A Theory of the Vibration of Japanese Hanging Bells

By

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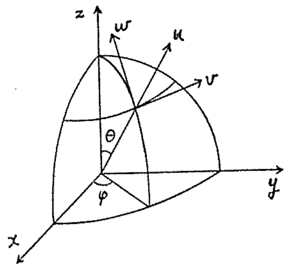
Abstract

The frequencies of the vibration of the Japanese hanging bell are calculated on the assumption that it is a circular cylinder with a hemispherical cap and no line traced upon the middle surface of the shell undergoes extension when the bell is sounding. It is found that the frequencies of the partial tones of a Japanese bell, of which the Poisson's ratio is $1/3$ and the ratio of the length of the cylindrical part to its radius is $3/2$, are in the ratios $359 : 981 : 1847 : \ldots\ldots$

The shape of the Japanese bell may roughly be represented by a circular cylinder with a hemispherical cap. But its thickness is not uniform everywhere, being very thin at the portion where the cylindrical part changes to the spherical form and very thick at the edge as well as at the top of the bell. Since an exact calculation is indeed scarcely to be hoped for, we assume, in the present paper, that the thickness of the shell is uniform throughout the bell. The vibration of a spherical shell[1] and of a cylindrical shell[2] are treated by Lord Rayleigh on the assumption that the shell is deformed by pure bending. Following Lord Rayleigh, we assume that any line on the middle surface of the shell undergoes no extension when the bell is sounding.

Let r, θ, φ be the spherical polar coordinates of any point of a hemispherical shell in the equilibrium state, and let (u, v, w) represent

1. Lord Rayleigh, Lond. Math. Soc. Proc. **13**, 4–6 (1881); Scientific Papers, vol. 1, pp. 551–562.
2. Lord Rayleigh, Roy. Soc. Proc., **45**, 105–123 (1888); Scientific Papers, vol. 3, pp. 217–232.



the displacement of that point in any state of vibration, u being the component along the radius vector, v that along the tangent to the parallel in the increasing direction of φ, and w that along the tangent to the meridian in the decreasing direction of θ. Then the most general possible forms for u, v, w, when the boundaries of the shell are two circles of latitude and the displacement is that of pure bending, are given by[1]

$$\left.\begin{aligned} u &= -\sum\Big\{(n+\cos\theta)A_n\tan^n\frac{\theta}{2}\cos(n\varphi+\alpha_n) \\ &\qquad -(n-\cos\theta)B_n\cot^n\frac{\theta}{2}\cos(n\varphi+\beta_n)\Big\}, \\ v &= sin\theta\sum\Big\{A_n\tan^n\frac{\theta}{2}\sin(n\varphi+\alpha_n)-B_n\cot^n\frac{\theta}{2}\sin(n\varphi+\beta_n)\Big\}, \\ w &= -\sin\theta\sum\Big\{A_n\tan^2\frac{\theta}{2}\cos(n\varphi+\alpha_n)+B_n\cot^n\frac{\theta}{2}\cos(n\varphi+\beta_n)\Big\}; \end{aligned}\right\} \quad (1)$$

where A_n and B_n are functions of time t, α_n and β_n pure constants, and the summation is taken with possible integral values of n from zero to infinity. But, since the terms for $n=0$ and $n=1$ correspond to the displacements of the shell when it moves as a rigid body, they may be omitted from the summation. Moreover to apply equations (1) to a hemispherical shell, since the displacement at the pole $\theta=0$ must be finite, we must reject the terms containing $\cot^n\frac{\theta}{2}$. Thus we get

$$\left.\begin{aligned} u &= -\sum(n+\cos\theta)A_n\tan^n\frac{\theta}{2}\cos(n\varphi+\alpha_n), \\ v &= \sin\theta\sum A_n\tan^n\frac{\theta}{2}\sin(n\varphi+\alpha_n), \\ w &= -\sin\theta\sum A_n\tan^n\frac{\theta}{2}\cos(n\varphi+\alpha_n). \end{aligned}\right\} \quad (2)$$

Next let a, φ, z be the cylindrical coordinates of any point on a circular cylinder of radius a and u, v, w be the components of dis-

