On the Development of the Slope Current and the Barometric Current in the Ocean

I. The Case of No Bottom-Current

By

Takaharu Nomitsu

(Received Feb. 25, 1933)

Abstract

The writer deals first with the development of a slope current theoretically and showed that the current will oscillate asymptotically around a steady value with a period of 12 pendulum hours and that the time required for it to become steady is substantially equal to that for the drift current.

The "barometric current" produced by a local difference in atmospheric pressure is next discussed, and it is emphasized that this may have in reality much more importance than is usually considered.

1. Introduction

When a surface slope exists in a sea, owing to any causes such as the wind action or the supply of tributary water from the land, etc., there must be produced a current, which is generally called the "Slope Current". The steady value of the current corresponding to a constant and uniform slope was investigated by Ekman[1]; but on the development of the slope current, there is hardly any report except for a rather popular explanation in the paper of Ekman.

The present writer[2] succeeded very easily in solving the rising stage of a drift current produced by wind, and he will first here show that a similar method may be applied to the problem of the development of slope current.

1) Ekman, V. W., On the influence of the earth's rotation on ocean current. Ark. f. Mat. Astr. o. Fys., Bd. 2 Nr. 11. 1905.

2) Nomitsu, T., A theory of the rising stage of drift current in the ocean. These Memoirs, A, **16.** 161 (1933).

Secondly, the importance of the oceanic current due to difference in atmospheric pressure seems to have been hitherto unduly undervalued. We can, indeed, scarcely find any literature on this subject except an account[1] of the current in the channels connecting the Baltic proper with the North Sea due to the difference in the barometric pressure over those two bodies of water. This is perhaps because there is yet no theoretical solution for the development of current corresponding to a rapid (such as diurnal) change in the atmospheric pressure, and also because it is probably presumed that a very slow variation such as an annual one, or still more the permanent gradient in the barometric pressure, might produce a reverse slope of the sea surface to balance each other and consequently it would not contribute to any appreciable ocean-current.

Careful consideration, however, shows that in real seas the effect of barometric pressure can not be so simply treated. If the water basin is comparatively small in size and the barometric gradient is uniform over the whole area, then a balancing slope of water surface will immediately be produced in the reverse direction and neutralise the effect of the atmospheric pressure so that no appreciable current may be seen, as is commonly assumed. But consider the case of an ocean of vast extent where the marked gradient of barometric pressure is either limited to only a narrow strip of the ocean surface so that the water can easily escape laterally, or extended over the whole ocean but changing in direction from place to place so as to facilitate a complete horizontal circulation of the water. In such cases, accumulation of water in one side of the ocean, and consequently a surface slope, will not easily arise, and thus it may sometimes be possible that the current due to the barometric gradient can keep its intensity. The writer[2] has discussed the wind current on similar lines and verified his contentions along the Japanese Coast. It is very desirable that this subject should be investigated practically by ocean steamers, to decide whether the same is true for the effect of atmospheric pressure or not.

At any rate, the writer proposes to give the name "*Barometric Current*" to that ocean current directly due to difference in barometric pressure and would emphasize the point that the barometric current

1. Krümmel, O., Handbuch der Ozeanographie, Bd. II, S. 514—519 (1911).

2. Nomitsu, T., The causes of the annual variation of the mean sea level etc., These Memoirs, A, **10**, 156 (1927).

may be as important as the drift current, the slope current or the convection current, especially when considered with respect to the circulation in deep layers of the ocean. Fortunately the mathematical treatment of it is entirely the same as that of the slope current, and the following theory can be applied to the barometric as well as the slope current.

