# On the Convection Current and the Surface Level of a Two-layer Ocean

By

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Abstract

In order to find what difference in height of sea-level should be produced by a local difference in the density of sea-water, one[1] of the present authors discussed theoretically in the previous paper the position of the horizontal isobaric layer in the convection current of a two-layer ocean, neglecting Coriolis' force.

The present paper deals with the same problem, taking the influence of the earth's rotation into account, and gives first the formulae for the convection current and then the depth of the horizontal isobaric surface in a rotating ocean of two-layers such as in the previous paper.

According to the result, the ratio of level difference in dynamical equilibrium to that in statical, and hence the depth of the horizontal isobaric layer measured as a fraction of the thickness of the heterogeneous upper part, is equal to unity far more nearly than in the previous case.

## Introduction

In the previous paper[2] one of the present authors discussed theoretically the depth of the horizontal isobaric layer in the convection current of a two-layer ocean, and indicated that the local difference in height of sea-level and its seasonal variation due to the difference or variation in the density of sea-water might be generally treated hydrostatically. In that case however, for simplicity, he treated the problem without consideration of the effect of Coriolis' force.

But since this problem is of great importance in the practical

1. Nomitsu, T., On the so-called "Grenzflaeche" in the Current due to the Difference of Density. These Memoirs, **10**, 111 (1929).

2. Loc. cit.

study of the sea-level, and also has a bearing even on Rosenhead's investigation[1] into the latitude variation, we shall again take up the same problem theoretically, taking the effect of Coriolis' force into account.

Now, for simplicity, let us consider an ocean such that it extends to infinity and consists of two-layers just as in the previous paper, i. e, an upper stratum of a small thickness where the density of water linearly differs from place to place in one horizontal direction, and the lower remaining stratum in which the water is homogeneous throughout.

To find the difference in height of the surface level of such an ocean after the dynamical equilibrium has been reached, it will be enough to determine the depth of the isobaric surface which lies horizontal in the sea. And in order to know the depth of it theoretically, the mathematical formulae for currents in the ocean must be established.

W. Ekman[2] and his collaborator E. Stefansen investigated the problem of the convection current of a two-layer ocean, and gave in the Ann. d. Hydrog. current-diagrams for some special cases without any mathematical illustration, but saying

"Die ziemlich komplizierten Rechnungen, die hier nicht beschreiben werden können, sind zum grossen Teil von Dr. Elizabeth Stefansen ausgefuhrt."

The diagram is reproduced in Krümmel's Handbuch d. Ozeanographie[3], but there also no information is given about it although the book generally contains so full bibliographies about other problems. Thus we do not know Ekman's own formulae corresponding to the diagram, and we rather think that they have perhaps not yet been published.

As preliminary to our final object, we have ourselves prepared the mathematical equations for the convection current and drawn the current-diagrams. Our diagrams thus obtained are very similar to those of Ekman, but show some differences. Our mathematical calculations also have no such complication as to be called "ziemlich kompliziert." Hence we conjecture there are some differences between

1. Geophys. Suppl. M. N. R. A. S., **2**, 140—170 (1929)
2. Ann. d. Hydrog, 574—575 (1906).
3. Krümmel, Handb. d. Ozeanographie. II, 468 (1911).

Ekman's theory and ours, with respect to either the constitution of the assumed ocean or the boundary conditions taken. For instance, even to the present case of a two-layer ocean, Ekman may have applied his formulae for an entirely heterogeneous sea obtained in the paper of 1905 on the assumption[1] that the horizontal layer of no-pressure-gradient coincides with the no-current layer; but according to our view[2] this is not the case generally, i. e., the horizontal isobaric layer does not always coincide with the no-current layer.

For the above reasons, in the present paper we shall first give our mathematical solutions of the convection currents in a two-layer ocean and then discuss the depth of the horizontal isobaric layer together with the difference of sea-level due to the difference in water-density.

## I. Convection Currents in a Two-layer Ocean

§ 1. *Equations for the Currents.*

Let $H$ denote the total depth of the ocean, and let $H_1$ and $H_2 = H - H_1$ be the thickness of the upper heterogeneous and of the lower homogeneous stratum respectively. In the mean free-surface take the $y$-axis parallel to the direction of the gradient of density and the $x$-axis perpendicular to it, and take the $z$-axis vertically downward.

We shall use the following notation:

$\rho$ = the density of sea-water (use $\rho_1$ for the upper stratum only, if desired),

$\mu$ = the viscosity coefficient,

$p$ = the pressure at a point,

$u, v$ = the component velocities of current in the direction of $x$ and $y$ respectively, and to distinguish the current in the upper heterogeneous and in the lower homogeneous stratum, we use the suffixes 1 and 2 respectively,

$t$ = the time,

$\omega$ = the angular velocity of the earth's rotation,

$\lambda$ = the latitude of the place under consideration,

$g$ = the acceleration due to gravity,

$b = g\dfrac{\partial \rho_1}{\partial y}$ (assume constant throughout the upper stratum),

1. Ekman, V. W., On the influence of the earth's rotation on ocean current. Ark. f. Math. Astr. o. Fys., Bd. 2, Nr. 11. 1905.

2. Nomitsu, On the so called "Grenzfläche" etc. Loc. cit.

$d$ = the depth of the horizontal isobaric layer.

