A Theory of the Rising Stage of Drift Current in the Ocean

II. The Case of No Bottom-Friction

By

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Abstract

Practical measurements of current in real seas, especially in shallow waters, as well as in rivers, generally show that it is more appropriate in reality to consider some "slip velocity" at the bottom than to assume "no bottom-current". Moreover, there are many phenomena, such as long waves or seiche motion, which may be practically explained even by the hydrodynamics of an *ideal fluid* that can slide freely on solid boundaries.

Hence the writer treats here again the rising stage of drift current on the supposition that the water is uniformly viscous but there is no friction for slipping on the bottom. Comparing the results obtained thus with those in the previous case of "on bottom-current", we see that:

1) For a very deep sea, the current coincides exactly with that in the previous case.

2) The shallower the sea, the stronger is the current and the larger becomes the angle of deflection from the direction of the wind, just contrary to the previous case.

3) The velocity of the current oscillates around the steady value with a period of 12 pendulum hours as before, but it does not attain a definite value even when $t=\infty$, and the hodograph has an assymptotic circle whose radius is inversely proportional to the depth of the sea.

4) The time required for the current to attain a quasi-steady state is about one-third the time required for it to become steady in the previous case.

The real sea of course must lie between the two extreme cases of "no bottom-velocity" and "no bottom-friction".

I. Introduction

In the hydrodynamics of a viscous fluid, it is customary to assume that no motion can exist at a solid wall, and so in discussing a sea motion most oceanographers take the sea to be motionless at the bottom. Such an assumption may be right for a laminar motion, but the oceanic motion is of course turbulent, and practical measurements in real seas, especially in shallow waters, show that the assumption does not generally hold. For instance, the currents observed on 2–3 July, 1931 by J. N. S. "Yodo"[1] in the Yellow Sea at 37° 37′ N., 122° 47′ E., were as shown in Table 1 and Fig. 1. The depth of the sea was 38 metres, and even at only 3 metres above the bottom the current was entirely comparable with or sometimes rather greater than at the surface.

Fig. 1

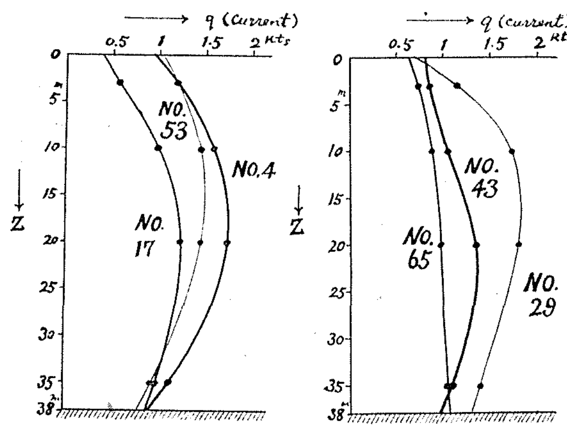


Table 1

Current observed in the Yellow Sea

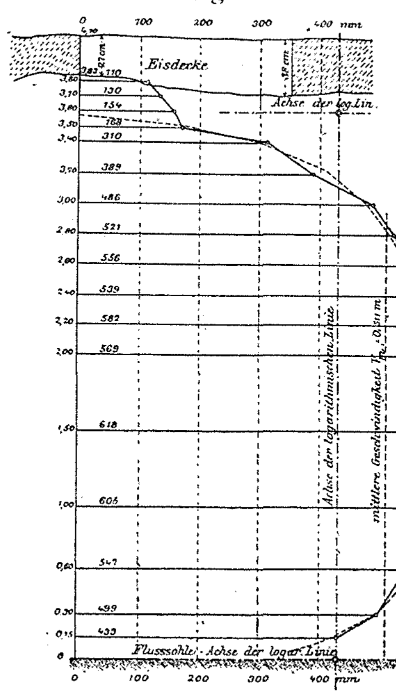
(Total depth=38 m; Dir. angles are measured from N. towards E.)

