On the Solutions of the Cosmological Field Equations and Some Models of the Universe with Annihilation of Matter

By

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Abstract

The aim of this paper is to classify all the possible solutions for relativistic cosmology and to construct a reasonable model of our actual universe. Using the line element

$$ds^2 = -R(t)^2\{d\vartheta^2 + \sin^2\vartheta d\varphi^2 + \sin^2\vartheta \sin^2\varphi d\psi^2\} + F(\vartheta, t)^2 dt^2,$$

the writer shows that the solutions of the amplified field-equations may be separated into two groups i.e. $\dot{R}=0$ and $\frac{\partial F}{\partial \vartheta}=0$. The former gives Einstein's and de Sitter's world, while the latter gives a unique solution when the pressure p vanishes, and many different solutions when p does not vanish. Important properties of a non-statical world, such as motion of a particle, Doppler shift and the form of the universe at eternity are described in § 4 and § 5. In the last section the writer introduces a new model of the universe and discusses its properties and then estimates the initial and present radius of that world.

One of the most interesting attempts in the general theory of relativity is to consider the shape and size of our universe as a whole. The inadequacy of interpreting our actual universe by Einstein's cylindrical or de Sitter's spherical world has come to be generally recognised. Many relativitists are inclined to consider our universe as a non-statical rather than a statical world on account of valuable contributions due to American astrophysicists. Adopting

$$ds^2 = -\frac{e^{g(t)}}{\left(1+\frac{r^2}{4R^2}\right)^2}(dx^2+dy^2+dz^2)+dt^2$$

as the line-element of our non-statical world, they[1] attempted to in-

1. R. C. Tolman: Proc. Nat. Acad. Sci., **16**, 320, 409, 511 (1930).
W. de Sitter: Proc. Nat. Acad. Sci., **16**, 474 (1930).
T. Takéuchi: Proc. Phys. Math. Soc. Jap., **13**, 166 (1931).

terpret many important observational facts, for example the correlation between distances and radial velocities of extragalactic nebulae, as the necessary consequences of our space-time world.

In the present paper, starting from a more general line-element for the universe, the writer tries first to show systematically how many solutions are possible for the cosmological problem, next to study what properties they have and finally to give a new model having reasonable properties in many respects.

§ 1. Field-equations with λ-term.

We assume that in our world at large there exist coordinate systems for which the forward velocity of light along any track is equal to the backward velocity along the same track and the 3-spaces are homogeneous and isotropic with respect to space-like coordinates. If $(\vartheta, \varphi, \psi, t)$ be a coordinate system having these properties, the line-element of our world can be written in the form

$$ds^2 = -R(t)^2\{d\vartheta^2 + \sin^2\vartheta d\varphi^2 + \sin^2\vartheta\sin^2\varphi d\psi^2\} + F(\vartheta, t)^2\, dt^2. \quad (1\cdot1)$$

From (1·1) it is obvious that our coordinate hypersurfaces form a 4-tuply orthogonal system. Accordingly we can find the Riemann-tensor R_{hijk} of our world by means of the well-known formulae[1]. Among 20 independent components of R_{hijk}, we obtain

$$\left.\begin{aligned}
R_{2112} &= R^2\sin^2\vartheta\left\{1+\left(\frac{\dot R}{F}\right)^2\right\},\\
R_{3113} &= R^2\sin^2\vartheta\sin^2\varphi\left\{1+\left(\frac{\dot R}{F}\right)^2\right\},\\
R_{4114} &= RF\left\{\frac{1}{R}\frac{\partial^2 F}{\partial\vartheta^2}-\frac{\partial}{\partial t}\left(\frac{\dot R}{F}\right)\right\},\\
R_{3223} &= R^2\sin^4\vartheta\sin^2\varphi\left\{1+\left(\frac{\dot R}{F}\right)^2\right\},\\
R_{4224} &= RF\sin\vartheta\left\{\frac{1}{R}\frac{\partial F}{\partial\vartheta}\cos\vartheta-\frac{\partial}{\partial t}\left(\frac{\dot R}{F}\right)\sin\vartheta\right\},\\
R_{4334} &= RF\sin\vartheta\sin^2\varphi\left\{\frac{1}{R}\frac{\partial F}{\partial\vartheta}\cos\vartheta-\frac{\partial}{\partial t}\left(\frac{\dot R}{F}\right)\sin\vartheta\right\},\\
R_{1224} &= \frac{R\dot R}{F}\frac{\partial F}{\partial\vartheta}\sin^2\vartheta,\\
R_{1334} &= \frac{R\dot R}{F}\frac{\partial F}{\partial\vartheta}\sin^2\vartheta\sin^2\varphi,
\end{aligned}\right\} \quad (1\cdot2)$$

1. L. P. Eisenhart: Riemannian geometry, p. 119. (1926).

as non-vanishing components, where R, F and $\dot{R}$ are the abridged notations of $R(t)$, $F(\vartheta, t)$ and $\frac{dR}{dt}$ respectively.