Then the horizontal pressure-gradients in the ocean are

$$-\frac{\partial p}{\partial x}=0,$$

and $$-\frac{\partial p}{\partial y}=b(d-z) \quad \text{for the upper stratum } 0<z<H_1$$
$$=b(d-H_1) \quad \text{for the lower stratum } H_1<z<H.$$

Hence the equations of motion may be written as follows :—

$$\left.\begin{array}{l} \text{For } 0<z<H_1 \\ \dfrac{\partial^2 u_1}{\partial z^2}=-2k^2 v_1, \quad \dfrac{\partial^2 v_1}{\partial z^2}=2k^2 u_1-a(d-z), \\ \text{for } H_1<z<H \\ \dfrac{\partial^2 u_2}{\partial z^2}=-2k^2 v_2, \quad \dfrac{\partial^2 v_2}{\partial z^2}=2k^2 u_2-a(d-H_1), \end{array}\right\} \quad \cdots\cdots\cdots (1)$$

where $$a=\frac{b}{\mu}=\frac{g}{\mu}\frac{\partial\rho_1}{\partial y} \quad \text{and} \quad k=\sqrt{\frac{\rho\omega\sin\lambda}{\mu}} \quad \cdots\cdots\cdots (2)$$

If we introduce a complex variable $w=u+iv$, then equation (1) becomes

$$\left.\begin{array}{ll} \dfrac{\partial^2 w_1}{\partial z^2}=2ik^2 w_1-ia(d-z) & 0<z<H_1 \\ \dfrac{\partial^2 w_2}{\partial z^2}=2ik^2 w_2-ia(d-H_1) & H_1<z<H \end{array}\right\} \quad \cdots\cdots\cdots (1')$$

The general solutions of (1′) are obviously

$$\left.\begin{array}{l} w_1=\dfrac{a}{4k^3}[K_1e^{(1+i)kz}+K_1'e^{-(1+i)kz}+2k(d-z)] \\ w_2=\dfrac{a}{4k^3}[K_2e^{(1+i)kz}+K_2'e^{-(1+i)kz}+2k(d-H_1)] \end{array}\right\} \quad \cdots\cdots\cdots (3)$$

where $K_1$, $K_1'$, $K_2$ and $K_2'$ are complex arbitrary constants.

Or let

$$\left.\begin{array}{ll} K_1=A_1-iB_1, & K_1'=C_1+iD_1 \\ K_2=A_2-iB_2, & K_2'=C_2+iD_2 \end{array}\right\} \quad \cdots\cdots\cdots (4)$$

then we get

$$\left.\begin{aligned}
u_1 &= \frac{a}{4k^3}[A_1e^{kz}\cos kz + B_1e^{kz}\sin kz + C_1e^{-kz}\cos kz + D_1e^{-kz}\sin kz + 2k(d-z)] \\
v_1 &= \frac{a}{4k^3}[A_1e^{kz}\sin kz - B_1e^{kz}\cos kz - C_1e^{-kz}\sin kz + D_1e^{-kz}\cos kz] \\
u_2 &= \frac{a}{4k^3}[A_2e^{kz}\cos kz + B_2e^{kz}\sin kz + C_2e^{-kz}\cos kz + D_2e^{-kz}\sin kz + 2k(d-H_1)] \\
v_2 &= \frac{a}{4k^3}[A_2e^{kz}\sin kz - B_2e^{kz}\cos kz - C_2e^{-kz}\sin kz + D_2e^{-kz}\cos kz]
\end{aligned}\right\} \qquad (3')$$

where $A$, $B$, $C$, $D$ are real constants.

We shall next determine these integration constants with the boundary conditions.

At the free surface ($z=0$), $\left|\frac{\partial w_1}{\partial z}\right|_{z=0}=0$, or from (3)

$$0=(1+i)k(K_1-K_1')-2k$$

$$\therefore \quad K_1-K_1'=\frac{2}{1+i}=(1-i) \qquad (5)$$

or

$$\left.\begin{aligned} C_1 &= A_1-1 \\ D_1 &= -B_1+1 \end{aligned}\right\} \qquad (5')$$

At the intermediate surface of discontinuity ($z=H_1$),

$$0=|w_1-w_2|_{z=H_1} \quad \text{or } 0=(K_1-K_2)e^{(1+i)kH_1}+(K_1'-K_2')e^{-(1+i)kH_1}$$

and

$$0=\left|\frac{\partial w_1}{\partial z}-\frac{\partial w_2}{\partial z}\right|_{z=H_1}$$

or $0=(K_1-K_2)e^{(1+i)kH_1}-(K_1'-K_2')e^{-(1+i)kH_1}-\frac{2}{1+i}$

$$\therefore \quad (K_1-K_2)e^{(1+i)kH_1}=-(K_1'-K_2')e^{-(1+i)kH_1}=\frac{1-i}{2} \qquad (6)$$