No.	z=3 m Time (h)	(min)	Current Speed (kt)	Dir.	z=10 m Time (h)	(min)	Current Speed (kt)	Dir.	z=20 m Time (h)	(min)	Current Speed (kt)	Dir.	z=35 m Time (h)	(min)	Current Speed (kt)	Dir.
4	11	32	1.19	154°	11	39	1.60	166°	11	45	1.75	163°	11	54	1.14	183°
17	18	2	0.58	341	18	8	0.99	0	18	12	1.24	329	18	19	1.00	340
29	0	1	1.17	145	0	7	1.77	170	0	12	1.79	160	0	19	1.40	171
43	7	2	0.88	332	7	7	1.04	330	7	12	1.35	330	7	20	1.12	320
53	12	2	1.18	162	12	7	1.48	177	12	12	1.47	205	12	19	0.96	172
65	18	2	0.72	0	18	6	0.88	320	18	12	0.98	341	18	18	1.06	336

1. 栗林今朝吉. 黃海に於ける潮汐觀測報告. 水路要報 第11年 第5號 第177頁.

In such cases it is obviously unnatural to take the bottom velocity as nil. In the case of rivers and canals also, early engineers[1] commonly assumed that no slip could happen at the bed; but modern hydrologists[2] have many observation-data which oblige them to consider a "bed current" at the water-bottom, and now they put much importance upon the slip velocity at the bottom as the direct cause of the transportation of sand and gravel and scouring of the bed. Fig. 2 is taken from Jasmund's book[3] on rivers, as a good example to show such slipping over solid boundaries. The currents observed on the North-Siberian shelf, which Sverdrup[4] and Fjeldstad[5] utilized to determine the coefficient of viscosity of a shallow sea, seem to me also to indicate some slip-velocity at the bottom, though it would not be so very absurd to assume the bottom velocity to be zero.

Fig. 2



On the other hand, in any treatise on pure hydrodynamics also, the greater part is taken up with the treatment of an ideal fluid and consequently water particles in a long wave or a seiche motion are considered to *slide freely* on the bottom with a horizontal velocity equal to that at the surface, and the results obtained with such an assumption hold good in many practical phenomena. This is in reality because the effect of viscosity will not appear even for a very viscous water, if the water basin is shallow and the bottom-friction is slight, so that the whole water moves almost

1. Such as Beyerhaus, Ringel, Lorenz.
2. Boussinesq, Bazin, Humphrey and Abbot, Hagen, Gerstner, Lippke, Jasmund, and many others.
3. Jasmund, R., Fliessende Gewässer. Handb. d. Ing. Wiss. III, Wasserbau (1929), S. 463.
4. Sverdrup, H. U., The Water on the North-Siberian Shelf. The Norw. N. Polar Exp. "Maud." Scientific Results, Vol. IV. No. 2 (1929).
5. Fjeldstad, J. E., Ein Beitrag z. Theorie d. Winderzeugten Meeresströmungen. Gerlands Beitr. z. Geophysik, **23**, 237 (1929).

equally from surface to bottom. Kaplan[1] showed by precise measurement of a stream in a rectangular wooden channel with Pitot-tubes that the bottom resistance in reality was very small compared with the internal viscosity.

The present writer published a paper[2] on the rising stage of drift current in the ocean assuming, as is commonly done, that no motion can exist at the sea-bottom. But the above considerations led him to think it very desirable to solve the same problem on the supposition that there is no friction at the sea bottom, so that slipping is free. The real sea, of course, will lie between the above two extreme cases. Moreover, the solution of the present case is very useful, because a steady current of any kind will be the sum of (1) a current of the same kind with "no bottom-friction" and (2) a drift current of the present case which would be produced by a tracting force acting at the bottom.

2. Rising Stage of Current under Constant Wind.

In the present case we must solve the differential equation

$$\frac{\partial w}{\partial t}=\nu\frac{\partial^2 w}{\partial z^2}-i2\bar{\omega}w, \tag{1}$$

with the conditions

$$\partial w/\partial z=-iT/\mu \quad \text{at} \quad z=0, \tag{2}$$

$$\partial w/\partial z=0 \quad \text{at} \quad z=H, \tag{3}$$

and

$$w=0 \quad \text{when } t=0. \tag{4}$$

Divide w, as before, into two parts: a steady part w_1, and a varying part w_2 such that

