A Theory of the Rising Stage of Drift Current in the Ocean

I. The Case of No Bottom-Current

By

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(Received December 24, 1932)

Abstract.

Fjeldstadt's solution of the rising stage of drift current in a sea of finite depth is too laborious and inconvenient for practical use. The present author has been able to solve the same problem in a way that is exceedingly easy and very rich in physical meaning and also very convenient for numerical calculation, as he exemplifies with several ocean depths D/10, D/4, D/2, D, and 2D, where D means the so-called "Reibungstiefe".

According to the results, when a constant wind begins to blow over a boundless sea, the current produced may be taken to reach a steady state in time, as shown below.

Depth of the sea	D/10	D/4	D/2	D	2D
Time in pendulum hours	1	6	18	48	120

In the course of his discussion, the writer obtains also the formula for the decay of the steady current when the wind suddenly stops, and shows that the subtraction of this decaying current from Ekman's steady value furnishes the developing stage of current required.

I. Introduction

The steady state of oceanic current which should be produced by a constant wind was beautifully solved by V. W. Ekman,[1] but he could not himself solve the problem of the developing stage of a wind current and reported only Fredholm's solution for an infinitely deep ocean.

1. Ark. f. Mat. Astr. och Fys., Bd. 2, Nr. 11, 1905.

The first paper which dealt with the rising stage of the wind current in a sea of finite depth was perhaps that of Van der Stock,[1] but unfortunately it contains a serious mistake in the calculation and consequently his result can not be used for the case aimed at in the problem. Indeed it may be said that Van der Stock brought together in one formula the two already-known results of Ekman for a steady current and that of Fredholm for the rising stage in an infinitely deep ocean, but no more.

Recently J. E. Fjeldstadt[2] succeeded in solving the remainig case of a wind current, but his method of deduction with integral equations is exceedingly complicated, and almost no numerical calculations are worked out, so that the real mode of development of a current in shallow waters has not been made concretely clear.

The present paper deals with the same problem in the case of constant wind as well as of varying wind, and the solution is obtained in such a simple manner that the whole precedure is probably only a tenth the length of that of Fjeldstadt's method. It is moreover far richer in physical significance, and indicates clearly the mode in which the current declines when the wind ceases, as well as the development when the wind begins to blow. Furthermore, the practical calculation with given numerical data can be easily carried out and some such calculations are given in tables and graphs as concrete examples.

After the present work was accomplished, the writer received a paper from Mr. K. Hidaka[3], who gives, however, simply a somewhat easier proof than Fjeldstadt's.

2. The Course of our Solution

Let us conceive, after Ekman, a rotating homogeneous ocean, which is boundless in horizontal extent and uniform in depth, and over which the wind is blowing uniformly throughout. Choose the coordinate axes such that the z-axis is directed vertically downwards and the y-axis lies on the sea surface and parallel to the direction of the wind.

Let the following notation be used:

T= the tractive force of the wind per unit area of the sea surface,

H= the depth of the sea,

ρ= the density of the sea water.

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1. Beiträge zu Geophysik. **11**, 106 (1911).
2. Zeits. f. angew. Math. u. Mech. **10**, 121 (1930).
3. Memoirs Imp. Mar. Observ. Kobe, **5**, 51 (1932).

μ=the coefficient of viscosity of the water,

$\nu=\mu/\rho$,

u, v=the component velocities of current in the direction of x and y respectively,

$w=u+iv$, i being $\sqrt{-1}$,

t=the time,

ω=the angular velocity of the earth's rotation,

λ=the latitude of the place under consideration,

$\varpi=\omega \sin \lambda$,

$k=\sqrt{\varpi/\nu}=\sqrt{\rho\omega\sin\lambda/\mu}$.

Then the developing stage of the current produced by the wind in a sea which was initially at rest will be represented by the solution of the eq.

$$\frac{\partial w}{\partial t}=\nu\frac{\partial^2 w}{\partial z^2}-2i\varpi w, \tag{1}$$

under the conditions

$$\frac{\partial w}{\partial z}=-i\frac{T}{\mu} \quad \text{at} \quad z=0, \tag{2}$$

$$w=0 \quad \text{at} \quad z=H, \tag{3}$$

$$w=0 \quad \text{when} \quad t=0. \tag{4}$$

It will, however, be very complicated to solve directly the general case in which the wind varies with the time, and so the writer attacks first the case where the wind action T is constant. If the case of constant wind is known, he can at once solve also the case of varying wind by his own mathematical theorem which will be published in the near future.