1. A. E. H. Love, Math. Theory of Elasticity, 4th ed., p. 508.

placement in the directions of increase of a, φ and z respectively. Then for a circular cylinder of finite length, when the displacement is restricted to pure bending, the most general possible forms for u, v, w are given by[1]

$$\left.\begin{aligned} u&=\sum n\{A'_n\sin(n\varphi+\alpha_n')+B_n'z\sin(n\varphi+\beta_n')\},\\ v&=\sum\{A_n'\cos(n\varphi+\alpha_n')+B'_nz\cos(n\varphi+\beta'_n)\},\\ w&=-\sum\frac{a}{n}B_n'\sin(n\varphi+\beta_n'),\end{aligned}\right\}\quad(3)$$

where A_n' and B_n' are functions of t, and α_n' and β_n' are pure constants.

If we consider the bell as made up by the hemisphere being connected with the cylinder, as its cap, since the displacements given by (2) and (3) must be equal at the connecting plane, which may be taken as $z=0$ passing through the center of the hemisphere, we get

$$\left.\begin{aligned} nA'_n\sin(n\varphi+\alpha_n')&=-nA_n\cos(n\varphi+\alpha n),\\ A_n'\cos(n\varphi+\alpha_n')&=A_n\sin(n\varphi+\alpha_n),\\ -\frac{a}{n}B_n'\sin(n\varphi+\beta_n')&=-A_n\cos(n\varphi+\alpha_n),\end{aligned}\right\}$$

or

$$\left.\begin{aligned} \alpha_n'&=\frac{\pi}{2}+\alpha_n, & A_n'&=-A_n,\\ \beta_n'&=\frac{\pi}{2}+\alpha_n, & B_n'&=\frac{n}{a}A_n.\end{aligned}\right\}\quad(4)$$

Substituting these values of A_n', B_n' α_n', β_n' given by (4) in (3), we obtain

$$\left.\begin{aligned} u&=-\sum n\left(1-\frac{n}{a}z\right)A_n\cos(n\varphi+\alpha_n),\\ v&=\sum\left(1-\frac{n}{a}z\right)A_n\sin(n\varphi+\alpha_n),\\ w&=-\sum A_n\cos(n\varphi+\alpha_n).\end{aligned}\right\}\quad(5)$$

In order to calculate the frequencies of vibration of the bell, first let us obtain its kinetic and potential energies. Let ρ be the density

1. A. E. H. Love, loc. cit. p. 506.

of the material and $2h$ be the thickness of the shell. Then the kinetic energy per unit area of the shell is evidently given by

$$\rho h\left\{\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right\}.$$

If the length of the cylindrical part is l, its kinetic energy T_c is

$$T_c=\int_{-l}^{0}\int_0^{2\pi}\rho h\left\{\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right\}ad\varphi dz.$$

When the values of u, v, w of (5) are substituted in the above equation, all the terms containing products of sines or cosines with different values of n vanish in the integration with respect to φ, as do also those which contain $\cos(n\varphi+\alpha_n)\sin(n\varphi+\alpha_n)$. Accordingly

$$T_c=\pi\rho alh\sum\left\{(n^2+2)+(n^2+1)\left(1+\frac{1}{3}\frac{nl}{a}\right)\frac{nl}{a}\right\}\left(\frac{dA_n}{dt}\right)^2. \quad (6)$$

The potential energy of bending per unit area of the cylindrical shell is[1]

$$\frac{1}{2}D\{k^2+2(1-\sigma)\tau^2\}, \qquad (7)$$

where k and τ are given by[2]

$$k\equiv-\frac{1}{a^2}\left(\frac{\partial^2 u}{\partial\varphi^2}+u\right),$$

$$\tau\equiv\frac{1}{a}\frac{\partial}{\partial z}\left(v-\frac{\partial u}{\partial\varphi}\right),$$

and D is the flexural rigidity

$$D\equiv\frac{2}{3}\frac{Eh^3}{1-\sigma^2}=\frac{4}{3}\frac{\mu h^3}{1-\sigma},$$