2. Rising Stage of Current under Constant Slope

Conceive, as in my previous paper[1] on drift current, a boundless sea of uniform depth and uniform density and suppose that the water is initially motionless, and that at the time $t=0$, a steady slope γ (or a barometric gradient) begins to come out in an invariable direction. Take that direction as the y-axis and for the rest use the same notation as before; then the horizontal pressure-gradient in the water will be

$$\frac{\partial p}{\partial x}=0, \qquad -\frac{\partial p}{\partial y}=g\rho\sin\gamma, \tag{1}$$

and the equation of the slope-current will become

$$\frac{\partial w}{\partial t}=\nu\frac{\partial^2 w}{\partial z^2}-i2\bar{\omega}w+ig\sin\gamma \tag{2}$$

together with the conditions

$$\frac{\partial w}{\partial z}=0 \quad \text{for} \quad z=0, \tag{3}$$

$$w=0 \qquad \text{for} \quad z=H, \tag{4}$$

and

$$w=0 \quad \text{when} \quad t=0. \tag{5}$$

Divide w into two parts such that

$$w=w_1-w_2, \tag{6}$$

where w_1 is a function of z but not of t, and

$$0=\frac{d^2w_1}{dz^2}-i2k^2w_1+i\frac{g\sin\gamma}{\nu}, \tag{2a}$$

$$\frac{dw_1}{dz}=0 \quad \text{for} \quad z=0, \tag{3a}$$

1. These Memoirs, A, **16**, 161 (1933).

$$w_1 = 0 \qquad \text{for} \quad z = H, \tag{4a}$$

and w_2 is a function of both z and t, and

$$\frac{\partial w_2}{\partial t} = \nu \frac{\partial^2 w_2}{\partial z^2} - i2\bar{\omega} w_2, \tag{2b}$$

$$\frac{\partial w_2}{\partial z} = 0 \quad \text{for} \quad z = 0, \tag{3b}$$

$$w_2 = 0 \qquad \text{for} \quad z = H, \tag{4b}$$

and

$$w_2 = w_1 \quad \text{when} \quad t = 0. \tag{5b}$$

Steady part w_1: Now the part w_1 is nothing but the steady current which Ekman solved, i. e.,

$$w_1 = \frac{g \sin \gamma}{2\bar{\omega}} \left\{ 1 - \frac{\cosh az}{\cosh aH} \right\},$$

where

$$a = (1 + i)k, \quad k = \sqrt{\frac{\bar{\omega}}{\nu}} = \sqrt{\frac{\omega \sin \lambda}{\nu}}. \tag{7}$$

As the mode of vertical distribution of the steady current can be seen in Ekman's diagram, here the surface value of it only will be a little more discussed. The surface current takes the value

$$\bar{w}_1 = \frac{g \sin \gamma}{2\bar{\omega}} \left\{ 1 - \frac{1}{\cosh aH} \right\} = \frac{g \sin \gamma}{2\,\omega \sin \lambda} \left\{ 1 - \frac{1}{\cosh(1+i)kH} \right\},$$

or in the x and the y direction separately

$$\bar{u}_1 = \frac{g \sin \gamma}{\omega \sin \lambda} \left\{ \frac{1}{2} - \frac{\cosh kH \cos kH}{\cosh 2k\,H + \cos 2kH} \right\},$$

$$\bar{v}_1 = \frac{g \sin \gamma}{\omega \sin \lambda} \cdot \frac{\sinh kH \sin kH}{\cosh 2kH + \cos 2kH}.$$

From these formulae we obtain Table 1 and the curve in Fig. 1, the coefficient $\frac{g \sin \gamma}{\omega \sin \lambda}$ (or twice the so-called "Gradient Current" in a deep sea) being assumed as unity.

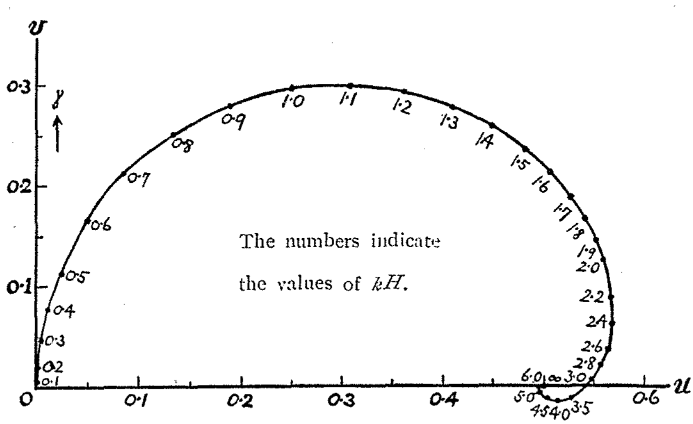
Table 1

Surface values of the steady current.