Now when the wind does not vary with the time, the problem may be reduced to two simpler cases, one of these being the case of steady current and the other the case of declining current. That is, we may put

$$w=w_1-w_2, \tag{5}$$

where w_1 is a function of z only, and satisfies

$$0=\nu\frac{d^2 w_1}{dz^2}-i\,2\varpi w_1, \tag{1$_a$}$$

$$\frac{dw_1}{dz}=-i\frac{T}{\mu} \quad \text{at} \quad z=0, \tag{2$_a$}$$

$$w_1 = 0 \qquad \text{at} \quad z = H, \tag{3$_a$}$$

and w_2 is a function of z and t, such that

$$\frac{\partial w_2}{\partial t} = \nu \frac{\partial^2 w_2}{\partial z^2} - i 2\bar{\omega} w_2, \tag{1$_b$}$$

$$\frac{\partial w_2}{\partial z} = 0 \qquad \text{at} \quad z = 0, \tag{2$_b$}$$

$$w_2 = 0 \qquad \text{at} \quad z = H, \tag{3$_b$}$$

$$w_2 = w_1 \qquad \text{when} \quad t = 0. \tag{4$_b$}$$

The first is nothing but the steady current which Ekman investigated, and the second corresponds to the current which would gradually decay if the wind suddenly stopped after the current had reached the steady value, and the vector difference of such two currents represents the required current in the developing stage.

3. The Steady Part w_1

Although the solution of this part has already been obtained by Ekman in real form, let us form it again with complex variables for later convenience.

Table 1.
Surface values of the steady current

kH	$\bar{u}$	$\bar{v}$
0	0	0
0.1	0.0006	0.1000
0.2	0.0053	0.1999
0.3	0.0179	0.2987
0.4	0.0420	0.3946
0.5	0.0801	0.4840
0.6	0.1329	0.5618
0.7	0.1977	0.6218
0.8	0.2700	0.6623
0.9	0.3417	0.6798
1.0	0.4061	0.6778
1.1	0.4584	0.6616
1.2	0.4970	0.6372
1.3	0.5226	0.6098
1.4	0.5374	0.5832
1.5	0.5440	0.5596
1.6	0.5450	0.5442
1.7	0.5424	0.5241
1.8	0.5377	0.5122
1.9	0.5320	0.5036
2.0	0.5257	0.4974
2.2	0.5154	0.4919
2.4	0.5074	0.4911
2.6	0.5022	0.4925
2.8	0.4995	0.4948
3.0	0.4983	0.4969
3.2	0.4985	0.4989
∞	0.5000	0.5000

The general solution of eq. (1$_a$) is

$$w_1 = Ae^{\alpha z} + Be^{-\alpha z}$$

where $\alpha = (1+i)k$.

The condition (3$_a$) requires

$$-Ae^{\alpha H} = Be^{-\alpha H} = K/2,$$

K being a new arbitrary constant, so that

$$\left. \begin{aligned} w_1 &= K \sinh\alpha(H-z), \\ \alpha &= (1+i)k. \end{aligned} \right\} \tag{6}$$

Next the condition (2$_a$) gives

$$K = \frac{iT}{\mu\alpha \cosh\alpha H} = \frac{(1+i)T}{2k\mu.\cosh(1+i)kH}. \tag{7}$$

Thus Eqs. (6) and (7) determine the steady current, of which the real and imaginary parts are of course identical with u and v in Ekman's formula.

The vertical distribution of w_1 in a sea of various depths can be seen in the diagram of Ekman, but the relation of the surface current to the depth of the sea has not yet been made sufficiently clear. Hence we shall add here a little on that point.

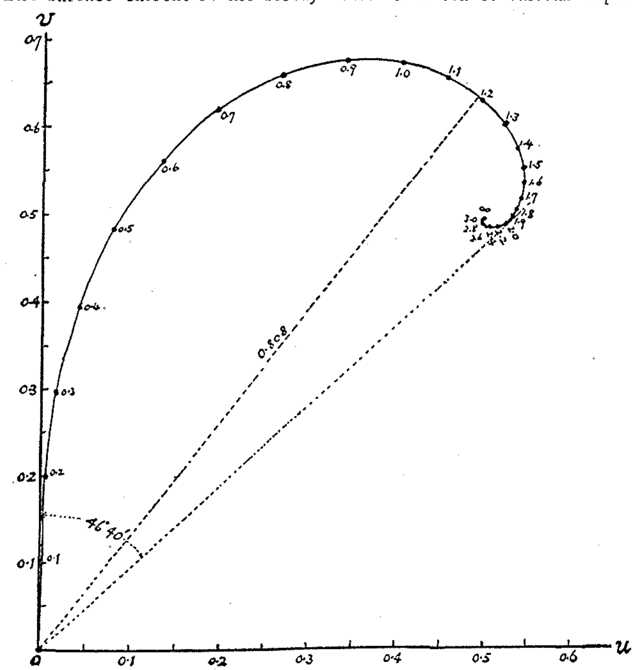
The surface value of a steady current is given by