Moreover we may assume that in our coordinate system the energy-momentum tensor T_{ij} which occurs in the Einstein's field-equations with λ-term, i. e.

$$G_{ij}-\lambda g_{ij}=-\varkappa\left(T_{ij}-\frac{1}{2}g_{ij}T\right), \tag{1·3}$$

is given by the following scheme

$$T_{ij}=\begin{matrix} -pg_{11} & 0 & 0 & 0 \\ 0 & -pg_{22} & 0 & 0 \\ 0 & 0 & -pg_{33} & 0 \\ 0 & 0 & 0 & g_{44}(\rho_0+3p), \end{matrix} \tag{1·4}$$

where both the pressure p and the microscopic density of matter ρ_0 are functions of t only according to the spacial homogeneity of our world. If we calculate G_{ij} by (1·1) and (1·2) and apply these values of G_{ij} and (1·4) to (1·3), we obtain

$$G_{11}=-2\left\{1+\left(\frac{\dot{R}}{F}\right)^2\right\}+\frac{R}{F}\left\{\frac{1}{R}\frac{\partial^2 F}{\partial\vartheta^2}-\frac{\partial}{\partial t}\left(\frac{\dot{R}}{F}\right)\right\}$$
$$=-R^2\left\{\lambda+\varkappa p+\frac{\varkappa\rho_0}{2}\right\}, \tag{1·5}$$

$$G_{22}=\frac{1}{F}\frac{\partial F}{\partial\vartheta}\sin\vartheta\cos\vartheta-\sin^2\vartheta\left[2\left\{1+\left(\frac{\dot{R}}{F}\right)^2\right\}\right.$$
$$\left.+\frac{R}{F}\frac{\partial}{\partial t}\left(\frac{\dot{R}}{F}\right)\right]=-R^2\sin^2\vartheta\left\{\lambda+\varkappa p+\frac{\varkappa\rho_0}{2}\right\}, \tag{1·6}$$

$$G_{44}=-\frac{F}{R}\left[\frac{1}{R}\frac{\partial^2 F}{\partial\vartheta^2}-\frac{\partial}{\partial t}\left(\frac{\dot{R}}{F}\right)+\frac{2}{\sin\vartheta}\left\{\frac{1}{R}\frac{\partial F}{\partial\vartheta}\cos\vartheta\right.\right.$$
$$\left.\left.-\frac{\partial}{\partial t}\left(\frac{\dot{R}}{F}\right)\sin\vartheta\right\}\right]=F^2\left\{\lambda-3\varkappa p-\frac{\varkappa\rho_0}{2}\right\}, \tag{1·7}$$

$$G_{14}=-2\frac{\dot{R}}{FR}\frac{\partial F}{\partial\vartheta}=0, \tag{1.8}$$

as the independent equations to be solved. From (1·8) it follows that the solutions of our problem can be classified into two groups :—
(I) $\dot{R}=0$, i. e. statical solutions in which R takes a constant value, say R_0.

(II) $\frac{\partial F}{\partial \vartheta}=0$, i. e. solutions in which we may put $F=1$ without loss of generality.

In accordance with this important result, we shall treat these two solutions separately.

§ 2. Statical solutions; $R=R_0$.

In this article we shall consider the statical solutions i. e. $R=R_0$. Thus the field-equations (1·5), (1·6) and (1·7) can be reduced to

$$\frac{1}{F}\frac{\partial^2 F}{\partial \vartheta^2}-2=-R_0^2\left(\lambda+\varkappa p+\frac{\varkappa\rho_0}{2}\right) \tag{2·1}$$

$$\frac{1}{F}\frac{\partial F}{\partial \vartheta}\sin\vartheta\cos\vartheta-2\sin^2\vartheta=-R_0^2\sin^2\vartheta\left(\lambda+\varkappa p+\frac{\varkappa\rho_0}{2}\right) \tag{2·2}$$

$$\frac{1}{R_0^2}\frac{\partial^2 F}{\partial \vartheta^2}+\frac{2}{R_0^2}\frac{\partial F}{\partial \vartheta}\cot\vartheta=-F\left(\lambda-3\varkappa p-\frac{\varkappa\rho_0}{2}\right). \tag{2·3}$$