Separating these values into real and imaginary parts, we have

$$(A_1-A_2)\cos kH_1+(B_1-B_2)\sin kH_1=\frac{1}{2}e^{-kH_1}$$

$$(A_1-A_2)\sin kH_1-(B_1-B_2)\cos kH_1=-\frac{1}{2}e^{-kH_1}$$

$$(C_1-C_2)\cos kH_1+(D_1-D_2)\sin kH_1=-\frac{1}{2}e^{kH_1}$$

$$(C_1-C_2)\sin kH_1-(D_1-D_2)\cos kH_1=-\frac{1}{2}e^{kH_1}$$

$$\therefore\quad \left.\begin{aligned} A_2 &= A_1-\frac{1}{2}e^{-kH_1}(\cos kH_1-\sin kH_1)\\ B_2 &= B_1-\frac{1}{2}e^{-kH_1}(\cos kH_1+\sin kH_1)\\ C_2 &= C_1+\frac{1}{2}e^{kH_1}(\cos kH_1+\sin kH_1)\\ D_2 &= D_1-\frac{1}{2}e^{kH_1}(\cos kH_1-\sin kH_1)\end{aligned}\right\}\quad\cdots\cdots(6')$$

Thus, if we know only two, $A_1$ and $B_1$, of these eight integration constants, then the other six can be at once calculated by equations (5′) and (6′).

Next, the bottom condition of the real ocean may be some intermediate one between that there is no bottom-current and that no bottom-friction exists, so it will be sufficient to treat these two extreme cases.

In this article we shall first treat the *case where no bottom-current exists*, that is

$$0=|w_2|_{z=H}\quad\text{or}\quad K_2e^{(1+i)kH}+K_2'e^{-(1+i)kH}+2k(d-H_1)=0\qquad(7)$$

Substitute (5) and (6) in (7), then

$$0=\left[K_1-\frac{(1-i)}{2}e^{-(1+i)kH_1}\right]e^{(1+i)kH}$$
$$+\left[K_1-(1-i)+\frac{(1-i)}{2}e^{(1+i)kH_1}\right]e^{-(1+i)kH}+2k(d-H_1)$$

$$\therefore\quad K_1=A_1-iB_1$$
$$=\frac{1}{2\cosh(1+i)kH}\left[(1-i)\left\{\sinh(1+i)k(H-H_1)+e^{-(1+i)kH}\right\}\right.$$
$$\left.+2k(H_1-d)\right]\quad\cdots\cdots(8)$$

But
$$\frac{1}{2\cosh(1+i)kH}=\frac{\cosh(1-i)kH}{2\cosh(1+i)kH\cosh(1-i)kH}$$
$$=\frac{\cosh kH\cos kH-i\sinh kH\sin kH}{\cosh 2kH+\cos 2kH}$$

$$(1-i)\cosh(1-i)kH = (\cosh kH \cos kH - \sinh kH \sin kH)$$
$$- i(\cosh kH \cos kH + \sinh kH \sin kH)$$

and $$\sinh(1+i)k(H-H_1) + e^{-(1+i)kH} = \sinh k(H-H_1)\cos k(H-H_1)$$
$$+ e^{-kH}\cos kH + i\left\{\cosh k(H-H_1)\sin k(H-H_1) - e^{-kH}\sin kH\right\}$$

$$\therefore \quad A_1 = \frac{1}{\cosh 2kH + \cos 2kH}\Big[2k(H_1-d)\cosh kH \cos kH$$
$$+ (\cosh kH \cos kH - \sinh kH \sin kH) \times$$
$$\{\sinh k(H-H_1)\cos k(H-H_1) + e^{-kH}\cos kH\}$$
$$+ (\cosh kH \cos kH + \sinh kH \sin kH) \times$$
$$\{\cosh k(H-H_1)\sin k(H-H_1) - e^{-kH}\sin kH\}\Big] \quad \cdots\cdots\cdots (9)$$

$$B_1 = \frac{1}{\cosh 2kH + \cos 2kH}\Big[2k(H_1-d)\sinh kH \sin kH$$
$$- (\cosh kH \cos kH - \sinh kH \sin kH) \times$$
$$\{\cosh k(H-H_1)\sin k(H-H_1) - e^{-kH}\sin kH\}$$
$$+ (\cosh kH \cos kH + \sinh kH \sin kH) \times$$
$$\{\sinh k(H-H_1)\cos k(H-H_1) + e^{-kH}\cos kH\}\Big] \quad \cdots\cdots\cdots (10)$$

Thus, all integration constants are expressed in the terms of $H$, $H_1$, and $k$ which define the constitution of the ocean.

Of the current represented by eq. (3′) together with (9) and (10), the part which corresponds to $d=0$ may be called "pure convection current", and the remaining part, which contains the terms of $d$ only, is nothing but a "slope current" given by Ekman.

§ 2. *Deep Sea in which* $e^{-k(H-H_1)} \ll 1$

In the real ocean, $H$ is very large while $H_1$ is very small compared with $H$, so that $e^{-k(H-H_1)}$ and $e^{-kH}$ may be neglected as compared with 1.