E, σ, and μ being Young's modulus, Poisson's ratio, and the modulus of rigidity respectively. Using the values of (5), we get

$$k=-\frac{1}{a^2}\sum n(n^2-1)\left(1-\frac{n}{a}z\right)A_n\cos(n\varphi+\alpha_n),$$

$$\tau=\frac{1}{a^2}\sum n(n^2-1)A_n\sin(n\varphi+\alpha_n).$$

1. A. E. H. Love, loc. cit. p. 503.
2. A. E. H. Love, loc. cit. p. 507.

The potential energy V_c of the cylindrical portion is obtained by integrating the expression (7) over the shell; and we get

$$V_c=\int_{-l}^{0}\int_{0}^{2\pi}\frac{1}{2}D\{k^2+2(1-\sigma)\tau^2\}ad\varphi dz$$

$$=\frac{1}{2}\frac{\pi Dl}{a^3}\sum n^2(n^2-1)^2\left\{(3-2\sigma)+\frac{nl}{a}+\frac{1}{3}\frac{n^2l^2}{a^2}\right\}A_n^2$$

$$=\frac{2}{3}\frac{\pi\mu lh^3}{(1-\sigma)a^3}\sum n^2(n^2-1)^2\left\{(3-2\sigma)+\frac{nl}{a}+\frac{1}{3}\frac{n^2l^2}{a^2}\right\}A_n^2. \quad (8)$$

The kinetic enegy T_s and the potential energy V_s of the hemispherical portion are given by[1]

$$T_s=\pi\rho a^2h\sum\left(\frac{dA_n}{dt}\right)^2\int_0^{\frac{\pi}{2}}\sin\theta\{2\sin^2\theta+(n+\cos\theta)^2\}\tan^{2n}\frac{\theta}{2}d\theta,$$

$$V_s=\frac{8}{3}\frac{\pi\mu h^3}{a^2}\sum n^2(n^2-1)^2A_n^2\int_0^{\frac{\pi}{2}}\tan^{2n}\frac{\theta}{2}\frac{d\theta}{\sin^3\theta}.$$

By the transformation of the variable $1+\cos\theta=x$, we easily get the relation

$$\int_0^{\frac{\pi}{2}}\sin\theta\{2\sin^2\theta+(n+\cos\theta)^2\}\tan^{2n}\frac{\theta}{2}d\theta$$

$$=\int_1^2\{(n-1)^2+2(n+1)x-x^2\}\frac{(2-x)^n}{x^n}dx;$$

and similarly by putting $\tan\frac{\theta}{2}=x$, we get

$$\int_0^{\frac{\pi}{2}}\tan^{2n}\frac{\theta}{2}\frac{d\theta}{\sin^3\theta}=\frac{1}{4}\frac{2n^2-1}{n(n^2-1)}.$$

Hence the expressions for T_s and V_s become

$$T_s=\pi\rho a^2h\sum\left(\frac{dA}{dt}\right)^2\int_1^2\{(n-1)^2+2(n+1)x-x^2\}\frac{(2-x)^n}{x^n}dx, \quad (9)$$

$$V_s=\frac{2}{3}\frac{\pi\mu h^3}{a^2}\sum n(n^2-1)(2n^2-1)A_n^2. \quad (10)$$

From (7), (8), (9) and (10), we obtain

1. A. E. H. Love, loc. cit. p. 513.

$$T=\pi\rho a^2 h\sum\Big[(n^2+2)\frac{l}{a}+(n^2+1)\Big(1+\frac{1}{3}\frac{nl}{a}\Big)\frac{nl^2}{a^2}$$
$$+\int_1^2\{(n-1)^2+2(n+1)x-x^2\}\frac{(2-x)^n}{x^n}dx\Big]\Big(\frac{dA_n}{dt}\Big)^2, \qquad (11)$$

$$V=\frac{2}{3}\frac{\pi\mu h^3}{a^2}\sum n^2(n^2-1)^2\Big[\frac{1}{1-\sigma}\Big\{(3-2\sigma)+\frac{nl}{a}$$
$$+\frac{1}{3}\frac{n^2l^2}{a^2}\Big\}\frac{l}{a}+\frac{2n^2-1}{n(n^2-1)}\Big]A_n^2 \qquad (12)$$