Eliminating $\left(\lambda+\varkappa p+\frac{\varkappa\rho_0}{2}\right)$ between (2·1) and (2·2), we obtain

$$\frac{\partial^2 F}{\partial \vartheta^2}-\frac{\partial F}{\partial \vartheta}\cot\vartheta=0.$$

Hence we get

$$F=T_2(t)+T_1(t)\cos\vartheta, \tag{2.4}$$

$T_2(t)$ and $T_1(t)$ being arbitrary functions of t only. Eliminating ρ_0 between (2·1) and (2·3) and then applying (2·4) to it, we get

$$\left(\frac{1}{R_0^2}-\lambda+\varkappa p\right)T_2(t)+\left(\frac{3}{R_0^2}-\lambda+\varkappa p\right)T_1(t)\cos\vartheta=0.$$

From this it follows that the present case should be separable into two classes :—

(i) $T_1=0$ and $\frac{1}{R_0^2}=\lambda-\varkappa p$.

If we introduce $\int T_2(t)dt$ as a new coordinate and denote it by t, the line-element can be put in the form

$$ds^2=-R_0^2(d\vartheta^2+\sin^2\vartheta d\varphi^2+\sin^2\vartheta\sin^2\varphi d\psi^2)+dt^2 \tag{2·5}$$

and the relations connecting p, ρ_0, λ and R_0 are evidently

$$\left.\begin{aligned}\frac{1}{R_0^2}&=\lambda-\varkappa p,\\ \frac{2}{\varkappa R_0^2}&=4p+\rho_0.\end{aligned}\right\} \tag{2·6}$$

It is well known that this is nothing but Einstein's cylindrical world.

(ii) $T_2=0$ and $\frac{3}{R_0^2}=\lambda-\varkappa p$.

If we introduce $\int T_1(t)dt$ as a new coordinate and denote it by t, the line-element can be written in the form

$$ds^2=-R_0^2(d\vartheta^2+\sin^2\vartheta d\varphi^2+\sin^2\vartheta\sin^2\varphi d\psi^2)+\cos^2\vartheta dt^2 \tag{2·7}$$

and the relations connecting p, ρ_0, λ and R_0 are obviously

$$\left.\begin{aligned}&\frac{3}{R_0^2}=\lambda-\varkappa p,\\&4p+\rho_0=0.\end{aligned}\right\} \tag{2·8}$$

As is well known, this is de Sitter's spherical world.

According to our nomenclature, when $R(t)$ takes a constant value the world is said to be a statical one. Strictly speaking, it is a statical world with respect to the coordinate system $(\vartheta, \varphi, \psi, t)$. From the above consideration it will be seen that a statical world belongs either to Einstein's cylindrical world or to de Sitter's spherical world. This corresponds to the theorem independently obtained by Tolman[1] and Robertson[2].

§ 3. Non-statical solutions

Hereafter let us confine ourselves to the consideration of the case $F=1$. Thus the field equations (1·5), (1·6) and (1·7) are reducible to the form

$$\frac{2\ddot{R}}{R}+\frac{1+\dot{R}^2}{R^2}=\lambda-\varkappa p, \tag{3·1}$$

$$\frac{1+\dot{R}^2}{R^2}=\frac{1}{3}(\lambda+\varkappa\rho_0+3\varkappa p). \tag{3·2}$$

Putting

$$\rho=\rho_0+3p \tag{3·3}$$

and calling it the macroscopic density of energy, we can write it in the form

$$\rho=\frac{f(R)}{R^3}, \tag{3·4}$$

1. R. C. Tolman: Proc. Nat. Acad. Sci., **15**, 297 (1929).
2. H. P. Robertson: Proc. Nat. Acad. Sci., **15**, 822 (1929).

where $f(R)$ is a function of R yet unknown. Applying (3·4) to (3·2), we obtain

$$\dot{R}=\sqrt{\frac{\lambda}{3}R^2-1+\frac{\varkappa f(R)}{3R}}. \tag{3·5}$$

The reasons for rejecting the negative root will become clear later.

From (3·5), we get

$$t=\int_{R_0}^{R}\frac{dx}{\sqrt{\frac{\lambda}{3}x^2-1+\frac{\varkappa f(x)}{3x}}}, \tag{3·6}$$

R_0 being the initial value of R.