$$\left.\begin{aligned}\bar{w}_1 &= K\sinh aH = \frac{(1+i)T}{2k\mu}\cdot\tanh(1+i)kH, \\ \text{i. e.,}\qquad \bar{u}_1 &= \frac{T}{k\mu}\left[\frac{1}{2}\cdot\frac{\sinh 2kH - \sin 2kH}{\cosh 2kH + \cos 2kH}\right] \\ \bar{v}_1 &= \frac{T}{k\mu}\left[\frac{1}{2}\cdot\frac{\sinh 2kH + \sin 2kH}{\cosh 2kH + \cos 2kH}\right]\end{aligned}\right\} \tag{8}$$

Table 1 and Fig. 1 were prepared by taking $\frac{T}{k\mu}$ as unit velocity.

Fig. 1.

The surface current in the steady state for a sea of various depths.



The greatest current occurs when $kH = 1.19$ and takes the value 0.808, while the maximum deviation from the wind-direction is 46° 40′ in a sea of $kH = 2.0$.

4. The Decaying Part w_2

Supposing there is a solution of eq. (1_b) in the form $e^{i\beta z + \gamma t}$, we must have

$$\gamma = -\nu\beta^2 - 2i\varpi.$$

Hence $\quad w_2 = (A\cos\beta z + B\sin\beta z)e^{-(\nu\beta^2+2i\varpi)t}$

is obviously a solution of the differential equation. Here β, A and B are all arbitrary constants and may be complex in general.

But condition (2_b) demands

$$B = 0,$$

and then condition (3_b) requires

$$\cos\beta H = 0,$$

or

$$\left.\begin{aligned} &\beta_n = \left(n+\frac{1}{2}\right)\frac{\pi}{H}, \qquad n = 0,\ 1,\ 2,\ 3,\ \ldots\ldots \\ &w_2 = \sum_{n=0}^{\infty} A_n\cos\beta_n z . e^{-(\nu\beta_n^2+i2\varpi)t} \end{aligned}\right\} \qquad (9)$$

and thus

satisfies eqs. (1_b), (2_b) and (3_b).

In order that eq. (9) may suit also the initial condition (4_b), it is sufficient to find A_n such that

$$\sum_{n=0}^{n=\infty} A_n\cos\beta_n z = w_1 = K.\sinh a(H-z). \qquad (10)$$

The left-hand side of this equation is not identical but very similar to Fourier's series, and the value of the coefficients may be determined in a similar way. For β_n as given by (9), we have

$$\beta_m - \beta_n = (m-n)\frac{\pi}{H}, \qquad \beta_m + \beta_n = (m+n+1)\frac{\pi}{H},$$

and so

$$\int_0^H \cos\beta_m x \cdot \cos\beta_n x \cdot dx = 0 \qquad \text{if } m \neq n,$$

$$= \frac{H}{2} \qquad \text{if } m = n.$$

Hence, in order to determine A_n, multiply $\cos\beta_n z \cdot dz$ into both sides of eq. (10) and integrate from $z=0$ to $z=H$. Then we have

$$A_n\int_0^H \cos^2\beta_n z . dz = \int_0^H K\sinh a(H-z).\cos\beta_n z . dz.$$

$$\therefore\ A_n = \frac{2K}{H}\int_0^H \sinh a(H-z)\cos\beta_n z dz$$

$$= \frac{2K}{H}\left| \frac{-a\cosh a(H-z)\cos\beta_n z + \beta_n\sinh a(H-z)\sin\beta_n z}{a^2+\beta_n^2} \right|_0^H$$

$$= \frac{2K.\ a\cosh aH}{H(a^2+\beta_n^2)} = \frac{2iT}{\mu(2ik^2+\beta_n^2)H}.$$

i. e.,

$$A_n = \frac{2T}{\mu H}\cdot\frac{2k^2+i\beta_n^2}{4k^4+\beta_n^4}. \qquad (11)$$