In such cases the formulae for the calculation of coefficients in the preceding Art. can be much simplified. That is to say, from (9), (10) and (5′) we obtain

$$\left.\begin{aligned} A_1 &\doteqdot \frac{1}{2}e^{-kH_1}(\cos kH_1 - \sin kH_1) \\ B_1 &\doteqdot \frac{1}{2}e^{-kH_1}(\cos kH_1 + \sin kH_1) \\ C_1 &= A_1 - 1 \\ D_1 &= -B_1 + 1 \end{aligned}\right\} \qquad (11)$$

and from (6′)

$$\left.\begin{aligned} A_2 &= 0 \\ B_2 &= 0 \\ C_2 &= C_1 + \frac{1}{2}e^{kH_1}(\cos kH_1 + \sin kH_1) \\ D_2 &= D_1 - \frac{1}{2}e^{kH_1}(\cos kH_1 - \sin kH_1) \end{aligned}\right\} \qquad (12)$$

Moreover since, as will be shown in the following Art.,[1] in such a deep sea the depth of the horizontal isobaric layer $d$ is nearly equal to $H_1$, the current equations (3′) will be reduced to

$$\left.\begin{aligned} u_1 &= \frac{a}{4k^3}\Big[2A_1\cosh kz\cos kz + 2B_1\sinh kz\sin kz + e^{-kz}(\sin kz - \cos kz) \\ &\qquad + 2k(H_1 - z)\Big] \\ v_1 &= \frac{a}{4k^3}\Big[2A_1\sinh kz\sin kz - 2B_1\cosh kz\cos kz + e^{-kz}(\cos kz + \sin kz)\Big] \\ u_2 &= \frac{a}{4k^3}\Big[C_2\cos kz + D_2\sin kz\Big]e^{-kz} \\ v_2 &= \frac{a}{4k^3}\Big[-C_2\sin kz + D_2\cos kz\Big]e^{-kz} \end{aligned}\right\} \qquad (13)$$

The above formulae (11), (12), (13) are all functions of $H_1$, but do not contain $H$. Thus it is to be noticed that the convection current in such a deep sea will be determined by the thickness of the heterogeneous stratum and the gradient of the density of sea-water only, but *it is independent of the total depth* $H$.

For example, let us consider the cases where the thickness of the heterogeneous stratum $H_1$ is equal to $\frac{1}{4}D$, $\frac{1}{2}D$, $D$ and $1.25D$ respectively, $D$ being the "depth of wind current" or "depth of the frictional influence" as it is called by Ekman.

1. See § 4, the case where $S_y = 0$.

Then $kH_1 = \frac{\pi}{4}, \frac{\pi}{2}, \pi, 1\frac{1}{4}\pi$, since $D = \frac{\pi}{k}$.

The corresponding values of coefficients in the current formulae are given by eqs. (11) and (12) as follows:—

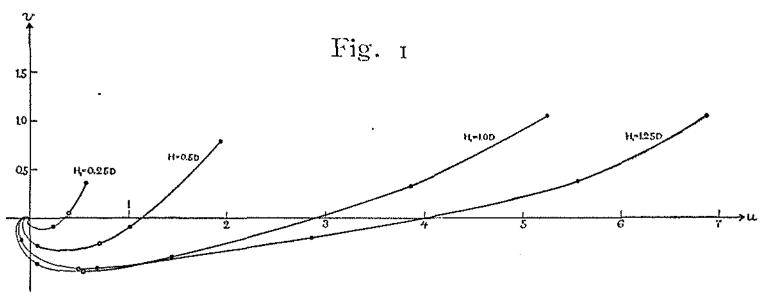
| $kH_1$ | $\pi/4$ | $\pi/2$ | $\pi$ | $1.25\pi$ |
|---|---|---|---|---|
| $A_1$ | 0 | −0.1039 | −0.0216 | 0 |
| $B_1$ | +0.3224 | +0.1039 | −0.0216 | −0.0139 |
| $C_1$ | −1 | −1.1039 | −1.0216 | −1 |
| $D_1$ | +0.6776 | +0.8961 | +1.0216 | +1.0139 |
| $C_2$ | +0.5509 | +1.3013 | −12.592 | −36.894 |
| $D_2$ | +0.6776 | +3.3014 | +12.592 | +1.0139 |

Substituting these in eq. (13), and calculating the values of $u$ and $v$ at $z=H_1$ and at the intervals of $0.4D$ from surface to depth $4D$, we get the following table, in which, for simplicity, $\frac{a}{4k^3}$ is taken as unit velocity.

| $kz$ \ $kH_1$ | $\frac{\pi}{4}$ | | $\frac{\pi}{2}$ | | $\pi$ | | $1\frac{1}{4}\pi$ | |
|---|---|---|---|---|---|---|---|---|
| | $u$ | $v$ | $u$ | $v$ | $u$ | $v$ | $u$ | $v$ |
| 0 | +0.571 | +0.355 | +1.934 | +0.792 | +5.240 | +1.043 | +6.854 | +1.028 |
| 0.4π | +0.232 | −0.090 | +1.008 | −0.082 | +3.861 | +0.318 | +5.591 | +0.375 |
| 0.8π | −0.004 | −0.071 | +0.072 | −0.278 | +1.431 | −0.391 | +2.841 | −0.158 |
| 1.2π | −0.020 | −0.005 | −0.069 | −0.044 | +0.064 | −0.407 | +0.674 | −0.520 |
| 1.6π | −0.003 | +0.005 | −0.018 | +0.015 | −0.105 | −0.054 | −0.080 | −0.230 |
| 2.0π | +0.001 | +0.001 | +0.003 | +0.006 | −0.024 | +0.024 | −0.070 | +0.002 |
| 2.4π | +0.000 | −0.000 | +0.002 | +0.000 | +0.003 | +0.008 | −0.007 | +0.015 |
| 2.8π | — | — | +0.000 | −0.000 | +0.003 | +0.000 | +0.001 | +0.001 |
| 3.2π | — | — | — | — | +0.000 | −0.000 | +0.000 | −0.000 |
| 3.6π | — | — | — | — | — | — | — | — |
| 4.0π | — | — | — | — | — | — | — | — |
| $kH_1$ | +0.396 | +0.041 | +0.686 | −0.270 | +0.544 | −0.544 | +0.499 | −0.527 |

Drawing hodographs with these values, we obtain Fig. 1.