Differentiating (3·5) with respect to t and applying (3·1) and (3·5) to it, we obtain

$$p=-\frac{f'(R)}{3R^2}. \tag{3·7}$$

If we assume the 3-space to be elliptical, the total invariant mass of matter in the world M_0 is given by

$$M_0=\pi^2R^3\rho_0=\pi^2\{f(R)+Rf'(R)\}. \tag{3·8}$$

From what has been said, it is clearly convenient to discuss all the non-statical solutions separately according to the following classification.

(i) $p=0$, i. e. zero-pressure solutions. In this case, as will easily be seen from (3·7), $f(R)$ is a constant. Denoting it by $\frac{3A}{\varkappa}$, we have, by (3·8),

$$M_0=\frac{3A\pi^2}{\varkappa}. \tag{3·9}$$

In the zero-pressure universe the total invariant mass is necessarily constant.

(ii) $p\neq 0$. We can divide this case into two sub-classes.

(a) $M_0=\text{const.}$ Denoting this constant by π^2D, (3·8) gives

$$f(R)+Rf'(R)=D.$$

Hence we get

$$f(R)=D+\frac{B}{R}, \tag{3·10}$$

where B is an integration constant.

(b) $M_0 \neq$ const. In this case, $f(R)$ is indeterminate.

§ 4. Some properties of a non-statical world.

Let us now consider the motion of a particle and the behaviour of light in a non-statical world. Without any loss of generality we may confine ourselves to the hypersurface $\varphi = \frac{\pi}{2}$, where the line element assumes the form

$$ds^2 = dt^2 - R(t)^2 \{d\vartheta^2 + \sin^2\vartheta d\psi^2\}. \tag{4·1}$$

(A) Motion of a particle.

Using the variational principle

$$\delta \int ds = 0$$

and (4·1), we find

$$\left.\begin{aligned} &\frac{d}{ds}(R^2 \sin^2\vartheta\, \psi') = 0, \\ &t'' + R\dot{R}(\vartheta'^2 + \sin^2\vartheta\, \psi'^2) = 0, \\ &R^2(\vartheta'^2 + \sin^2\vartheta\, \psi'^2) = t'^2 - 1, \end{aligned}\right\} \tag{4·2}$$

for the differential equations of the path, where the accent denotes $\frac{d}{ds}$. Eliminating t and s, the spacial path of the particle is determined by the differential equation

$$\frac{d^2\vartheta}{d\psi^2} - 2\cot\vartheta\left(\frac{d\vartheta}{d\psi}\right)^2 - \sin\vartheta\cos\vartheta = 0.$$

It can easily be shown that this defines a geodesic in the 3-space whose line element dl is given by

$$dl^2 = R(t)^2 (d\vartheta^2 + \sin^2\vartheta d\psi^2). \tag{4·3}$$

From the last two equations of (4·2), we obtain the first integral

$$ds = \frac{R dt}{\sqrt{a^2 + R^2}}, \tag{4·4}$$

where a is an integration constant and the positive sense of s is taken for the direction in which t increases. If we define the squared velocity v^2 of a particle by $\left(\frac{dl}{dt}\right)^2$, we find, from (4·1), (4·3) and (4·4),

$$v^2 = \left(\frac{dl}{dt}\right)^2 = \frac{a^2}{a^2 + R^2}. \tag{4·5}$$

Consequently the velocity of a free particle decreases only when $\dot{R}>0$. On the other hand, the velocity increases with time when $\dot{R}<0$, so that the stable model of the universe will not be obtained, as Tolman has pointed out.

(B) Behaviour of light.

By a slight modification of the argument given in (A), we can easily show that the spacial path of a light-ray is a geodesic in the 3-space $t=\text{const.}$ Next let us suppose a light source situated in (ϑ_1, ψ_1) at an instant t_1. At this moment the source sends a pair of light-signals separated by the time Δt_1, during which its position changes from (ϑ_1, ψ_1) to $(\vartheta_1+\Delta\vartheta_1, \psi_1+\Delta\psi_1)$, to the stationary observer at the origin of the coordinate system. If t_2 and $t_2+\Delta t_2$ are respectively the instants receiving these signals, we have

$$\int_{t_1}^{t_2}\frac{dt}{R(t)}=\int_{(0,\,0)}^{(\vartheta_1,\,\psi_1)}\sqrt{d\vartheta^2+\sin^2\vartheta d\psi^2}$$

and

$$\int_{t_1+\Delta t_1}^{t_2+\Delta t_2}\frac{dt}{R(t)}=\int_{(0,\,0)}^{(\vartheta_1+\Delta\vartheta_1,\,\psi_1+\Delta\psi_1)}\sqrt{d\vartheta^2+\sin^2\vartheta d\psi^2}\,.$$