kH	$\bar{u}$	$\bar{v}$
0	0	0
0.1	0.0000	0.0050
0.2	0.0003	0.0200
0.3	0.0033	0.0475
0.4	0.0106	0.0787
0.5	0.0251	0.1199
0.6	0.0497	0.1654
0.7	0.0864	0.2106
0.8	0.1343	0.2500
0.9	0.1899	0.2792
1.0	0.2508	0.2955
1.1	0.3098	0.2991
1.2	0.3639	0.2919
1.3	0.4108	0.2768
1.4	0.4500	0.2567
1.5	0.4817	0.2334
1.6	0.5067	0.2103
1.7	0.5260	0.1870
1.8	0.5405	0.1645
1.9	0.5512	0.1434
2.0	0.5587	0.1236
2.2	0.5665	0.0891
2.4	0.5673	0.0607
2.6	0.5637	0.0379
2.8	0.5572	0.0202
3.0	0.5492	0.0070
3.5	0.5283	−0.0106
4.0	0.5120	−0.0138
4.5	0.5023	−0.0109
5.0	0.4981	−0.0065
6.0	0.4953	−0.0011
∞	0.5000	0

Fig. 1

Surface values of the steady current according to depth of sea.



Decaying part w_2: The second part w_2 corresponds to a case where the above steady current will decay if the slope suddenly disappears, and it differs from the decaying part of the drift current only in its initial value of w_1.

Hence, just as in the previous paper, the solution which satisfies eqs. (2_b), (3_b) and (4_b) may be put as

$$\left.\begin{aligned} w_2 &= \sum_{n=0}^{\infty} A_n e^{-(\nu\beta_n^2+i2\varpi)t} \cdot \cos\beta_n z, \\ \text{where} & \\ \beta_n &= \left(n+\frac{1}{2}\right)\frac{\pi}{H}. \end{aligned}\right\} \tag{8}$$

In order to let eq. (8) be also in conformity with the remaining condition (5_b), we must have

$$A_n = \frac{2}{H}\int_0^H w_1 \cos\beta_n z\cdot dz = \frac{2}{H}\cdot\frac{g\sin\gamma}{2\varpi}\int_0^H\left(1-\frac{\cosh az}{\cosh aH}\right)\cos\beta_n z\cdot dz.$$

But

$$\int_0^H\left(1-\frac{\cosh az}{\cosh aH}\right)\cos\beta_n z dz = \left|\frac{\sin\beta_n z}{\beta_n} - \frac{a\sinh az\cos\beta_n z+\beta_n\cosh az\sin\beta_n z}{(a^2+\beta_n^2)\cosh aH}\right|_0^H$$

$$= (-1)^n\left[\frac{1}{\beta_n} - \frac{\beta_n}{a^2+\beta_n^2}\right] = (-1)^n\frac{a^2}{\beta_n(a^2+\beta_n^2)}$$

$$= (-1)^n\frac{2ik^2}{\beta_n(2ik^2+\beta_n^2)},$$

$$\therefore \quad A_n = \frac{g\sin\gamma}{\varpi}\cdot\frac{(-1)^n}{\beta^n H}\cdot\frac{1+i\dfrac{\beta_n^2}{2k^2}}{1+\left(\dfrac{\beta_n^2}{2k^2}\right)^2}. \tag{9}$$

Thus we get

$$w_2 = \sum_{n=0}^{\infty} A_n e^{-(\nu\beta_n^2+i2\varpi)t}\cdot\cos\beta_n z$$

$$= \frac{g\sin\gamma}{\omega\sin\lambda}\cdot\frac{1}{\pi}(\cos 2\varpi t - i\sin 2\varpi t)\times$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\dfrac{1}{2}\right)}\cdot\frac{1+i\dfrac{\beta_n^2}{2k^2}}{1+\left(\dfrac{\beta_n^2}{2k^2}\right)^2}e^{-\nu\beta_n^2 t}\cos\beta_n z, \tag{10}$$