Introducing this in eq. (9), there finally results

$$\left.\begin{aligned} w_2 &= \frac{T}{\mu H}\sum_{n=0}^{\infty}\frac{4k^2+2i\beta_n^2}{4k^4+\beta_n^4}\cdot\cos\beta_n z e^{-(\nu\beta_n^2+i2\varpi)t} \\ &= \frac{T}{\mu H}(\cos 2\varpi t - i\sin 2\varpi t)\sum_{n=0}^{\infty}\frac{4k^2+2i\beta_n^2}{4k^4+\beta_n^4}\cdot\cos\beta_n z\, e^{-\nu\beta_n^2 t} \\ &\text{where} \\ \beta_n &= \left(n+\frac{1}{2}\right)\frac{\pi}{H}, \qquad n=0,\ 1,\ 2,\ 3,\ldots\ldots\ldots \end{aligned}\right\}(12)$$

This series is uniformly convergent for all values of z and t as far as $t \geqq o$, and therefore can be taken as the required solution of the problem.

Separating the above equation into real and imaginary parts and noting that the "Reibungstiefe" of Ekman is $D=\frac{\pi}{k}$, we get

$$\left.\begin{aligned} u_2 &= \frac{T}{\mu k}\cdot\frac{k}{H}\left\{\cos 2\varpi t\sum_{n=0}^{\infty}\frac{4k^2}{4k^4+\beta_n^4}\cdot\cos\beta_n z\, e^{-\nu\beta_n^2 t}\right. \\ &\qquad\left.+\sin 2\varpi t\sum_{n=0}^{\infty}\frac{2\beta_n^2}{4k^4+\beta_n^4}\cos\beta_n z\, e^{-\nu\beta_n^2 t}\right\} \\ &= \frac{T}{\mu k}\cdot\frac{D}{\pi H}\sum_{n=0}^{\infty}\frac{1}{\sqrt{1+(\beta_n^2/2k^2)^2}}\cdot\cos\beta_n z.\, e^{-\nu\beta_n^2 t} \\ &\qquad\times\sin(2\varpi t+\varphi_n), \\ v_2 &= \frac{T}{\mu k}\cdot\frac{k}{H}\left\{\cos 2\varpi t\sum_{n=0}^{\infty}\frac{2\beta_n^2}{4k^4+\beta_n^4}\cos\beta_n z\, e^{-\nu\beta_n^2 t}\right. \\ &\qquad\left.-\sin 2\varpi t\sum_{n=0}^{\infty}\frac{4k^2}{4k^4+\beta_n^4}\cdot\cos\beta_n z\, e^{-\nu\beta_n^2 t}\right\} \\ &= \frac{T}{\mu k}\cdot\frac{D}{\pi H}\sum_{n=0}^{\infty}\frac{1}{\sqrt{1+(\beta_n^2/2k^2)^2}}\cdot\cos\beta_n z\, e^{-\nu\beta_n^2 t} \\ &\qquad\times\cos(2\varpi t+\varphi_n), \end{aligned}\right\}(13)$$

where
$$\left.\begin{aligned} \beta_n &= \left(n+\frac{1}{2}\right)\frac{\pi}{H}, \\ \cot\varphi_n &= \frac{\beta_n^2}{2k^2} = \frac{1}{2}\left(n+\frac{1}{2}\right)^2\frac{D^2}{H^2}. \end{aligned}\right\} \qquad (14)$$

Thus the main feature of a decaying current can be seen at a glance from these formulae, namely

i) The current w_2 approaches zero, oscillating with a period

$$\frac{\pi}{\omega\sin\lambda} = 12 \text{ "pendulum hours".} \qquad (15)$$

ii) Since the convergency factor is

$$e^{-\nu\beta_n^2 t} = e^{-\nu\left(n+\frac{1}{2}\right)^2\frac{\pi^2}{H^2}\cdot t}, \qquad (16)$$

the decaying is very rapid for shallow waters and for larger n.