Hence we get

$$\int_{t_2}^{t_2+\Delta t_2}-\int_{t_1}^{t_1+\Delta t_1}=\Delta\vartheta_1.$$

But, according to Einstein's principle of the permanence of atoms, this gives

$$\frac{\Lambda+\delta\Lambda}{\Lambda}=\frac{R(t_2)}{R(t_1)}\cdot\frac{1+(v_r)_1}{\left(\frac{ds}{dt}\right)_1},$$

where Λ and $\Lambda+\delta\Lambda$ are the wave-lengths of a definite line in the stellar and terrestrial spectrum and $(v_r)_1$ is the radial velocity of the source. Since the velocity v_1 of the source is given by

$$\left(\frac{ds}{dt}\right)_1^2=1-v_1^2,$$

we obtain, from (4·6), the general formula for the Doppler-effect

$$V=\frac{\delta\Lambda}{\Lambda}=\frac{R(t_2)}{R(t_1)}\cdot\frac{1+(v_r)_1}{\sqrt{1-v_1^2}}-1, \tag{4·7}$$

V being the radial velocity of the source. Neglecting v_1^2 for 1, (4·7) can be put in the form

$$V=\frac{R(t_2)}{R(t_1)}\{1+(v_r)_1\}-1. \tag{4·8}$$

Furthermore, if we neglect the proper motion of the source relative to our coordinate system, we finally get

$$V=\frac{R(t_2)}{R(t_1)}-1. \tag{4·9}$$

This shows us that $\dot{R}$ must be positive in order to interpret the red shift as the necessary consequence of our space-time world.

§ 5. Property of non-statical world when t tends to $+\infty$.

According to Riemannian geometry, the necessary and sufficient condition that a space V_n be of constant curvature K_0 is that the components of the fundamental tensor g_{ij} satisfy the conditions

$$R_{hijk}=K_0(g_{hj}g_{ik}-g_{hk}g_{ij}). \tag{5·1}$$

If we calculate R_{hijk} in order to apply this theorem to our non-statical world whose line element is given by

$$ds^2=-R(t)^2\{d\vartheta^2+\sin^2\vartheta d\varphi^2+\sin^2\vartheta\sin^2\varphi d\psi^2\}+dt^2,$$

we find, by (1·2)

$$\begin{aligned}
R_{2112}&=R^2\sin^2\vartheta(1+\dot{R}^2), & R_{3223}&=R^2\sin^4\vartheta\sin^2\varphi(1+\dot{R}^2),\\
R_{3113}&=R^2\sin^2\vartheta\sin^2\varphi(1+\dot{R}^2), & R_{4224}&=-R\ddot{R}\sin^2\vartheta,\\
R_{4114}&=-R\ddot{R}, & R_{4334}&=-R\ddot{R}\sin^2\vartheta\sin^2\varphi,
\end{aligned}$$

for non-vanishing components. By means of (3·1), (3·2), (5·1) and these values of R_{hijk}, we find that the necessary and sufficient condition that our non-statical world be of constant curvature K_0 is that the ratios

$$\begin{aligned}
\frac{R_{2112}}{g_{12}g_{12}-g_{11}g_{22}}&=-\frac{1+\dot{R}^2}{R^2}=-\frac{\lambda}{3}-\frac{\varkappa\rho_0}{3}-\varkappa p,\\
\frac{R_{3113}}{g_{13}g_{13}-g_{11}g_{33}}&=-\frac{1+\dot{R}^2}{R^2}=-\frac{\lambda}{3}-\frac{\varkappa\rho_0}{3}-\varkappa p,\\
\frac{R_{4114}}{g_{14}g_{14}-g_{11}g_{44}}&=-\frac{\ddot{R}}{R}=-\frac{\lambda}{3}+\frac{\varkappa\rho_0}{6}+\varkappa p,\\
\frac{R_{3223}}{g_{23}g_{23}-g_{22}g_{33}}&=-\frac{1+\dot{R}^2}{R^2}=-\frac{\lambda}{3}-\frac{\varkappa\rho_0}{3}-\varkappa p,
\end{aligned} \tag{5·2}$$

$$\frac{R_{4224}}{g_{24}g_{24}-g_{22}g_{44}}=-\frac{\ddot{R}}{R}=-\frac{\lambda}{3}+\frac{\varkappa\rho_0}{6}+\varkappa p,$$

$$\frac{R_{4334}}{g_{34}g_{34}-g_{33}g_{44}}=-\frac{\ddot{R}}{R}=-\frac{\lambda}{3}+\frac{\varkappa\rho_0}{6}+\varkappa p,$$

must have the common value K_0. Hence if ρ_0 and p tend to zero when t tends to $+\infty$, all the ratios of the left-hand side in (5·2) converge to the limiting value $-\frac{\lambda}{3}$. Thus we have the theorem: The non-statical world such that its pressure and microscopic density of matter tend to zero when the time tends to positive infinity converges to the de Sitter's spherical world having $\sqrt{\frac{3}{\lambda}}$ as its curvature-radius if the time becomes positive infinity.

