{n-1}}{x-\sqrt{x^2-1}}\right)\left(\frac{x-\sqrt{x^2-1}}{2}\right)^n.\end{aligned}$$

To satisfy this equality, we may choose H_n and K_{n-1} in several ways. But assume for example

$$\left.\begin{aligned} H_n+\frac{2K_{n-1}}{x+\sqrt{x^2-1}}&=\frac{n}{\sqrt{x^2-1}} \\ H_n+\frac{2K_{n-1}}{x-\sqrt{x^2-1}}&=-\frac{n}{\sqrt{x^2-1}} \end{aligned}\right\},$$

then we have

$$H_n=\frac{nx}{x^2-1}, \quad K_{n-1}=-\frac{n}{2(x^2-1)},$$

so that we have

$$\left.\begin{aligned} T'_{n+1}&=\frac{(n+1)x}{x^2-1}T_{n+1}-\frac{n+1}{2(x^2-1)}T_n, \quad (n\geqslant 1) \\ \text{and hence}\quad T'_n&=\frac{2n}{x^2-1}T_{n+1}-\frac{nx}{x^2-1}T_n, \quad (n\geqslant 0) \end{aligned}\right\}. \tag{7}$$

Here we have

$$L_{n+1}=\frac{2n}{x^2-1}, \quad K_n=-\frac{n+1}{2(x^2-1)},$$

such that $L_{n+1} : K_n$ is constant with respect to x. Therefore, as we have proved, $\alpha_n=\alpha_{n+1}$. In our present case, the legitimacy of the equalities (7) can easily be verified, since the differential equation is previously given. But if as we have proposed, the sequence of functions $\{F_n(x)\}$ be only defined and the differential equations to be satisfied by them are not given, it seems to be useful to test the ratio $L_{n+1} : K_n$ for the verification of $\alpha_n=\alpha_{n+1}$.

6. Hitherto we have considered the coefficient α_n of the differential equation to be independent of n. The functions which satisfy the differential equations with such a condition seem to play a special rôle just as in the previous examples the functions commonly known satisfy the differential equations with the same condition.

If α_n is independent of n, several special properties would perhaps be displayed. In the following I shall consider the zeros of the functions $F_n(x)$ supposed to satisfy the differential equations

$$\frac{d^2F_n}{dx^2}+\alpha\frac{dF_n}{dx}+\beta_nF_n=0 \tag{8}$$

respectively where $\alpha(x)$ is independent of n and β_n is an analytical expression of x, n real. We shall conclude that often the zeros of $F_n(x)$ change monotonely with n. This seems to be peculiar to the class of these sequences of functions.

Putting $u(x)\equiv e^{\frac{1}{2}\int a\,dx}$,

$$V_n\equiv uF_n, \tag{9}$$

V_n satisfy the following differential equation transformed of (8)

$$\left.\begin{aligned}&\frac{d^2V_n}{dx^2}+\varphi_n(x)V_n=0,\\ \text{where}\quad &\varphi_n(x)\equiv\beta_n-\frac{a'}{2}-\frac{a^2}{4}.\end{aligned}\right\} \tag{10}$$

Differentiating (10) with respect to n,

$$\frac{d^2}{dx^2}\left(\frac{\partial V_n}{\partial n}\right)+\varphi_n\frac{\partial V_n}{\partial n}+\frac{\partial\varphi_n}{\partial n}V_n=0. \tag{11}$$

Multiplying (10) by $\frac{\partial V_n}{\partial n}$, (11) by V_n and substructing side by side, we have

$$\frac{\partial V_n}{\partial n}\frac{d^2V_n}{dx^2}-V_n\frac{d^2}{dx^2}\left(\frac{\partial V_n}{\partial n}\right)=\frac{\partial\varphi_n}{\partial n}V_n^2,$$

or

$$\frac{d}{dx}\left\{V_n\frac{d}{dx}\left(\frac{\partial V_n}{\partial n}\right)-\frac{\partial V_n}{\partial n}\frac{dV_n}{dx}\right\}=-\frac{\partial\varphi_n}{\partial n}V_n^2.$$

If we put $f_n(x)\equiv\frac{\partial F_n(x)}{\partial n}$, $f_n'(x)\equiv\frac{\partial^2 F_n(x)}{\partial n\partial x}$,

and remember (9) after integration, the above equality becomes

$$u^2(F_nf_n'-f_nF_n')\Big|_a^b=-\int_a^b\frac{\partial\varphi_n}{\partial n}u^2F_n^2dx, \tag{12}$$

with a proper choice of the limits of the integration, in which the integral of the right-hand side is supposed to exist. Assume $u(0)=0$ or $F_n(0)=F_n'(0)=0$; let $x_0>0$ be a simple root of $F_n(x)=0$, where $u(x_0)\neq0$, then we have by (12)

$$-u^2(x_0)f_n(x_0)F_n'(x_0)=-\int_0^{x_0}\frac{\partial\varphi_n}{\partial n}u^2F_n^2dx.$$

Since by (10)

$$\frac{\partial\varphi_n}{\partial n}=\frac{\partial\beta_n}{\partial n},$$

we have $$f_n(x_0)=\frac{1}{u^2(x_0)F_n'(x_0)}\int_0^{x_0}\frac{\partial\beta_n}{\partial n}u^2F_n^2dx.$$

From $F_n(x_0)=0$, we have

$$\frac{dx_0}{dn}=-\frac{f_n(x_0)}{F'_n(x_0)}.$$

Hence we have

$$\frac{dx_0}{dn}=-\frac{1}{u^2(x_0)F'^2_n(x_0)}\int_0^{x_0}\frac{\partial\beta_n}{\partial n}u^2F_n^2dx.$$

When $\frac{\partial\beta_n}{\partial n}<0$, we may conclude that x_0 increases with n.

The above reasoning is of course inadequate, yet we are convinced that a special rôle is played by the property that $u(x)$ and hence $a(x)$ is independent of n.