Table 2 Decaying part w_2

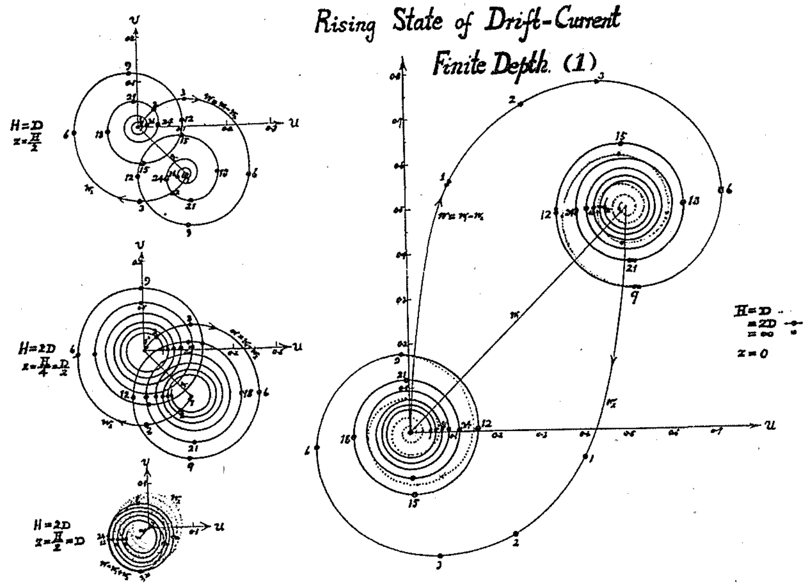
	$H=D/2$				$H=D/4$			
z	0		$H/2$		0		$H/2$	
t	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2
0^h	+0.5451	+0.5451	+0.3369	+0.0698	+0.2595	+0.6573	+0.1770	+0.3028
1	+0.4464	−0.0166	+0.3031	−0.0254	+0.1667	+0.1101	+0.1179	+0.0779
2	+0.2827	−0.1855	+0.1981	−0.1317	+0.0700	+0.0042	+0.0495	+0.0030
3	+0.1162	−0.2323	+0.0820	−0.1642	+0.0220	−0.0110	+0.0156	−0.0078
6	−0.1059	−0.0529	−0.0749	−0.0374	−0.0005	−0.0010	−0.0003	−0.0007
9	−0.0242	+0.0483	−0.0171	+0.0342	−0.0000	+0.0000	−0.0000	−0.0000
12	+0.0220	+0.0110	+0.0156	+0.0078				
15	+0.0050	−0.0101	+0.0036	−0.0071				
18	−0.0046	−0.0023	−0.0032	−0.0016				
21	−0.0010	+0.0021	−0.0007	+0.0015				
24	+0.0010	+0.0005	+0.0004	+0.0007				
36	+0.0000	+0.0000	+0.0000	+0.0000				

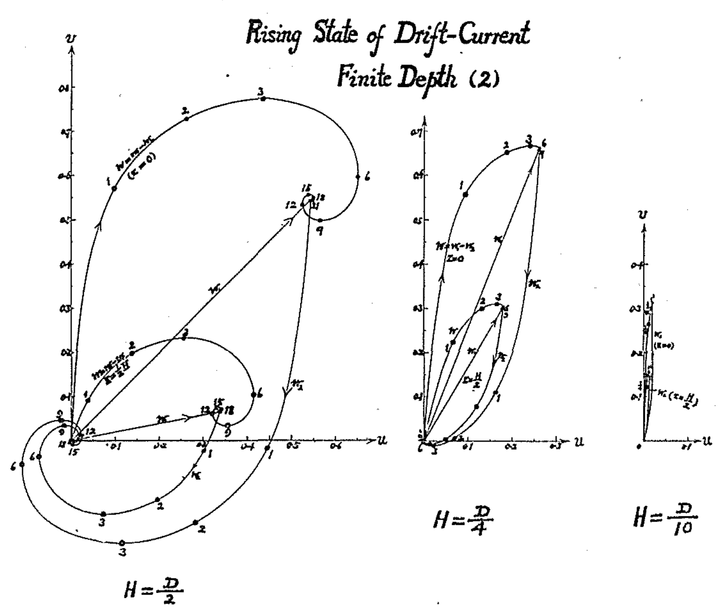
	$H=D/10$			
z	0		$H/2$	
t	u_2	v_2	u_2	v_2
0^h	+0.0166	+0.3123	+0.0141	+0.1559
$1/4$	+0.0101	+0.0484	+0.0072	+0.0342
$1/2$	+0.0032	+0.0092	+0.0023	+0.0064
1^h	+0.0002	+0.0003	+0.0002	+0.0002

Table 2. Decaying part w_2 (Continued)