Fig. 1



In the diagram, the thick lines correspond to currents in the upper heterogeneous stratum and the thin lines to currents in the lower homogeneous part. Comparing this diagram with that of Ekman and Stefansen, we perceive some difference in them notwithstanding much resemblance in the main features. We can not judge, however, whether the difference originates from the fundamental assumptions in the theory or merely from the process of approximation in the numerical calculation, because we do not know Ekman's current formulae.

From the above table or graph, we get also the shallowest depth at which velocity $v$ becomes nil, as follows :—

| $H_1$ | 0.25 $D$ | 0.5 $D$ | 1.0 $D$ | 1.25 $D$ |
|---|---|---|---|---|
| $z$ where $v=0$ | 1.20 $H_1$ | 0.74 $H_1$ | 0.58 $H_1$ | 0.60 $H_1$ |

Thus we notice that the layer of no-current in $y$-direction will be tolerably different from the horizontal isobaric surface explained in the next Chapter.

## II. Depth of the Horizontal Isobaric-layer

§ 3. *Total Flow S*

In order to determine the depth of the isobaric layer which lies horizontal, we must prepare the integration of current from surface to bottom i. e., the total flow of current.

Let $S_x$ and $S_y$ be the total flow in the direction of $x$ and $y$ respectively, that is

$$S_x = \int_0^{H_1} u_1 dz + \int_{H_1}^{H} u_2 dz,$$

$$S_y = \int_0^{H_1} v_1 dz + \int_{H_1}^{H} v_2 dz.$$

And let $S$ denote the resultant of the above two components, then

$$S = S_x + iS_y = \int_0^{H_1} w_1 dz + \int_{H_1}^{H} w_2 dz$$

$$= \frac{a}{4k^3}\left[\frac{1}{(1+i)k}\left(K_1 e^{(1+i)kz} - K_1' e^{-(1+i)kz}\right) - k(d-z)^2\right]_0^{H_1}$$

$$+\frac{a}{4k^3}\Big[\frac{1}{(1+i)k}(K_2e^{(1+i)kz}-K_2'e^{-(1+i)kz})$$

$$+2k(d-H_1)z\Big]_{H_1}^{H}$$

$$=\frac{a}{4k^4}\Big[\frac{1-i}{2}\Big\{K_2e^{(1+i)kH}-K_2'e^{-(1+i)kH}$$

$$+(K_1-K_2)e^{(1+i)kH_1}$$

$$-(K_1'-K_2')e^{-(1+i)kH_1}-(K_1-K_1')\Big\}$$

$$+k^2\Big\{2(d-H_1)(H-H_1)-(d-H_1)^2+d^2\Big\}\Big] \qquad\text{(14)}$$

Substituting (5) and (6) in this, we finally get[1]

$$S=\frac{a}{4k^4}\Big[(1-i)K_1\sinh(1+i)kH+i\Big\{\cosh(1+i)k(H-H_1)-e^{-(1+i)kH}\Big\}$$

$$+k^2\Big\{H_1^2-2H(H_1-d)\Big\}\Big] \qquad\text{(15)}$$

Or putting $A_1-iB_1$ instead of $K_1$, and separating the equation into two parts, real and imaginary, we have

$$S_x=\frac{a}{4k^4}\Big[A_1(\cosh kH\sin kH+\sinh kH\cos kH)$$

$$+B_1(\cosh kH\sin kH-\sinh kH\cos kH)$$

$$-\Big\{\sinh k(H-H_1)\sin k(H-H_1)+e^{-kH}\sin kH\Big\}$$

$$+k^2\Big\{H_1^2-2H(H_1-d)\Big\}\Big] \qquad\text{(16)}$$

$$S_y=\frac{a}{4k^4}\Big[A_1(\cosh kH\sin kH-\sinh kH\cos kH)$$

$$-B_1(\cosh kH\sin kH+\sinh kH\cos kH)$$

$$+\Big\{\cosh k(H-H_1)\cos k(H-H_1)-e^{-kH}\cos kH\Big\}\Big] \qquad\text{(17)}$$

If the sea is so deep that $e^{-k(H-H_1)}$ is negligible compared with 1, the above equations reduce to

1. Though $S$, $S_x$, $S_y$ can be expressed in simpler form by using equation (7), we adopt the form of (15) because it is rather convenient for calculation of the horizontal isobaric layer.

$$S_x = \frac{a}{4k^2}\left\{H_1^2 - 2H(H_1 - d)\right\} \quad \text{.......................................} (16')$$

$$S_y = 0 \quad \text{.......................................} (17')$$

§ 4. *Determination of d corresponding to* $S_y = 0$

Suppose a case where the earth is covered throughout with an ocean of uniform depth, and the density of sea-water varies from equator to pole according to latitude only. Or conceive another case in which a long straight coast exists and from every part of it land-water spreads equally far off. In both cases the total flow in the direction of gradient of density must be zero when the sea has reached a steady state.