or

$$
\left.
\begin{aligned}
u_2 = \frac{g \sin \gamma}{\omega \sin \lambda} \cdot \frac{1}{\pi} \Bigg[& \cos 2\varpi t \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)} \cdot \frac{e^{-\nu\beta_n^2 t}}{1+\left(\frac{\beta_n^2}{2k^2}\right)^2} \cos \beta_n z \\
& + \sin 2\varpi t \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)} \cdot \frac{\frac{\beta_n^2}{2k^2} \cdot e^{-\nu\beta_n^2 t}}{1+\left(\frac{\beta_n^2}{2k^2}\right)^2} \cos \beta_n z \Bigg], \\
v_2 = \frac{g \sin \gamma}{\omega \sin \lambda} \cdot \frac{1}{\pi} \Bigg[& \cos 2\varpi t \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)} \cdot \frac{\frac{\beta_n^2}{2k^2} \cdot e^{-\nu\beta_n^2 t}}{1+\left(\frac{\beta_n^2}{2k^2}\right)^2} \cos \beta_n z \\
& - \sin 2\varpi t \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)} \cdot \frac{e^{-\nu\beta_n^2 t}}{1+\left(\frac{\beta_n^2}{2k^2}\right)^2} \cos \beta_n z \Bigg].
\end{aligned}
\right\} \quad (11)
$$

Development of the current w: Having obtained the values of w_1 and w_2, we can at once calculate numerically the value of w with the relation

$$w = w_1 - w_2.$$

Table 2 and the curves in Fig. 2 are calculated from these equations and represented quite in the same way as in the case of the development of drift current. Here also the coefficient $g \sin\gamma/\omega \sin \lambda$ is taken as unity.

Table 2

Decaying part w_2.

	$H=\infty$		$H=D$				$H=D/2$			
z	0		0		$H/2$		0		$H/2$	
t	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2
0^h	+0.5000	0	+0.5431	+0.0000	+0.5000	+0.0992	+0.5000	+0.2173	+0.3657	+0.2035
1	+0.4330	−0.2500	+0.4762	−0.2501	+0.4337	−0.1490	+0.4346	−0.0295	+0.3114	−0.0163
2	+0.2500	−0.4330	+0.3007	−0.4329	+0.2655	−0.3195	+0.2811	−0.1860	+0.1993	−0.1313
3	0	−0.5000	+0.0467	−0.4992	+0.0581	−0.3756	+0.1161	−0.2322	+0.0826	−0.1642
6	−0.5000	0	−0.4205	−0.0498	−0.3012	−0.0396	−0.1059	−0.0529	−0.0749	−0.0374
9	0	+0.5000	−0.0428	+0.3473	−0.0312	+0.2463	−0.0241	+0.0483	−0.0170	+0.0342
12	+0.5000	0	+0.2857	+0.0356	+0.2021	+0.0251	+0.0220	+0.0110	+0.0156	+0.0078
15	0	−0.5000	+0.0293	−0.2348	+0.0208	−0.1661	+0.0050	−0.0100	+0.0035	−0.0071
18	−0.5000	0	−0.1930	−0.0241	−0.1365	−0.0171	−0.0045	−0.0023	−0.0041	−0.0016
21	0	+0.5000	−0.0198	+0.1586	−0.0140	+0.1121	−0.0012	+0.0024	−0.0008	+0.0017
24	+0.5000	0	+0.1303	+0.0163	+0.0923	+0.0115	+0.0012	+0.0006	+0.0008	+0.0003
36	+0.5000	0	+0.0595	+0.0074	+0.0420	+0.0053	+0.0000	+0.0000	+0.0000	+0.0000

z \ t	$H=D/4$			
	0		$H/2$	
t	u_2	v_2	u_2	v_2
0^h	+0.1268	+0.2447	+0.5905	+0.1862
1	+0.0834	+0.0550	+0.0590	+0.0389
2	+0.0350	+0.0021	+0.0247	+0.0014
3	+0.0110	−0.0055	+0.0078	−0.0039
6	−0.0003	−0.0005	−0.0002	−0.0004
9	−0.0000	+0.0000	−0.0000	+0.0000

z \ t	$H=D/10$			
	0		$H/2$	
t	u_2	v_2	u_2	v_2
0^h	+0.0040	+0.0490	+0.0027	+0.0368
1/4	+0.0087	+0.0088	+0.0062	+0.0062
1/2	+0.0007	+0.0018	+0.0005	+0.0014
1	+0.0000	+0.0001	+0.0000	+0.0000

Fig. 2, a

Developing stage of slope current.