$$w=w_1-w_2. \tag{5}$$

Steady part w_1. This part must satisfy the equations

$$0=\nu\frac{d^2 w_1}{dz^2}-i2\bar{\omega}w_1, \tag{1a}$$

$$dw_1/dz=-iT/\mu \quad \text{at} \quad z=0, \tag{2a}$$

and

$$dw_1/dz=0 \quad \text{at} \quad z=H. \tag{3a}$$

The solution of Eq. (1_a) under the condition (3_a) is obviously

1. Kaplan, V., Die Gesetze der Flüssigkeitsströmung usw., Zeits. d. Ver. Deut. Ing. **56**, 1578 (1912).

2. These Memoirs, A, **16**, 161 (1933).

$$
\left.\begin{aligned}
&w_1 = K\cosh a(H-z),\\
\text{where}\qquad &a=(1+i)k,\qquad k=\sqrt{\frac{\varpi}{\nu}}=\sqrt{\frac{\rho\omega\sin\lambda}{\mu}},
\end{aligned}\right\}\qquad (6)
$$

and condition (2_a) gives

$$
K=\frac{iT}{a\mu\sinh aH}=\frac{(1+i)T}{2k\mu\sinh(1+i)kH}.\qquad (7)
$$

Thus writing in real forms we get

$$
\left.\begin{aligned}
u_1&=A\cosh kz'\cos kz'-B\sinh kz'\sin kz'\\
v_1&=A\sinh kz'\sin kz'+B\cosh kz'\cos kz'
\end{aligned}\right\}\qquad (6')
$$

where

$$
\left.\begin{aligned}
&z'=H-z,\\
\text{and}\qquad &A=\frac{T}{k\mu}\cdot\frac{\sinh kH\cos kH+\cosh kH\sin kH}{\cosh 2kH-\cos 2kH},\\
&B=\frac{T}{k\mu}\cdot\frac{\sinh kH\cos kH-\cosh kH\sin kH}{\cosh 2kH-\cos 2kH}.
\end{aligned}\right\}\qquad (7')
$$

In order to understand concretely the mode of vertical distribution of the steady current, we calculated the values of u_1 and v_1 at every tenth of the total depth for the four cases H=D/10, D/4, D/2 and 1.25 D. The results are shown in Table 2 and in four curves of Fig. 3, $T/\mu k$ being taken as unity as before. Next, to facilitate the estimation of current in a sea of any depth, we evaluated the surface and the bottom current according to the depth of the sea by the following formulae:

For surface current

$$
\bar{u}_1=\frac{T}{k\mu}\left[\frac{1}{2}\cdot\frac{\sinh 2kH+\sin 2kH}{\cosh 2kH-\cos 2kH}\right],
$$

$$
\bar{v}_1=\frac{T}{k\mu}\left[\frac{1}{2}\cdot\frac{\sinh 2kH-\sin 2kH}{\cosh 2kH-\cos 2kH}\right],
$$

and for bottom current

$$
\underline{u}_1=A,\quad \underline{v}_1=B.
$$

Table 3 and the upper and the lower curve in Fig. 3 represent the result. The current anywhere in a sea of any depth can be approximately obtained by combining this set of two curves and the former set of four curves.

Table 2

Vertical distribution of drift-current,
No bottom-friction.

	$H=D/10$		$H=\frac{D}{4}$		$H=D/2$		$H=1\frac{1}{4}D$		$H=D$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	1.5954	0.1050	0.6578	0.2594	0.4586	0.4586	0.5006	0.5006	0.5019	0.5019
0.1	1.5945	0.0752	0.6565	0.1851	0.4486	0.3128	0.4420	0.1828		
0.2	1.5936	0.0484	0.6527	0.1095	0.4243	0.1903	0.3243	0		
0.3	1.5927	0.0214	0.6485	0.0604	0.3880	0.0864	0.2022	−0.0831		
0.4	1.5921	0.0002	0.6416	0.0098	0.3486	0.0024	0.1050	−0.1031		
0.5	1.5916	−0.0128	0.6355	−0.0326	0.3098	−0.0636	0.0388	−0.0900	0.0996	−0.0996
0.6	1.5911	−0.0270	0.6286	−0.0672	0.2726	−0.1156	0	−0.0642		
0.7	1.5908	−0.0378	0.6250	−0.0940	0.2419	−0.1535	−0.0190	−0.0379		
0.8	1.5905	−0.0498	0.6214	−0.1132	0.2187	−0.1793	−0.0261	−0.0172		
0.9	1.5904	−0.0504	0.6190	−0.1246	0.2042	−0.1944	−0.0278	−0.0043		
1.0	1.5903	−0.0521	0.6182	−0.1284	0.1993	−0.1993	−0.0279	0	−0.0433	−0.0433

