A Note on the Green Function of the Differential Equation $\Phi(y)+\lambda\cdot y=0$

By

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We consider the following self-adjointed equation:

(1) $\Phi(y)\equiv(P\cdot y')'+Q\cdot y=0,$

where the coefficients $P(x)$ and $Q(x)$ are analytic in the interval $a\leqq x\leqq b$.

When $G(x, y)$ is the Green function of equation (1), which satisfies the boundary conditions:

(L) $y(a)=y(b)=0,$

and when $\Gamma(x, y)$ is the Green function of the equation:

(2) $\Psi(y)\equiv\Phi(y)+\lambda\cdot y=0,$

where λ is a parameter, $\Gamma(x, y)$ satisfies the same boundary conditions (L), and it is known that $-\Gamma(x, y)$ *is the reciprocal kernel, considering* $G(x, y)$ *as a kernel.* Generally, in order to find the Fredholm determinant $D(\lambda)$ of $G(x, y)$, we must first find $\Gamma(x, y)$ by calculating iterated kernels of $G(x, y)$, and then from the formula

$$-\int_a^b \Gamma(x, x)dx=D'(\lambda)/D(\lambda)$$

drive it out, but the calculation is very complicated[1].

If we denote by $\varphi(x;\lambda_i)$ the integral of equation (2), corresponding to $\lambda=\lambda_i$, with its first and second derivatives continuous in the

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1. Vivanti-Schwank, Integralgleichungen, 1927, pp. 203-204.

interval $x(a, b)$, and satisfying the boundary conditions (L), then $\varphi(x\,;\lambda_i)$ satisfies the following integral equation:

$$(3) \qquad \varphi(x\,;\lambda_i)=\lambda_i\int_a^b G(x,y)\varphi(y\,;\lambda_i)dy.$$

As to the boundary conditions (L), we have

$$\varphi(a\,;\lambda_i)=\varphi(b\,;\lambda_i)=0.$$

On the other hand, we know that the solutions of (3) can be obtained when we substitute λ_i for λ in the linear function:

$$\sum_{h=1}^{r} c_h \Delta(x_1\ldots\ldots;x_{h-1},x,x_{h+1},\ldots\ldots x_r\,;y_1\ldots\ldots,y_r\,;\lambda)^1,$$

where the coefficients c_s' are arbitrary and r is the rank of the characteristic constant. But from the definition of the function $\Delta(x_1, x_2,\ldots\ldots, x_r\,; y_1, y_2,\ldots\ldots, y_r\,;\lambda)$, we see at once that for any given value of x in the interval, $\varphi(x\,;\lambda)$ becomes an integral function of λ, and depends only on the kernel $G(x, y)$, the constant λ, the rank r and c_s'.

Suppose that $\varphi(b\,;\lambda)$ (or $\varphi(a\,;\lambda)$) involves λ explicitly and that it does not vanish identically for λ, then $\varphi(b\,;\lambda)$ is an integral function of λ, and clearly one of its zeros is λ_i. If we now give the one determinate value r throughout the ranks of the characteristic constants $\lambda_1, \lambda_2, \lambda_3,\ldots\ldots, \lambda_n,\ldots\ldots$, and one pair of values of c_s', the solutions of (3) depend only on λ; then the zeros of the integral function $\varphi(b\,;\lambda)$ are

$$\lambda_1, \lambda_2, \lambda_3,\ldots\ldots, \lambda_n,\ldots\ldots .$$

As the differential equation (2) has at most two independent solutions, we have $r\leqq 2$; accordingly we can conclude that *when all the characteristic constants have always the same rank 2, for the suitable choice of two constants c_1, c_2, the zeros of $\varphi(b\,;\lambda)$ will be identical with those of the determinant $D(\lambda)$ under the exclusion of their multiplicities.*

Now we have seen that the series:

$$(4) \qquad \sum_{h=1}^{+\infty}\frac{1}{\lambda_h^2}$$

converges absolutely, and the determinant may be written as follows:

$$(5) \qquad D(\lambda)=e^{\alpha\lambda}\prod_{h=1}^{+\infty}\left(1-\frac{\lambda}{\lambda_h}\right)e^{\frac{\lambda}{\lambda_h}}.$$

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1. Vivanti-Schwank, loc. cit., pp. 97–107.