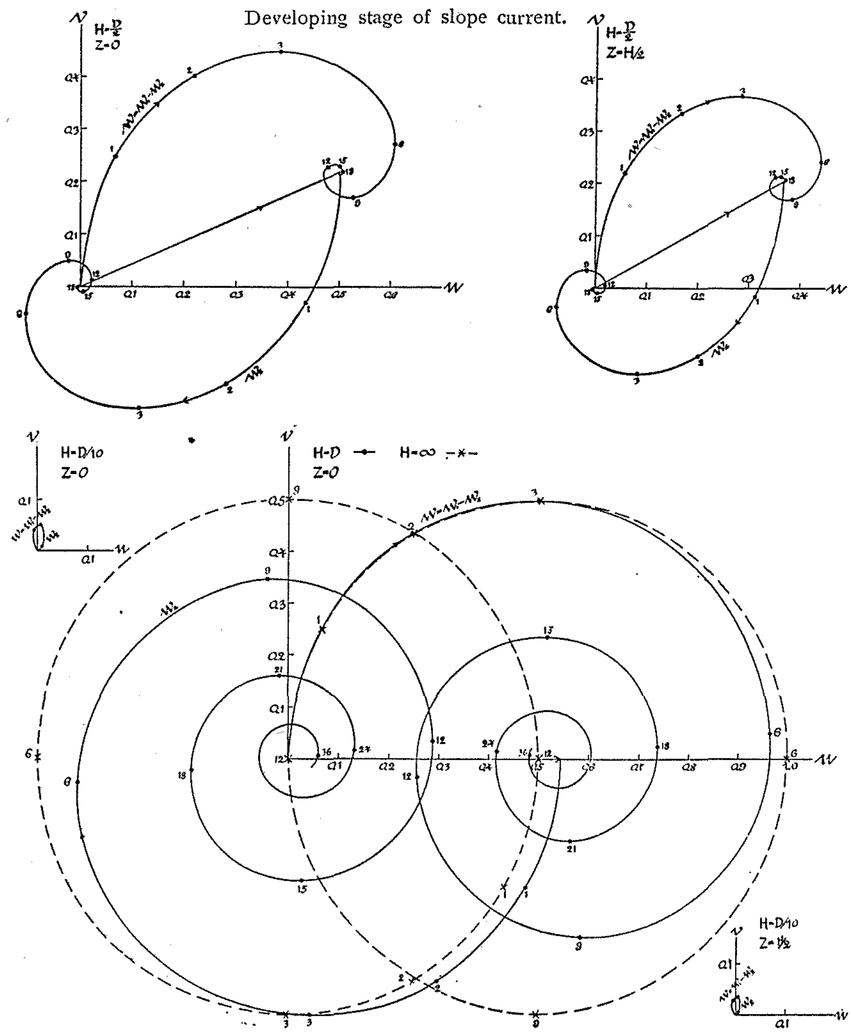
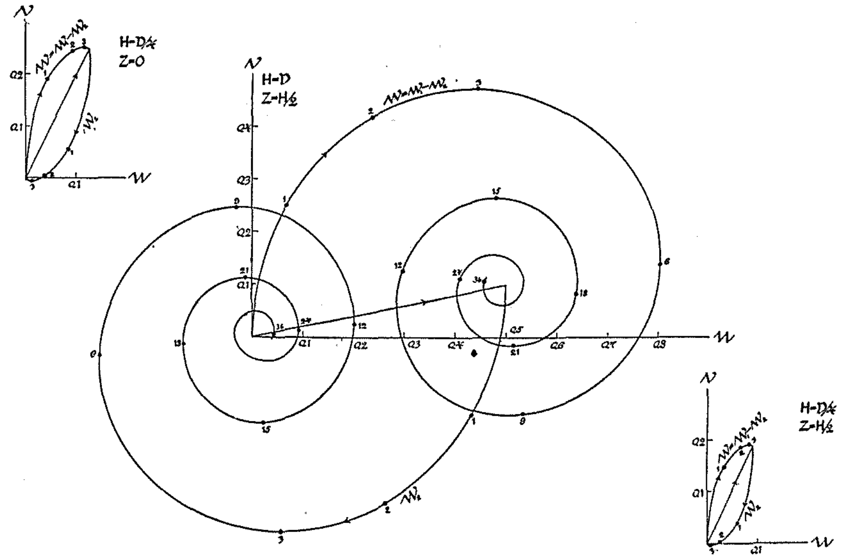