Fig. 3

Vertical distribution of steady current; and the surface and the bottom current according to the values of kH.
(No bottom-friction)

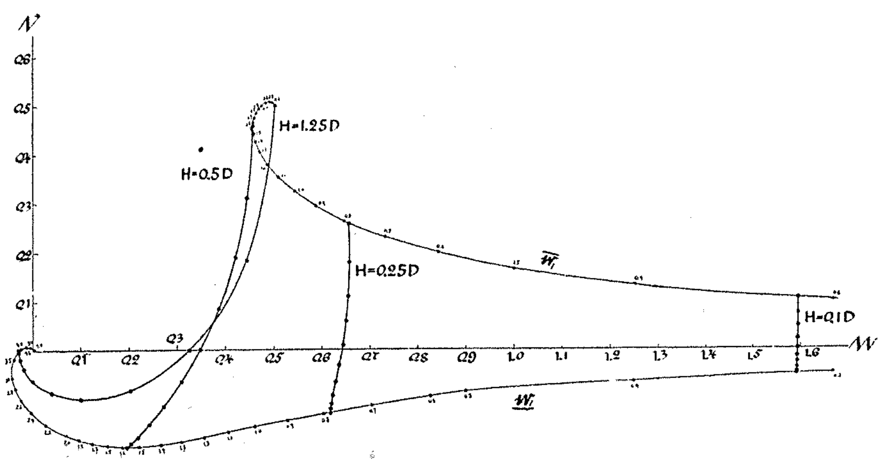


Table 3

The surface and the bottom values of the steady current
(No bottom-friction)

kH	Surface current		Bottom current	
	$\bar{u}_1$	$\bar{v}_1$	$\underline{u}_1$	$\underline{v}_1$
0	$\frac{1}{kH}=\infty$	0	$\frac{1}{kH}=\infty$	0
0.1	5.0000	0.0334	5.0000	−0.0075
0.2	2.5003	0.0666	2.5000	−0.0338
0.3	1.6676	0.1000	1.6656	−0.0500
0.4	1.2529	0.1333	1.2475	−0.0667
0.5	1.0055	0.1664	0.9951	−0.0831
0.6	0.8429	0.1994	0.8250	−0.0994
0.7	0.7294	0.2320	0.7011	−0.1151
0.8	0.6474	0.2640	0.6016	−0.1308
0.9	0.5872	0.2952	0.5280	−0.1452
1.0	0.5428	0.3252	0.4627	−0.1586
1.1	0.5106	0.3538	0.4059	−0.1706
1.2	0.4879	0.3806	0.3545	−0.1814
1.3	0.4728	0.4052	0.3083	−0.1896
1.4	0.4637	0.4273	0.2655	−0.1952
1.5	0.4594	0.4466	0.2205	−0.1990
1.6	0.4625	0.4631	0.1889	−0.2023
1.7	0.4609	0.4769	0.1545	−0.1970
1.8	0.4644	0.4875	0.1225	−0.1922
1.9	0.4692	0.4957	0.0940	−0.1854
2.0	0.4748	0.5019	0.0682	−0.1762
2.2	0.4845	0.5077	0.0260	−0.1535
2.4	0.4925	0.5089	−0.0046	−0.1281
2.6	0.4977	0.5075	−0.0249	−0.1022
2.8	0.5055	0.5052	−0.0368	−0.0779
3.0	0.5017	0.5031	−0.0423	−0.0565
3.5			−0.0389	−0.0177
4.0			−0.0258	+0.0019
4.5			−0.0132	+0.0085
5.0			−0.0045	+0.0083
6.0			+0.0034	+0.0061
∞	0.5000	0.5000	0	0

Comparing the results with those obtained in the case of "no bottom-current", we notice here the following points:

1) The shallower the sea of "no bottom-friction", the stronger is the drift current and the larger becomes the deviation-angle from the wind, just contrary to the previous case.