Thus in many important cases we may take the equation of continuity as

$$S_y = \int_0^{H_1} v_1 dz + \int_{H_1}^{H} v_2 dz = 0 \quad \text{.......................................} (18)$$

The calculation of this equation may be executed by substituting eqs. (9) and (10) for $A_1$ and $B_1$ in eq. (17), but it is easier to use the complex eqs. (15) and (8). That is, the terms which contain $A_1$ and $B_1$ in eq. (17) correspond to the imaginary parts in the following expression

$$(1-i)K_1 \sinh(1+i)kH = \frac{\sinh(1+i)kH \cosh(1-i)kH}{2\cosh(1+i)kH \cosh(1-i)kH} \times$$

$$\left[\begin{array}{l}(1-i)^2\{\sinh(1+i)k(H-H_1) + e^{-(1+i)kH}\} \\ +(1-i)2k(H_1-d)\end{array}\right]$$

$$= \frac{\sinh 2kH + i \sin 2kH}{\cosh 2kH + \cos 2kH} \times$$

$$\left[\begin{array}{l}-i\{\sinh k(H-H_1)\cos k(H-H_1) + e^{-kH}\cos kH\} \\ +\cosh k(H-H_1)\sin k(H-H_1) - e^{-kH}\sin kH + (1-i)k(H_1-d)\end{array}\right].$$

Hence eq. (18) becomes

$$0 = -\sinh 2kH\left\{\sinh k(H-H_1)\cos k(H-H_1) + e^{-kH}\cos kH\right\}$$

$$+\sin 2kH\left\{\cosh k(H-H_1)\sin k(H-H_1) - e^{-kH}\sin kH\right\}$$

$$-k(H_1-d)(\sinh 2kH - \sin 2kH)$$

$$+(\cosh 2kH+\cos 2kH)\Big\{\cosh k(H-H_1)\cos k(H-H_1) - e^{-kH}\cos kH\Big\}.$$

From this, the formula for the depth of the horizontal isobaric layer is given by

$$\frac{H_1-d}{H_1}=$$

$$\frac{\cosh k(H+H_1)\cos k(H-H_1)+\cosh k(H-H_1)\cos k(H+H_1)-2\cosh kH\cos kH}{kH_1(\sinh 2kH-\sin 2kH)}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots\cdots(19)$$

Especially when the ocean is very deep but its heterogeneous stratum is very thin as compared with the whole depth, the above equation may be written as

$$\frac{H_1-d}{H_1}\doteq 2\frac{(\cosh kH_1\cos kH_1-1)\cos kH+\sinh kH_1\sin kH_1\sin kH}{kH_1e^{kH}}, \quad (19')$$

and hence $d$ is very nearly equal to $H_1$, and even $(H_1-d)H$ tends to zero as $H$ increases indefinitely.

On the other hand, if the heterogeneous stratum extends to the bottom i. e., $H=H_1$, we have

$$\frac{H_1-d}{H_1}=\frac{\cosh 2kH+\cos 2kH-2\cosh kH\cos kH}{kH(\sinh 2kH-\sin 2kH)}, \quad \cdots\cdots\cdots(19'')$$

and in this case $d/H_1$ is nearly equal to $\frac{3}{8}=0.375$ for a very shallow sea, and approaches to 1 for a very deep ocean.

The general tendency of the values of $d/H_1$ will be shown in the following table and Fig. 2.

From the table we see that as the total depth $H$ increases in comparison with $D$ or $H_1$, the value of $d/H_1$ approaches to unity more rapidly than in the previous paper which neglected the effect of Coriolis' force.

Now since $d/H_1$ corresponds to the ratio of level difference due to the density inequality in dynamical equilibrium to that in statical, we can conclude as follows.

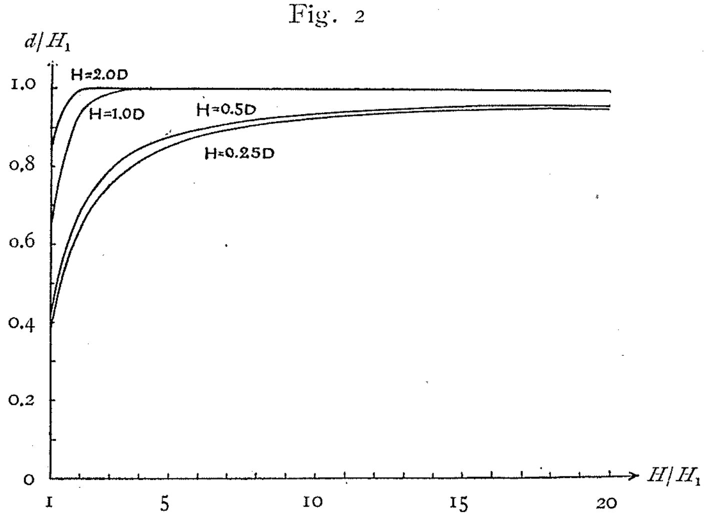
The rule that difference or variation in the height of sea-level due to the local inequality or variation of the density of sea-water