By the absolute convergency of series (4), *the integral function* $\varphi(b;\lambda)$ *is the function of the first class with the rank 1, and thus the* λ-*index*[1] *of* $\varphi(b;\lambda)$ *is not greater than 2.* But on the other hand, from Hadamard's theorem, we get

$$\left| G\begin{pmatrix} x_1, x_2, \dots\dots\dots, & x_n \\ y_1, y_2, \dots\dots\dots, & y_n \end{pmatrix} \right| \leqq n^{\frac{n}{2}} N^n,$$

where N is the maximum value of $|G(x,y)|$ throughout the square domain $(a \leqq x \leqq b\,;\, a \leqq y \leqq b)$, and

$$G\begin{pmatrix} x_1, x_2, \dots\dots\dots, & x_n \\ y_1, y_2, \dots\dots\dots, & y_n \end{pmatrix} \equiv |\,G(x_i, y_j)\,|,$$

$$i, j = 1,\ 2,\ 3, \dots\dots n.$$

Thus by observing the maximum absolute value of the coefficient of λ^n in the expansion of $\Delta(x_1, x_2, \dots\dots, x_r\,;\, y_1, y_2, \dots\dots,\ y_r\,;\,\lambda)$ in powers of λ, we obtain the conclusion that *the* μ-*index of the integral function* $\varphi(b;\lambda)$ *is not smaller than* $1/2$, *and therefore the* ν-*index is not greater than 2.*

From these results we have

$$\text{(6)} \qquad \varphi(b\,;\,\lambda) = e^{\alpha' + \beta\lambda + \gamma\lambda^2} \prod_{h=1}^{+\infty} \left(1 - \frac{\lambda}{\lambda_h}\right) e^{\frac{\lambda}{\lambda_h}}.$$

By equating (5) and (6), under the exclusion of multiplicities, we have

$$\text{(7)} \qquad D(\lambda) = e^{p + q\lambda + r\lambda^2} \cdot \varphi(b\,;\,\lambda)\,;\; e^p \varphi(b\,;\,0) = 1.$$

Particularily, *when the differential equation* (2) *has one and only one integral* $\varphi(x\,;\,\lambda_i)$, *corresponding to every characteristic constant* $\lambda_i (i = 1, 2, 3, \dots\dots)$, *which satisfies our conditions* (L), *then the rank of* λ_i *becomes 1, and accordingly its multiplicity is also 1*[2].

Consequently, *if the integral function* $\varphi(b\,;\,\lambda)$ *has the simple zeros only, the above formula* (7) *will be as follows:*

$$\text{(7')} \qquad D(\lambda) = e^{p + q\lambda + r\lambda^2} \cdot \varphi(b\,;\,\lambda),$$

where $e^p \cdot \varphi(b\,;\,0) = 1$.

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1. It is to be noticed that this number is quite different from the characteristic constant; see Vivanti-Schwank, loc. cit., p. 12.

2. Our kernel $G(x, y)$ is symmetric.

And moreover, when the following series:

(8) $$\sum_{i=1}^{+\infty}\frac{x_i^2\varphi(x\,;\lambda_i)\varphi(y\,;\lambda_i)}{\lambda_i}$$

converges uniformly in the domain $(a\leqq x\leqq b\,;a\leqq y\leqq b)$, where the coefficients are so chosen that the system $\{x_i\varphi(x\,;\lambda_i)\}, i=1, 2, 3, \ldots\ldots$ is normalized, both integral functions will become functions of the first class with the rank 0.

Consequently, we can conclude that *if series* (8) *is uniformly convergent, and equation* (2) *has one and only one integral, and if all the zeros of the integral function* $\varphi(b\,;\lambda)$ *are simple, then the equality* (7′) *will become*

(7″) $$D(\lambda)=e^{p+q\lambda+r\lambda^2}.\varphi(b\,;\lambda),$$

where the determinant can be written:

$$D(\lambda)=\prod_{h=1}^{+\infty}\left(1-\frac{\lambda}{\lambda_h}\right).$$

This results from the assumption:

$$\varphi(b\,;0)\neq 0.$$

But if $\varphi(b\,;0)=0$, we need a little modification.