2) In very shallow water, the current will be nearly uniform from surface to bottom, and it deviates almost 90° from the wind.

3) The deeper the sea, the nearer the current approaches to that in the case of "no bottom-current"; and if the sea is infinitely deep, the currents in both cases are reduced to exactly the same, namely

$$w_1 = \frac{T}{\mu k\sqrt{2}} e^{-kz}\left\{\cos\left(\frac{\pi}{2} - kz\right) + i\sin\left(\frac{\pi}{2} - kz\right)\right\}.$$

4) The current in the bottom layer always has a component velocity v_1 opposite to the direction of wind, while in the previous case it is directed always leeward if the sea is shallower than $D/2$. This is probably of great importance in connection with the transportation of bottom materials.

Varying part w_2. This part must satisfy the equations

$$\frac{\partial w_2}{\partial t} = \nu \frac{\partial^2 w_2}{\partial z^2} - i2\varpi w_2, \qquad (1_b)$$

$$\partial w_2/\partial z = 0 \quad \text{at} \quad z=0 \quad \text{and} \quad z=H, \qquad (2_b) \text{ and } (3_b)$$

and

$$w_2 = w_1 \quad \text{when} \quad t=0. \qquad (4_b)$$

A solution of the differential equation (1_b) under the surface condition (2_b) is as in the previous paper

$$w_2 = C\cos\beta z \cdot e^{-(\nu\beta^2 + i2\varpi)t}.$$

But the bottom condition (3_b) in the present case requires

$$\sin\beta H = 0$$

or

$$\left.\begin{aligned} &\beta = n\pi/H, \qquad n = 0, 1, 2, 3, \ldots\ldots\ldots \\ &\text{so that} \quad w_2 = \sum_{n=0}^{\infty} C_n \cos\beta_n z\, e^{-(\nu\beta_n^2 + i2\varpi)t}. \end{aligned}\right\} \qquad (8)$$

Finally, the initial condition (4_b) requires

$$\sum_{n=0}^{\infty} C_n \cos\beta_n z = w_1 \equiv K\cosh(1+i)k(H-z),$$

i. e., the coefficients C_n are nothing but the coefficients of Fourier's cosine series for $K\cosh(1+i)k(H-z)$.

Hence for $n\neq 0$, we have

$$C_n=\frac{2K}{H}\int_0^H\cosh(1+i)k(H-z)\cos\beta_n z dz=i\frac{2T}{\mu H}\cdot\frac{1}{2ik^2+\beta_n^2}$$

$$=\frac{T}{\mu k^2H}\cdot\frac{1+i\beta_n^2/2k^2}{1+(\beta_n^2/2k)^2},$$

and for $n=0$,

$$C_0=\frac{1}{2}\cdot\frac{T}{\mu k^2H}=\frac{1}{2}\cdot\frac{T}{\rho H\omega\sin\lambda}.$$

Thus we get for the varying part

$$w_2=\frac{T}{\mu k^2H}e^{-i2\varpi t}\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1+i\beta_n^2/2k^2}{1+(\beta_n^2/2k^2)^2}\cos\beta_n z\cdot e^{-\nu\beta_n^2 t}\right\},\quad (9)$$

or

$$\left.\begin{aligned}
u_2&=\frac{T}{\mu k}\cdot\frac{D}{\pi H}\Bigg[\cos 2\varpi t\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1+(\beta_n^2/2k^2)^2}\cos\beta_n z\cdot e^{-\nu\beta_n^2 t}\right\}\\
&\qquad+\sin 2\varpi t\sum_{n=1}^{\infty}\frac{\beta_n^2/2k^2}{1+(\beta_n^2/2k^2)^2}\cos\beta_n z\cdot e^{-\nu\beta_n^2 t}\Bigg]\\
v_2&=\frac{T}{\mu k}\cdot\frac{D}{\pi H}\Bigg[\cos 2\varpi t\sum_{n=1}^{\infty}\frac{\beta_n^2/2k^2}{1+(\beta_n^2/2k^2)^2}\cos\beta_n z\cdot e^{-\nu\beta_n^2 t}\\
&\qquad-\sin 2\varpi t\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1+(\beta_n^2/2k^2)^2}\cos\beta_n z\cdot e^{-\nu\beta_n^2 t}\right\}\Bigg]
\end{aligned}\right\}\quad (10)$$