Values of $d/H_1$

| $H/H_1$ \ $H/D$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 6 | 8 | 10 | 20 | $\rightarrow\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.378 | 0.417 | 0.653 | 0.841 | 0.920 | 0.947 | 0.960 | 0.968 | 0.984 | $1.000 = 1 - \frac{1}{kH}$ |
| 2 | 0.639 | 0.688 | 0.945 | 1.014 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | |
| 4 | 0.812 | 0.844 | 0.993 | 1.002 | | | | | | |
| 6 | 0.877 | 0.896 | 0.998 | 1.000 | | | | | | |
| 8 | 0.906 | 0.922 | 0.999 | | | | | | | |
| 10 | 0.927 | 0.937 | 1.000 | | | | | | | |
| 12 | 0.939 | 0.948 | | | | | | | | |
| 14 | 0.946 | 0.955 | | | | | | | | |
| 16 | 0.953 | 0.961 | | | | | | | | |
| 18 | 0.958 | 0.965 | | | | | | | | |
| 20 | 0.962 | 0.969 | | | | | | | | |
| $\infty$ | 1.000 | 1.000 | | | | | | | | |

All the blanks = 1.000

Fig. 2




may be calculated statically, is much more strongly confirmed by taking Coriolis' force into account than by neglecting it.

There remains a matter that calls for a little more explanation. On rare occasions, according to the values of $H$ and $H_1$, e. g., when $H = 2D$ and $H/H_1 = 2$ or 4 in the above table, the formula (18) gives a value of $d/H_1$ which exceeds unity by a very small amount. Mathematically this is possible and forms no objection to the foregoing treatment, as far as the said boundary conditions and differential equation (2) hold.

From the physical point of view, however, since $d/H_1$ must not become greater than unity, such an ocean would not strictly reach the steady state indicated by differential equation (2) and the boundary conditions.

Nevertheless, if we notice that the excess over unity is very slight and that such a case always lies between possible cases in which $d/H_1$ is smaller than unity, we may say that even though such a sea might not exactly reach a steady state, it will fluctuate so near the state corresponding to the above solution that the deviation may be neglected.

Lastly for a very deep sea, since $(H_1-d)H$ tends to zero, eq. (16′) may be written as

$$S=S_x=\frac{a}{4k^2}H_1^2=\frac{g\frac{\partial\rho_1}{\partial y}}{4\rho\omega\sin\lambda}H_1^2 \qquad\qquad (16'')$$

§ 5. *Case of an Enclosed Sea.*

If an enclosed sea is in a steady state, the total flow must be zero in the direction of $x$ as well as $y$. In this case a surface-slope $\gamma_x$ will generally be produced in the direction of $x$ also, in addition to the slope, $\gamma_y=\frac{1}{\rho}\frac{\partial\rho_1}{\partial y}\cdot d$, in the $y$-direction corresponding to $d$. Consequently there should result a slope-current, of which total flow is given by Ekman[1] as follows :—

$$S_x'=\frac{g\cdot\sin\gamma_x}{4k\omega\sin\lambda}\cdot\frac{\sinh 2kH-\sin 2kH}{\cosh 2kH+\cos 2kH} \qquad\qquad (20)$$

$$S'_y=\frac{g\cdot\sin\gamma_x}{4k\omega\sin\lambda}\cdot\left(\frac{\sinh 2kH+\sin 2kH}{\cosh 2kH+\cos 2kH}-2kH\right) \qquad\qquad (21)$$

Then the equations of continuity for an enclosed sea will be

$$S_x+S'_x=0 \qquad\qquad (22)$$

$$S_y+S'_y=0 \qquad\qquad (23)$$

where $S_x$ and $S_y$ are to be taken from eqs. (16) and (17).

By these two equations (22) and (23), we can easily determine the two quantities $\gamma_x$ and $d$.

1. Loc. cit.

If we put for simplicity

$$\left.\begin{aligned}
F&=\sinh 2kH-\sin 2kH\\
G&=\sinh 2kH+\sin 2kH-2kH(\cosh 2kH+\cos 2kH)\\
P&=\sinh k(H+H_1)\sin k(H-H_1)+\sinh k(H-H_1)\sin k(H+H_1)\\
&\quad-2\sinh kH\sin kH+k^2H_1^2(\cosh 2kH+\cos 2kH)\\
Q&=\cosh k(H+H_1)\cos k(H-H_1)+\cosh k(H-H_1)\cos k(H+H_1)\\
&\quad-2\cosh kH\cos kH\\
m&=\frac{b}{g\rho k}=\frac{1}{\rho}\frac{\partial\rho_1}{k\partial y},
\end{aligned}\right\}\qquad(24)$$

then eqs. (22) and (23) take form

$$\left.\begin{aligned}
F.\sin\gamma_x+m[P+G.k(H_1-d)]&=0\\
G.\sin\gamma_x+m[Q-F.k(H_1-d)]&=0
\end{aligned}\right\},$$

from which we get

$$\left.\begin{aligned}
k(H_1-d)&=\frac{FQ-GP}{F^2+G^2}\\
\sin\gamma_x&=-m\frac{FP+GQ}{F^2+G^2}
\end{aligned}\right\}\qquad(25)$$

Especially when the sea is very deep ($e^{-kH}\ll 1$), the above equations give

$$\left.\begin{aligned}
&\gamma_x=0\\
&H_1-d=\frac{H_1^2}{2H},\quad\text{or}\quad\frac{d}{H_1}=1-\frac{H_1}{2H}
\end{aligned}\right\}\qquad(25')$$

## III. Case of no Bottom-friction

§ 6. *The Currents*

Hitherto we have assumed the bottom-current zero as the bottom-condition. Next let us consider what difference will occur if we adopt the condition that there is no friction at the sea-bottom.