	$H=2D$						$H=D$			
z	0		$H/4$		$H/2$		0		$H/2$	
t	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2
0^h	+0.5000	+0.5000	+0.1039	−0.1039	−0.0216	−0.0216	+0.4981	+0.4981	+0.1082	−0.1082
1	+0.4011	−0.0617	+0.1008	−0.1109	−0.0215	−0.0216	+0.3993	−0.0636	+0.1051	−0.1152
2	+0.2366	−0.2314	+0.0763	−0.1375	−0.0220	−0.0219	+0.2347	−0.2333	+0.0707	−0.1475
3	+0.0617	−0.2799	−0.0011	−0.1607	−0.0262	−0.0228	+0.0599	−0.2818	+0.0032	−0.1650
6	−0.2140	−0.0289	−0.1529	−0.0076	−0.0531	+0.0132	−0.2157	−0.0311	−0.1467	−0.0015
9	−0.0173	+0.1789	−0.0074	+0.1408	−0.0016	+0.0789	−0.0218	+0.1746	−0.0153	+0.1225
12	+0.1566	+0.0102	+0.1301	+0.0013	+0.0741	−0.0025	+0.1430	+0.0180	+0.1010	+0.0125
15	+0.0086	−0.1408	+0.0047	−0.1211	−0.0006	−0.0764	+0.0147	−0.1174	+0.0104	−0.0830
18	−0.1289	−0.0067	−0.1134	−0.0045	−0.0763	−0.0005	−0.0965	−0.0121	−0.0682	−0.0085
21	−0.0054	+0.1195	−0.0040	+0.1067	−0.0012	+0.0750	−0.0099	+0.0793	−0.0070	+0.0561
24	+0.1117	+0.0046	+0.1009	+0.0036	+0.0729	+0.0015	+0.0652	+0.0082	+0.0461	+0.0058
36	+0.0890	+0.0030	+0.0818	+0.0026	+0.0619	+0.0018	+0.0297	+0.0037	+0.0201	+0.0026
48	+0.0726	+0.0023	+0.0670	+0.0021	+0.0507	+0.0016	+0.0135	+0.0017	+0.0096	+0.0012
60	+0.0596	+0.0019	+0.0551	+0.0017	+0.0421	+0.0013				
72^h	+0.0490	+0.0015	+0.0452	+0.0014	+0.0346	+0.0011				

Fig. 2.


Rising State of Drift-Current
Finite Depth. (1)
H=D
z=H/2
H=2D
z=H/4=D/2
H=2D
z=H/2=D
z=0


Rising State of Drift-Current
Finite Depth (2)
H=D/2
H=D/4
H=D/10

In order to make these points clear in detail we calculated the surface and the mid-layer currents at several pendulum hours in seas of depth

$$H = 2D,\ D,\ D/2,\ D/4,\ D/10$$

respectively, and tabulated them in Table 2. $T/\mu k$ is taken as unit velocity as before.

Plotting these values in hodographs, we get the curves indicated by the letter w_2 in Fig. 2.

4. The Development of a Current by a Constant Wind

Using the values of w_1 and w_2 obtained in the preceding articles, we can calculate the current

$$w = w_1 - w_2, \tag{5}$$

which will be produced when a constant wind suddenly begins to blow over a boundless sea initially at rest.

In Fig. 2 the curves marked by the letter w are the hodographs of such currents in the rising stage. They are identical in shape with w_2 and different from it only in position.

From these hodographs we see that:

i) The shallower the sea, the weaker is the drift current.

ii) The velocity of the current develops, oscillating around the final steady value w_2, with a period of 12 pendulum hours.

iii) The shallower the sea, the shorter the time required to reach the steady current. If the sea is 10 or 20 metres deep, the required time will be only 1 or 2 pendulum hours, and even for the continental shelf of one or two hundred metres, the current will reach the steady value in a few days.

iv) If the depth of the sea is more than $2D$, the development of the current in the first few days is almost the same as that in an infinitely deep sea, and though it will take a considerable number of days to reach the steady state completely, the daily mean will be nearly equal to the final value from the first day.