We ought to replace the above stated integral function of $\varphi(b\,;\lambda)$[1] by $\varphi(b\,;\lambda)/\lambda^m$ by choosing such a number m as

$$\lim_{\lambda\to 0}\frac{\varphi(b\,;\lambda)}{\lambda^m}=1.$$

Thus the equality (7″) will become

(7‴) $$D(\lambda)=\frac{1}{\lambda^m}e^{q\lambda+r\lambda^2}.\varphi(b\,;\lambda),$$

where $D(0)=1$.

As applications we consider the following two cases.

I. We take the Bessel equation:

$$\Phi(y)\equiv y''+\frac{1}{4x^2}y=0,$$

where $(L): y(0)=y(1)=0$,

1. Note that the function $\varphi(b\,;\lambda)$ in $\varphi(b\,;\lambda)/\lambda^m$ may not be always an integral function of λ.

then the Green function is

$$G(x,\xi)=\begin{cases}-\sqrt{x\xi}\log\xi \\ -\sqrt{x\xi}\log x\end{cases} \quad \text{for} \quad x \lesseqqgtr \xi.$$

Hence $\quad \Psi(y)\equiv y''+\dfrac{1}{4x^2}y+\lambda y=0.$

For the only solution of $\Psi(y)=0$, we take

$$\varphi(x\,;\lambda)=\sqrt{x}\,.J(x\sqrt{\lambda}\,).$$

By (7′)

$$D(\lambda)=e^{q\lambda+r\lambda^2}.J(\sqrt{\lambda}\,),$$

where $\varphi(1\,;0)=1$.

Now the series, corresponding to (8),

$$2\sqrt{x\xi}\,.\sum_{h=1}^{+\infty}\frac{J(x\sqrt{\lambda_h}\,).J(\xi\sqrt{\lambda_h}\,)}{\lambda_h.J'^2(\sqrt{\lambda_h}\,)}$$

converges uniformly in the square domain $(0\leqq x\leqq 1\,;\ 0\leqq\xi\leqq 1)$, hence

$$D(\lambda)=\prod_{h=1}^{+\infty}\left(1-\frac{\lambda}{\lambda_h}\right).$$

If we now rewrite the Bessel function $J(\sqrt{\lambda}\,)$ into Weierstrass' product-formula, we have

$$J(\sqrt{\lambda}\,)=\prod_{h=1}^{+\infty}\left(1-\frac{\lambda}{\lambda_h}\right).$$

Comparing these, we have $q=r=0$, then

$$D(\lambda)=J(\sqrt{\lambda}\,).$$

II. Next we take

$$\Phi(y)\equiv y''=0,$$

where the boundary conditions (L) are $y(0)=y(1)=0$, then the Green function is

$$G(x,\xi)=\begin{cases}\xi(1-x) \\ x(1-\xi)\end{cases} \quad \text{for} \quad x \lesseqqgtr \xi.$$

Hence we have $\quad \Psi(y)\equiv y''+\lambda y=0.$

For the only solution of $\Psi(y)=0$, we must take

$$\varphi(x\,;\lambda)=\frac{\sin\sqrt{\lambda}\,.\,x}{\sqrt{\lambda}},$$

where $\varphi(1\,;0)=\lim_{\lambda\to 0}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}=1.$

Since

$$\varphi(1\,;\lambda)=\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}},$$

from (7′), we have

$$D(\lambda)=e^{q\lambda+r\lambda^2}.\,\varphi(1\,;\lambda),$$

where all the zeros of $\varphi(1\,;\lambda)$ are simple.

On the other hand, the series:

$$\frac{2}{\pi^2}\sum_{m=1}^{+\infty}\frac{\sin m\pi x\,.\,\sin m\pi y}{m^2}$$

converges uniformly in the square domain $(0\leqq x\leqq 1\ ;\ 0\leqq y\leqq 1)$, and hence from (7‴), we have

$$D(\lambda)=e^{q\lambda+r\lambda^2}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}.$$

Thus, just as in example I above, we can easily conclude that

$$D(\lambda)=\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}. \qquad \text{Q. E. D.}$$

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