§ 6. Zero-pressure solution.

When $p=0$, from (3·6) and (3·9), the relation between R and t is given by the integral

$$t=\int_{R_0}^{R}\sqrt{\frac{x}{A-x+\frac{\lambda}{3}x^3}}\,dx, \tag{6·1}$$

where A is positive since M_0 must be positive. For simplicity we shall denote this integral by $I(R)$.

(a) Case: $A-\frac{2}{3\sqrt{\lambda}}>0$. $I(R)$ has a finite lower limit, because there exists no singularity in the integrand and R does not become negative.

(b) Case: $A-\frac{2}{3\sqrt{\lambda}}<0$. The denominator of the integrand in (6·1) has two distinct zero-points denoted by R_1 and R_2 $(>R_1)$ respectively. If $0\leqq R_0\leqq R_1$, R can not vary beyond R_1, for $I(R)$ is imaginary when $R_1<R$. But since the order of infinity at $R=R_1$ is equal to $\frac{1}{2}$, it is clear that $I(R)$ is a limited function of R. If $R_2\leqq R_0$, $I(R)$ has evidently a finite lower limit.

(c) Case: $A-\frac{2}{3\sqrt{\lambda}}=0$. The denominator of the integrand in (6·1) has a double zero-point $\frac{1}{\sqrt{\lambda}}$. If $\frac{1}{\sqrt{\lambda}}>R_0\geqq 0$ and $\frac{1}{\sqrt{\lambda}}>R\geqq 0$, it can easily be shown that $I(R)$ has a definite lower limit. If $R_0>\frac{1}{\sqrt{\lambda}}$ and $R>\frac{1}{\sqrt{\lambda}}$, then $I(R)$ can take any preassigned real value.

From these it follows that (a), (b) and the first part of (c) are unadmissible cases and the second part of (c) is admissible by considering the physical meaning of t. In this unique zero-pressure solution, by the relation $A=\frac{2}{3\sqrt{\lambda}}$, we can perform the integration in (6·1) as follows,

$$\frac{(R-R_{-\infty})\{R+R_{-\infty}+\sqrt{R(R+2R_{-\infty})}\}^{\sqrt{3}}}{2R+R_{-\infty}+\sqrt{3R(R+2R_{-\infty})}}=Ce^{\frac{t}{R_{-\infty}}}, \qquad (6\cdot 2)$$

where $R_{-\infty}$ and C represent $\frac{1}{\sqrt{\lambda}}$ and the value at $t=0$ of the left-hand side of (6·2) respectively[1].

From (6·2) and (1·1), it follows that R approaches $R_{-\infty}$ and the line-element tends to

$$ds^2=-R^2_{-\infty}(d\vartheta^2+\sin^2\vartheta d\varphi^2+\sin^2\vartheta\sin^2\varphi d\psi^2)+dt^2,$$

when t converges to $-\infty$. That is, we can say that the zero-pressure world is an Einstein's cylindrical world having $R_{-\infty}$ as its radius in an indefinitely long past. If t increases gradually from $-\infty$ to $+\infty$, R increases also from the definite value $R_{-\infty}$ to $+\infty$, as can easily be seen from (6·2). Therefore the density

$$\rho=\rho_0=\frac{2R_{-\infty}}{\varkappa R^3}$$

approaches to zero when $t\to+\infty$. According to § 5, we can conclude that our zero-pressure world will become a de Sitter's spherical one having $\sqrt{3}\,R_{-\infty}$ as its curvature radius, if the time elapses indefinitely. Since its total invariant mass remains unchanged i. e. it takes a constant value $\frac{2\pi^2R_{-\infty}}{\varkappa}$ during this process, our zero-pressure model of the universe has a somewhat different character from the actual universe in which an enormous quantity of matter is being transformed into radiating energy. But as for the red displacement of light emitted from distant objects, the explanation by this model is very satisfactory. If $R(0)$ and $R(t_1)$ are assumed to be greater than $R_{-\infty}$ in such an order that the squares of $\frac{R_{-\infty}}{R(0)}$ and $\frac{R_{-\infty}}{R(t_1)}$ can be neglected for unity, t_1 being the emitting time instant, we obtain, from (6·2),

$$\frac{R(t_1)}{R(0)}=e^{-\frac{t_1}{\sqrt{3}R_{-\infty}}}\doteqdot 1-\frac{t_1}{\sqrt{3}\,R_{-\infty}}. \qquad (6\cdot 3)$$