Lastly, we here notice that Eq. (5), with the aid of (9), (10) and (11), may be written as follows:

$$w = \sum_{n=0}^{\infty} A_n \cos\beta_n z \left[1 - e^{-(\nu\beta_n^2 + 2i\varpi)t}\right]$$

$$= \frac{2iT}{\rho H} \sum_{n=0}^{\infty} \cos\beta_n z . \int_0^t e^{-(\nu\beta_n^2 + i2\varpi)t} . dt. \tag{17}$$

5. The Case of Varying Wind

Knowing the current which should be produced by a constant wind, we can at once derive the formula for the current when the wind varies with the time and T is a function of t such that $T=T(t)$. Indeed by my newly established theorem[1], we can write down directly, from Eq. (17), the current required in the present case as:

$$w=\frac{2i}{\rho H}\sum_{n=0}^{\infty}\cos\beta_n z\int_0^t T(\tau)d\tau . e^{-(\nu\beta_n^2+i2\varpi)(t-\tau)} \tag{18}$$

Let us further consider a few examples of practical importance.

1) *Periodic wind.* If $T(t)=T_0\sin pt$, where p and T_0 are constants, then

$$w=\frac{2iT_0}{\rho H}\sum_{n=0}^{\infty}\cos\beta_n z . \frac{(\nu\beta_n^2+2i\varpi)\sin pt-p.\cos pt+pe^{-(\nu\beta_n^2+i2\varpi)t}}{(\nu^2\beta_n^4-4\varpi^2+p^2)+4i\nu\beta_n^2\varpi}. \tag{19}$$

In the case of a monsoon, we may take $t=\infty$ and assume the current to be in a quasi-steady state, so that

$$w=\frac{2iT_0}{\rho H}\sum_{n=0}^{\infty}\frac{(\nu\beta_n^2+2i\varpi)\sin pt-p\cos pt}{(\nu^2\beta_n^4-4\varpi^2+p^2)+4i\nu\beta_n^2\varpi}\cos\beta_n z. \tag{19'}$$

Thus, the hodograph will be an ellipse or a straight line.

2) *Uniformly increasing wind.* If $T(t)=T_0 \cdot t$, we have

$$w=\frac{2iT_0}{\rho H}\sum_{n=0}^{\infty}\frac{(\nu\beta_n^2+2i\varpi)t-e^{-(\nu\beta_n^2+i2\varpi)t}-1}{(\nu\beta_n^2+2i\varpi)^2}\cos\beta_n z. \tag{20}$$

3) *Exponentially varying wind.* In the case of $T(t)=T_0 e^{\pm pt}$,

$$w=\frac{2iT_0}{\rho H}\sum_{n=0}^{\infty}\frac{e^{\pm pt}-e^{-(\nu\beta_n^2+i2\varpi)t}}{\nu\beta_n^2+2i\varpi\pm p}\cos\beta_n z. \tag{21}$$

4) *Asymptotically increasing wind.* When $T(t)=T_0(1-e^{-pt})$, the current will be equal to the difference between the current due to a constant wind T_0 and the current just obtained in 3).

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1. It will be published soon later in these memoirs or in the Japanese Journal of Physics.

Appendix

1. The Deduction of Fredholm's Formula

Fredholm's formula for an infinitely deep ocean is reported by Ekman[1], but without any proof of how it is derived. It may be obtained by the method of Van der Stock, putting $H=\infty$ from the beginning.

Our formula (17) also will give the same result very simply. For, if the depth H is indefinitely large, we may put

$$\beta_n=\left(\frac{1}{2}+n\right)\frac{\pi}{H}=\beta, \qquad d\beta=\frac{\pi}{H},$$

and write Eq. (17) in the form

$$w=i\frac{2T}{\pi\rho}\int_0^\infty \cos\beta z.\, d\beta\int_0^t e^{-(\nu\beta^2+i2\varpi)t}.dt=i\frac{2T}{\pi\rho}\int_0^t dt e^{-i2\varpi t}\int_0^\infty \cos\beta z.e^{-\nu\beta^2 t}.d\beta.$$

But since $\int_0^\infty \cos\ ax\, e^{-bx^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{b}}e^{-\frac{a^2}{4b}}$,

we have $\quad w=\frac{iT}{\sqrt{\pi\mu\rho}}\int_0\frac{e^{-2i\varpi t}}{\sqrt{t}}.\ e^{-\frac{z^2}{4\nu t}}.dt, \qquad (17')$

which, when separated into real and imaginary parts, gives the formulae of Fredholm.

2. The Mistake of Van der Stock

In the paper of Van der Stock[2] we find a serious mistake in the course of the calculation of the integral

$$\int_0^t \frac{zF+(2H-z)F'}{t}.\ e^{-iat}.dt \qquad \text{at} \quad z=0,$$

where $\quad F=\frac{e^{-\frac{z^2}{4\nu t}}}{\sqrt{t}}, \qquad F'=\frac{e^{-\frac{(z-2H)^2}{4\nu t}}}{\sqrt{t}},$

and $\quad a=2\varpi \quad$ in our notation.