All the equations up to (6) are also applicable to this case, but eq. (7) must be replaced by

$$0=\left|\frac{\partial w_2}{\partial z}\right|_{z=H}=K_2e^{(1+i)kH}-K_2'e^{-(1+i)kH}\qquad(26)$$

Substitute (5) and (6) in (26), then we have

$$K_1 = A_1 - iB_1$$
$$= \frac{(1-i)\{\cosh(1+i)k(H-H_1) - e^{-(1+i)kH}\}}{2\sinh(1+i)kH} \qquad \text{.....................(27)}$$

or

$$A_1 = \frac{1}{(\cosh 2kH - \cos 2kH)}\Big[(\sinh kH \cos kH - \cosh kH \sin kH) \times$$
$$\Big\{\cosh k(H-H_1)\cos k(H-H_1) - e^{-kH}\cos kH\Big\}$$
$$+ (\sinh kH \cos kH + \cosh kH \sin kH) \times$$
$$\Big\{\sinh k(H-H_1)\sin k(H-H_1) + e^{-kH}\sin kH\Big\}\Big] \qquad \text{...........(28)}$$

$$B_1 = \frac{1}{(\cosh 2kH - \cos 2kH)}\Big[(\sinh kH \cos kH + \cosh kH \sin kH) \times$$
$$\Big\{\cosh k(H-H_1)\cos k(H-H_1) - e^{-kH}\cos kH\Big\}$$
$$- (\sinh kH \cos kH - \cosh kH \sin kH) \times$$
$$\Big\{\sinh k(H-H_1)\sin k(H-H_1) + e^{-kH}\sin kH\Big\}\Big] \qquad \text{...........(29)}$$

Combining these equations with (5′), (6′) and (3′), we can determine the currents $u$ and $v$.

§ 7. *The Total Flow and the Horizontal Isobaric-layer*

Let us now calculate the total flow $S = S_x + iS_y$. In the present case, all the terms containing $K$s in eq. (14) will cancel one another, and there remains

$$S = S_x + iS_y = \frac{a}{4k^2}\Big\{H_1^2 - 2H(H_1 - d)\Big\}$$

$$\therefore \quad \left.\begin{aligned} S_x &= \frac{a}{4k^2}\Big[H_1^2 - 2H(H_1 - d)\Big] \\ S_y &= 0 \end{aligned}\right\} \qquad \text{....................................(30)}$$

Hence, the condition $S_y = 0$ can not determine the depth of the horizontal isobaric layer.

If, however, there is any circumstance such as to make $S_x = 0$, e. g. as in an enclosed sea, the corresponding value of $d$ will be given by

$$\frac{d}{H_1} = 1 - \frac{H_1}{2H} \qquad (31)$$

This coincides with the result[1] obtained in the previous paper for the same bottom-condition but without taking Coriolis' force into account.

It must be here noticed that eqs. (31) and (25′) are identical in form but quite different in nature. There is no limitation of $H$ and $H_1$ for eq. (31), while eq. (25′) is restricted to a very deep sea with a comparatively thin heterogeneous upper stratum. For example eq. (31) can be used even when $H_1 = H$, and it gives

$$\frac{d}{H_1} = \frac{1}{2} = \frac{4}{8}$$

## Summary

We have investigated the currents and the horizontal isobaric surface in a two-layer ocean, taking the effect of the earth's rotation into account, and the chief results obtained are as follows.

1. Though the currents are similar to the diagram given by Ekman and Stefansen, there seems to be some difference. Since Ekman's mathemetical formulae are not known, we can not find where the origin of the difference will lie.

2. Generally speaking, the layer in which the velocity ($v$) in the direction of the gradient of density is nil, does not coincide with the layer in which the horizontal pressure-gradient is nil.

3. In a tolerably deep sea the ratio of the depth of the horizontal isobaric layer ($d$) to the thickness of the heterogeneous stratum ($H_1$) will be practically equal to unity, except when $H_1$ is nearly equal to the total depth $H$.

This rule is made more decisive and conspicuous by the action of Coriolis' force.

4. When local inequality of density extends from surface to bottom i.e. $H = H_1$, the ratio $d/H_1$ becomes minimum and may attain a value from $\frac{3}{8}$ to $\frac{4}{8}$ according to the bottom condition and the depth of the sea. This least value of $d/H_1$ coincides with that in the case where Coriolis' force is neglected.

5. It follows from the third item that in a deep ocean the level

1. Nomitsu, loc. cit. p. 118, eq. (16).

difference or its variation due to the difference in density of sea-water can be treated statically. The rule proves more decisive and more reliable in the present paper than in the previous paper.

6. The only exception is such a sea that the depth is very small and the difference in density or its variation extends nearly to the sea bottom. The level-difference or variation calculated statically will be too large to represent the actual state. The actual value will sometimes be only $\frac{3}{8}$ to $\frac{4}{8}$ of the statical one. The effect of the earth's rotation in this case hardly exists.

7. If the sea is sufficiently deep, and if there is no circumstance for producing a surface-slope in the $x$-direction and the equation of continuity demands only $S_y = 0$, then the total flow in the direction perpendicular to the gradient of water-density will take the value

$$S_x = \frac{g\frac{\partial \rho_1}{\partial y}}{4\rho\omega\sin\lambda} H_1^2,$$

where $\lambda$ is the latitude and $\rho_1$ the density of water in heterogeneous part.