1. This solution was found in a different way by Lemaître. (Cf. R. C. Tolman: Proc. Nat. Acad. Sci., **16**, 582 (1930).

By (6·2) and (4·9), we get the formula

$$V = -\frac{t_1}{\sqrt{3}\,R_{-\infty}}, \tag{6·4}$$

which says that the Doppler shift of extragalactic nebulae is proportional to their distances, provided that their proper motions are neglected. Taking a step further, let us consider n extragalactic nebulae having equal distances from the origin. Then, by (4·8), we get

$$\frac{\Sigma V}{n} = -\frac{t_1}{\sqrt{3}\,R_{-\infty}}, \tag{6·5}$$

where ΣV is the sum of their radial velocities due to Doppler shift; for $\Sigma(v_r)_1$ vanishes on account of the non-existence of systematic proper motions in such a world. If we select 8 nebulae of this kind from Hubble's table[1], our estimation of $R_{-\infty}$ by means of (6·5) gives the value

$$R_{-\infty} = 1.15 \times 10^{27}\text{cms}. \tag{6·6}$$

Hence we get

$$M_0 = \frac{2\pi^2 R_{-\infty}}{\varkappa} = 1.21 \times 10^{55}\text{ gms}. \tag{6·7}$$

§ 7. Non zero-pressure solution.

When the total invariant mass is constant, we have seen that $f(R)$ is of the form

$$f(R) = D + \frac{B}{R}. \tag{3·10}$$

Here we will not consider this type of solution, since de Sitter's research[2] in 1930 corresponds to this and the annihilation of matter does not take place in such a world.

There is no ground to determine the form of $f(R)$, when M_0 depends on t. For the purpose of this determination, first let us assume that the world tends to an Einstein's cylindrical one having $R_{-\infty}$ as its radius when t decreases indefinitely, that is to say, our first assumption claims that the cosmological constant appearing in the amplified field-equations means the reciprocal of the squared radius of the Einstein's cylindrical world which was the form of our idealized actual universe at the beginning of eternity.

1. E. Hubble: Proc. Nat. Acad. Sci., **15**, 168 (1929).
2. W. de Sitter: loc. cit.

In consequence of this, if we denote

$$\Psi(x)=\frac{\lambda}{3}x^2-1+\frac{\varkappa f(x)}{3x},$$

we get, from (3·6),

$$\Psi(R_{-\infty})=0, \quad \Psi'(R_{-\infty})=0.$$

Since these are reducible to

$$\varkappa f(R_{-\infty})=2R_{-\infty}, \quad f'(R_{-\infty})=0,$$

$f(R)$ must be of the form

$$\varkappa f(R)=2R_{-\infty}+\left(\frac{1}{R_{-\infty}}-\frac{1}{R}\right)^2\Phi(R), \tag{7·1 a}$$

$\Phi(R)$ being assumed to be expanded in the series

$$\Phi(R)=a_0+a_1\left(\frac{R_{-\infty}}{R}\right)+a_2\left(\frac{R_{-\infty}}{R}\right)^2+\dots\dots\dots, \tag{7·1 b}$$

which is convergent at $R=R_{-\infty}$.

Second let us assume that if t becomes infinitely great R also becomes so and the ratio $\frac{M_0}{(M_0)_{t=-\infty}}$ tends to a constant $\alpha(0\leqq\alpha<1)$, that is to say, our second assumption claims that at the end of eternity the radius of our world becomes indefinitely great and its pressure becomes zero, while its total invariant mass decreases to an amount α times as great as that of the beginning. Then $\Phi(R)$ can be put in the form

$$\Phi(R)=-2R^3_{-\infty}(1-\alpha)+a_1\left(\frac{R_{-\infty}}{R}\right)+a_2\left(\frac{R_{-\infty}}{R}\right)^2+\dots\dots, \tag{7·2}$$

by (3·8), (7·1)a and (7·1)b.