It is here to be noticed that, on account of the term corresponding to $n=0$, the part w_2 does not generally vanish even when $t=\infty$, but finally becomes a circular motion which is entirely the same from surface to bottom; that is to say, its hodograph has an asymptotic circle of radius

$$\frac{T}{2\mu k^2H}=\frac{1}{2}\cdot\frac{D}{\pi H}\cdot\frac{T}{\mu k}=\frac{T}{2\rho H\omega\sin\lambda},\qquad (11)$$

for the whole water of the sea.

The circle will be very large when the sea is very shallow, and it becomes zero only when the sea is infinitely deep.

Table 4 and the curves indicated by the letter w_2 in Fig. 4 are the results as calculated by Eq. (10), $T/\mu k$ being taken as unity.

Table 4

Decaying part w_2 (No bottom-friction)

	$H=D$				$H=\frac{D}{2}$				$H=\frac{D}{4}$			
z	0		$H/2$		0		$H/2$		0		$H/2$	
t	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2	u_2	v_2
0^h	+0.5019	+0.5019	+0.0996	−0.0996	+0.4586	+0.4586	+0.3098	−0.0636	+0.6578	+0.2594	+0.6355	−0.0326
1	+0.4032	−0.0599	+0.0965	−0.1064	+0.3599	−0.1030	+0.2750	−0.1601	+0.5528	−0.3164	+0.5514	−0.3184
2	+0.2375	−0.2292	+0.0621	−0.1389	+0.1942	−0.2736	+0.1592	−0.2757	+0.3184	−0.5514	+0.3184	−0.5514
3	+0.0629	−0.2777	−0.0060	−0.1567	+0.0119	−0.3239	−0.0000	−0.3184	+0.0000	−0.6366	−0.0000	−0.6366
6	−0.2122	−0.0267	−0.1591	−0.0003	−0.3187	−0.0005						
9	−0.0121	+0.1833	−0.0000	+0.1592	−0.0000	+0.3184						
12	+0.1702	+0.0055										
15	+0.0025	−0.1642										
18	−0.1615	−0.0011										
21	−0.0005	+0.1603										
24	+0.1597	+0.0003										
36	+0.1594	+0.0001										
48	+0.1592	+0.0000										

Fig. 4

Rising stage of drift current (No bottom-friction)