He put $\quad \int_0^t \frac{z\cdot F}{t}e^{-iat}\cdot dt=2\sqrt{\nu\pi}$,

and $\quad \int_0^t \frac{2H-z}{t}.\ F'.\ e^{-iat}.dt=2\sqrt{\nu\pi}.e^{-2(1+i)\sqrt{\frac{a}{2\nu}}H}.$

1. Loc. cit.
2. Loc. cit. p. 110.

The former of these integrals is correct, but the latter is wrong. For, by a substitution

$$\xi^2=\frac{(2H-z)^2}{4\nu t},\qquad d\xi=-\frac{2H-z}{2\sqrt{\nu}}\cdot\frac{dt}{2\sqrt{t^3}},$$

we have $\displaystyle\int_o^t\frac{2H-z}{t}.F e^{-iat}.dt=-4\sqrt{\nu}\int_{\infty}^{\frac{2H-z}{2\sqrt{\nu t}}}e^{-\xi^2-i\frac{a(2H-z)^2}{4\nu\xi^2}}.d\xi,$

which, when $z=o$, becomes

$$=4\sqrt{\nu}\int_o^{\infty}e^{-\xi^2-i\frac{aH^2}{\nu\xi^2}}.d\xi-4\sqrt{\nu}\int_o^{\frac{H}{\sqrt{\nu t}}}e^{-\xi^2-i\frac{aH^2}{\nu\xi^2}}.d\xi$$

$$=2\sqrt{\nu\pi}\,.e^{-2(1+i)\sqrt{\frac{a}{2\nu}}H}-4\sqrt{\nu}\int_o^{\frac{H}{\sqrt{\nu t}}}e^{-\xi^2-i\frac{aH^2}{\nu\xi^2}}.d\xi.$$

The last term is not zero, except when $H=\infty$, or when $t=\infty$ for a finite value of H. Consequently the result of Van der Stock only holds for the cases treated already by Ekman and Fredholm, but no more.

I think that Van der Stock went wrong in changing the upper limit in intergrating from t to ξ. For $z=o$ and $t=t$, he perhaps put $\xi=o$ instead of $\xi=H/\sqrt{\nu t}$.

3. The Assumption of Fjeldstadt for an Ice-covered Sea

Fjeldstadt conceived a sea which is covered with ice throughout and assumed the resistance of ice proportional to the surface current of water, so that he put the surface condition of the drift-current produced by wind as

$$\frac{\partial w}{\partial z}-rw=-\frac{iT}{\mu}\qquad\text{for}\quad z=o,\qquad(2')$$

instead of Eq. (2).

If we accept this condition, our method of treating the problem will give with equal facility an entirely similar solution of the current as before, i. e, the steady part in this case is

$$w_1=K.\sinh a(H-z),\qquad a=(1+i)k,$$

$$K=\frac{iT}{\mu(a\cosh aH-r\sinh aH)},\qquad(7')$$

and the decaying part is

$$w_2 = \sum_{n=1}^{\infty} A_n \sin\beta_n(H-z) . \; e^{-(\nu\beta_n^2 + i2\varpi)t} \,, \tag{10'}$$

where

$$A_n = \frac{2}{H} \cdot \frac{\beta_n^2 H^2 + r^2 H^2}{\beta_n^2 H^2 + (1 + rH) rH} \int_0^H K . \sinh\alpha\xi . \sin\beta_n\xi \cdot d\xi \tag{11'}$$

and β_n must be the roots of

$$\tan\beta H = -\beta/r = -\frac{\beta H}{rH} \tag{9'}$$

But according to my view, assumption (2′) is not right, because the wind action T minus the skin resistance of the water $\mu\frac{\partial w}{\partial z}$ must produce acceleration of the covering ice and when the steady state is reached the ice acts only as a means of transmitting the action of the wind to the water without any diminution. Thus condition (2′) should be replaced by

$$\mu\frac{\partial w}{\partial z} - \sigma\frac{\partial w}{\partial t} = -iT \qquad \text{at} \quad z=0, \tag{2''}$$

σ being the areal density of ice.

Under this condition, however, we can not apply the method described in the present paper as well as Fjeldstadt's, and some other procedure must be contrived.