as the total energies of the composite shell. The kinetic energy T is a series of $\dot{A}_n^2$ and the potential energy V is a series of A_n^2, so that the coefficients A_n are really the normal coordinates of the system and, by substituting (11) and (12) in Lagrange's equation of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{A}_n}=-\frac{\partial V}{\partial A_n},$$

it follows that A_n is proportional to a simple harmonic function of the time with a frequency $p_n/2\pi$ and p_n is given by

$$p_n^2=\frac{2\mu h^2}{3\rho a^4}\frac{P}{Q},$$

where

$$P\equiv n^2(n^2-1)^2\Big[\frac{1}{1-\sigma}\Big\{(3-2\sigma)+\frac{nl}{a}+\frac{1}{3}\frac{n^2l^2}{a^2}\Big\}\frac{l}{a}+\frac{2n^2-1}{n(n^2-1)}\Big],$$

$$Q\equiv(n^2+2)\frac{l}{a}+(n^2+1)\Big(1+\frac{1}{3}\frac{nl}{a}\Big)\frac{nl^2}{a^2}+\int_1^2\{(n-1)^2+2(n+1)x-x^2\}\frac{(2-x)^n}{x^n}dx. \qquad (13)$$

The integral

$$f(n)\equiv\int_1^2\{(n-1)^2+2(n+1)x-x^2\}\frac{(2-x)^n}{x^n}dx$$

can be evaluated for any integral value of n. $f(2)$, $f(3)$ and $f(4)$ have been calculated by Lord Rayleigh[1] as follows:

1. Lord Rayleigh, Scientific Papers, vol. 1, p. 557; Theory of Sound, 2nd ed., vol. 1, p. 430.

$f(2)=1.52961,$

$f(3)=1.88156,$

$f(4)=2.29609.$

To obtain the ratios of the frequencies of vibration of the Japanese bell, if we assume

$$\frac{l}{a}=\frac{3}{2} \qquad \text{and} \qquad \sigma=\frac{1}{3}$$

as the reasonable values, we find from (13)

$$p_2^2=\frac{2\mu h^2}{3\rho a^4}\times 12.91,$$

$$p_3^2=\frac{2\mu h^2}{3\rho a^4}\times 96.25,$$

$$p_4^2=\frac{2\mu h^2}{3\rho a^4}\times 341.10,$$

and therefore we get

$$p_2 : p_3 : p_4=\sqrt{12.91} : \sqrt{96.25} : \sqrt{341.10}$$

$$=359 : 981 : 1847$$

as the ratios of the frequencies of partial tones.

Equation (13) may be written in the form

$$p_n^2=\frac{\mu h^2}{\rho a^4}F\left(\frac{l}{a}, \sigma, n\right).$$

Therefore we see that the frequencies of corresponding vibrations of similar bells made of the same material are directly proportional to their thickness and inversely to the squares of their radii. If the similarity extend also to the thickness, the frequencies are inversely proportional to their linear dimensions. Hence, if the dimensions are halved, all the tones should rise in pitch by an exact octave.

In equation (13), if we make the length l infinitely small, we obtain

$$\lim_{l\to 0} p_n^2=\frac{2\mu h^2}{3\rho a^4}\,\frac{n(n^2-1)(2n^2-1)}{\int_1^2\left\{(n-1)^2+2(n+1)x-x^2\right\}\frac{(2-x)^n}{x^n}dx}.$$

This is exactly the same as the equation obtained by Lord Rayleigh[1] for the frequencies of vibration of a hemispherical shell. On the other hand, if we make the length l of the cylinder infinitely long in comparison with the diameter, the frequencies are given by

$$\lim_{l\to\infty} p_n^2 = \frac{2\mu h^2}{3\rho(1-\sigma)a^4}\,\frac{n^2(n^2-1)^2}{n^2+1}$$
$$= \frac{Eh^2}{3\rho(1-\sigma^2)a^4}\,\frac{n^2(n^2-1)^2}{n^2+1}.$$

These are also identical with those of a cylindrical shell obtaind by Lord Rayleigh[2].

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1. Lord Rayleigh, Scientific Papers, vol. 1, p. 557; Theory of Sound, 2nd ed., vol. 1, p. 429.
2. Lord Rayleigh, Scientific Papers, vol. 3, p. 231; Theory of Sound, 2nd ed., vol. 1, p. 417.