Fig. 2, b



Glancing at the results we see that:

1) The slope current also oscillates asymptotically around the steady value with a period of 12 pendulum hours.

2) The time required for it to become steady is substantially the same as that for the drift current, as shown below.

Depth of sea	D/10	D/4	D/2	D
Time required (hours)	1/2	3	18	48

3) If the depth of the sea is more than $2D$, the development of the current in the first few days is almost the same as that in an infinitely deep sea, and though it will take a considerable number of days to reach the steady state completely, the daily mean will be nearly the same as the final value from the first day.

4) The shallower the sea, the weaker is the current.

It is here to be noticed that the mathematical formula of w may conveniently be written in the form:

$$w=\sum_{n=0}^{\infty}A_n\cos\beta_n z[1-e^{-(\nu\beta_n^2+i2\varpi)t}]$$

$$=\sum_{n=0}^{\infty}A_n\cos\beta_n z(\nu\beta_n^2+i2\varpi)\int_0^t e^{-(\nu\beta_n^2+i2\varpi)t}\,dt$$

$$=\sum_{n=0}^{\infty}\frac{g\sin\gamma}{\varpi}\cdot\frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi}2ik^2\nu\cdot\cos\beta_n z\int_0^t e^{-(\nu\beta_n^2+i2\varpi)t}\,dt$$

$$=\frac{i2g\sin\gamma}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\cos\beta_n z\int_0^t e^{-(\nu\beta_n^2+i2\varpi)t}\,dt. \quad (12)$$

Infinitely deep ocean. If the depth H is infinite, we put

$$\beta_n=\left(n+\frac{1}{2}\right)\frac{\pi}{H}=\beta, \quad \frac{\pi}{H}=d\beta,$$

and

$$w=\frac{2ig\sin\gamma}{\pi}\int_0^t dt\, e^{-i2\varpi t}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\cos\beta z e^{-\nu\beta^2 t}.$$

Except when $t=\infty$ and $z=\infty$, the above expression can be very much simplified as follows.

Let ε be such a *small finite* value of β that it makes $\cos\beta z\, e^{-\nu\beta^2 t}\doteqdot 1$, and let the corresponding value of n be $\bar{n}$ (a great number), i. e.,

$$\left(\bar{n}+\frac{1}{2}\right)\frac{\pi}{H}=\varepsilon.$$

In order to evaluate $\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\cos\beta z e^{-\nu\beta^2 t}$, we separate it into two parts $\sum_{n=0}^{\bar{n}}+\sum_{n=\bar{n}}^{\infty}$.

Then, the first part $=\sum_{n=0}^{\bar{n}}\frac{(-1)^n\cos\beta z}{\left(n+\frac{1}{2}\right)}e^{-\nu\beta^2 t}=\sum_{n=0}^{\bar{n}}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}$

$$=2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots\ldots+\text{infinite terms}\right)$$

$$=4\left(\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\ldots\ldots\ldots\ldots+\quad\text{,,}\qquad\text{,,}\quad\right)$$

$$=\frac{\pi}{2}$$

and

the second part $=\sum_{n=\bar{n}}^{\infty}\frac{(-1)^n\cos\beta z}{\left(n+\frac{1}{2}\right)}e^{-\nu\beta^2 t}=-\frac{\pi}{2H}\int_{\varepsilon}^{\infty}\frac{d}{d\beta}\left(\frac{\cos\beta z\cdot e^{-\nu\beta^2 t}}{\beta}\right)d\beta$

$$=\frac{\pi}{2H}\cdot\frac{1}{\varepsilon}=0, \text{ since } H=\infty \text{ and } \varepsilon=\text{finite}.$$