By (7·1)a, (7·2) and the theorem obtained in § 5, we can show that the world tends to the de Sitter's spherical one when $t\to+\infty$. In the following we shall treat the most simple case contained in it i. e. $a_1=a_2=\dots\dots=0$. Applying (7·1)a and (7·2) to (3 6), (3·7) and (3·8), we get

$$t=\int_{R_0}^{R}\frac{dx}{\left(\frac{1}{R_{-\infty}}-\frac{1}{x}\right)\sqrt{\frac{1}{3x}\{x^3+2R_{-\infty}x^2-2R^3_{-\infty}(1-\alpha)\}}}, \tag{7·3}$$

$$p=\frac{4R^3_{-\infty}(1-\alpha)}{3\varkappa R^4}\left(\frac{1}{R_{-\infty}}-\frac{1}{R}\right), \tag{7·4}$$

$$M_0=\frac{2\pi^2 R_{-\infty}}{\varkappa}\left\{a+\frac{R^2_{-\infty}}{R^2}(1-a)\right\}. \tag{7·5}$$

If $R(0)$ and $R(t_1)$ are assumed to be greater than $R_{-\infty}$ in such an order that the squares of $\frac{R_{-\infty}}{R(0)}$ and $\frac{R_{-\infty}}{R(t_1)}$ can be neglected for unity, we obtain, from (7·3), the same equation as (6·3). For this reason we can give a satisfactory explanation for the linear relation between Doppler shifts and the distances of extragalactic nebulae and also obtain the value

$$R_{-\infty}=1.15\times 10^{27}\ \text{cms.}, \tag{6·6}$$

as can easily be seen from § 6.

Differentiating (7·5) with respect to t, we get

$$\dot{M}_0=-\frac{4\pi^2 R^3_{-\infty}(1-a)}{\varkappa R^3}\dot{R}, \tag{7·6}$$

where $\dot{R}$ is, of course, given by (7·3). Although, if we use these formulae (7·3), (7·4), (7·5) and (7·6), we can deduce the expression for annihilation of matter, we cannot estimate the percentage rate for annihilation owing to a wide range of values (observational) depending on the type of stars considered. Therefore we will try to estimate it for several values of a lying between $\frac{1}{10}$ and 1. In this case, by means of (7·5), (7·6) and (7·3), we get approximately

$$[\dot{M}_0]_{t=0}=\pi^2 R(0)^3[\rho_0]_{t=0}=a\frac{2\pi^2 R_{-\infty}}{\varkappa}, \tag{7·7}$$

$$\left[-\frac{1}{M_0}\frac{dM_0}{dt}\right]_{t=0}=\frac{2(1-a)R_{-\infty}}{\sqrt{3}\,aR(0)^2}. \tag{7·8}$$

Using Hubble's value of density

$$[\rho_0]_{t=0}=1.5\times 10^{-31}\frac{\text{gm.}}{\text{cm.}},$$

we can determine $R(0)$ by (7·7) and then estimate the percentage rate for annihilation of matter in our universe by (7·8).

For $a=\frac{11}{27}$, we get $\left[-\frac{\dot{M}_0}{M_0}\right]_{t=0}=8.1\times 10^{-12}$ (years)$^{-1}$ and $R(0)=1.5\times 10^{28}$ cms. In this case, the elliptic integral in (7·3) being expressed by elementary functions only, the rate is nearly equal to that of Betelgeuse and the present radius in 13 times as great as $R_{-\infty}$.

For $\alpha = \frac{31}{32}$, we get $\left[-\frac{\dot{M}_0}{M_0}\right]_{t=0} = 1.0 \times 10^{-12}$ (years)$^{-1}$ and $R(0) = 2 \times 10^{28}$ cms. This rate is just equal to that of Sirius A.

For $\alpha = 1 - 5.4 \times 10^{-4}$, we get $\left[-\frac{\dot{M}_0}{M_0}\right]_{t=0} = 1.7 \times 10^{-15}$ (years)$^{-1}$ and $R(0) = 2 \times 10^{28}$ cms. In this case, for which the ratio $\frac{(M_0)_{t=-\infty} - (M_0)_{t=+\infty}}{(M_0)_{t=-\infty}}$ is equal to that of electron and proton, the percentage rate is nearly twice as great as that of 60 Kruger B.

For $\alpha = 1 - 2.3 \times 10^{-4}$, we get $\left[-\frac{\dot{M}_0}{M_0}\right]_{t=0} = 7.0 \times 10^{-16}$ (years)$^{-1}$ and $R(0) = 2 \times 10^{28}$ cms. This value of rate is just equal to that of 60 Kruger B.

Thus we see that this non-statical universe serves as a reasonable model in many respects.

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