Thus we get

$$\left.\begin{aligned} w &= \frac{2ig\sin\gamma}{\pi}\int_0^t dt\cdot e^{-2i\bar{\omega}t}\frac{\pi}{2} = \frac{g\sin\gamma}{2\bar{\omega}}(1-e^{-2i\bar{\omega}t}), \\ \text{or} \qquad u &= \frac{g\sin\gamma}{2\omega\sin\lambda}(1-\cos 2\bar{\omega}t), \\ v &= \frac{g\sin\gamma}{2\omega\sin\lambda}\sin 2\bar{\omega}t, \end{aligned}\right\} \quad (13)$$

which shows that the hodograph is a circle.

3. Development of Current with Varying Slope

When the slope varies with the time and γ is a function of t such that $\gamma=\gamma(t)$, applying the writer's newly established theorem[1] to eq. (12), we get

$$w = \frac{i2g}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\cos\beta_n z\int_0^t \sin\gamma(\tau)\cdot e^{-(\nu\beta_n^2+i2\bar{\omega})(t-\tau)}\cdot d\tau. \quad (14)$$

Now since γ is always very small in reality, we may put $\sin\gamma \doteqdot \gamma$. Hence if $\gamma(t)$ is linearly increasing, or exponentially or periodically varying, the above integral (14) can be easily executed just as in the case of drift current under a similarly changing wind[2].

4. Barometric Current

A local difference in atmospheric pressure over a sea will also become a cause of a horizontal pressure gradient in the sea water, and consequently of a current which is to be called a *pure barometric current* when considered apart from the action of wind and surface slope.

If there is no surface slope, but a uniform barometric gradient in the y-direction prevails over the whole sea, then the barometric current must obviously be given by putting $-\frac{1}{\rho}\frac{\partial p}{\partial y}$ instead of $g\sin\gamma$ in the slope current formulae in the foregoing articles.

For practical convenience in calculating the pure barometric current from a isobaric chart drawn with pressures in mm. of mercury,

1. It will be published in the near future in these memoirs or in the Japanese Journal of Physics.

2. These Memoirs, A, **16**, 172 (1933).

let us here evaluate the coefficient $\frac{1}{\rho}\frac{\partial p}{\partial y}/\omega \sin\lambda$. If the pressure gradient is G mm. per 1 km. of distance, i. e.,

$$-\frac{\partial p}{\partial y}=G \text{ mm. per km.},$$

and if we take the specific gravity of mercury compared with the normal sea water as 13.2, then the required coefficient (which has a dimension of velocity) will be

$$-\frac{\frac{1}{\rho}\frac{\partial p}{\partial y}}{\omega \sin\lambda}=\frac{13.2\times 980\times 10^{-6}\cdot G}{0.0000729\times \sin\lambda}=178\,\frac{G}{\sin\lambda} \text{ in cms./sec.}$$

$$=3.46\frac{G}{\sin\lambda} \text{ in knots.} \qquad (15)$$

The product of this with Table 1 or 2 gives the values of a barometric current in its steady or decaying state. For instance, on a sea of depth $H>2D$ and of latitude $\lambda=35°$, a uniform gradient of pressure $G=1$ mm./km. will produce a current whose surface velocity in the steady state takes the value

$$u=0.5\times 3.46/0.5736=3.02 \text{ kts.}, \quad v=0.$$

Quite near a continent or in a small region of ring-isobars of a cyclone or an anticyclone, the current will be checked by a reverse surface-slope arising in a few hours to balance the barometric gradient and the fluctuation of sea level due to the variation of atmospheric pressure may be considered statically. But in an ocean of vast extent the circumstances are very different. The water there will be able to flow so freely that the sea surface may keep its level for a long time, notwithstanding the local difference in atmospheric pressure. Thus in an open ocean it seems to the writer possible that the so-called "pressure factor" for sea level may be considerably smaller than 13.2, and that a barometric current can actually coexist with a wind current, even when a long-standing barometric gradient acts.

The writer is glad to hear from Dr. T. Okada that the Japanese captains of some ocean-liners are intending to investigate practically the barometric current, because they have too often found that the current data hitherto considered only are quite inadequate for ocean navigation.