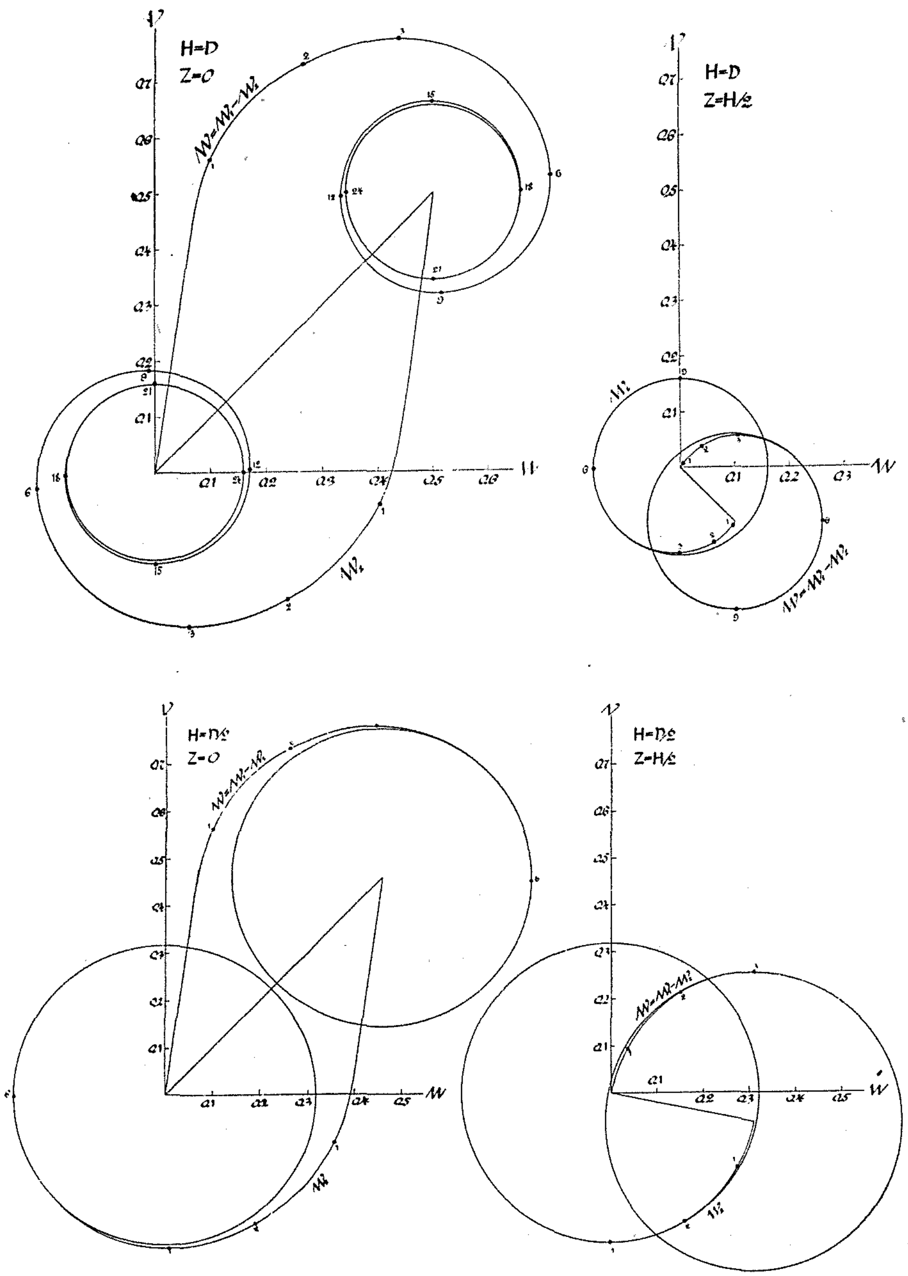
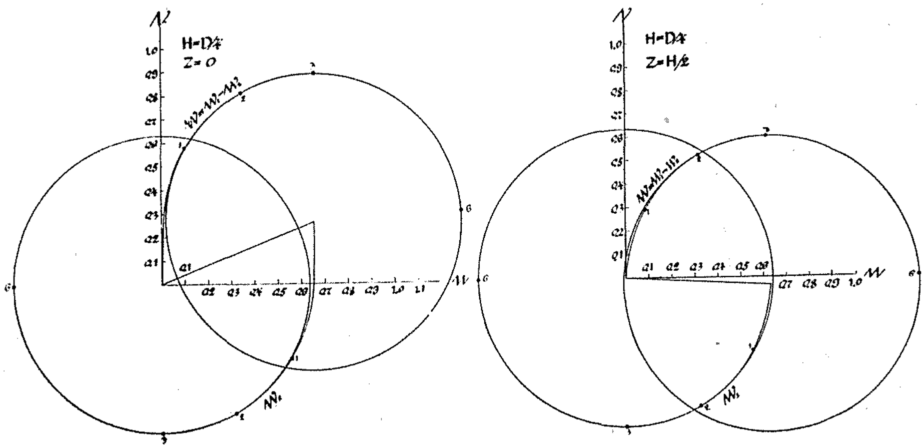


Fig. 4 (continued)

Rising State of Drift-Current: no Bottom friction



Developing stage of current w. Having obtained the values of w_1 and w_2, we can at once calculate numerically the value w with the relation

$$w = w_1 - w_2,$$

which is represented in the curves marked by the letter w in Fig. 4.

From these hodographs we see that:

i) The velocity of the current oscillates around the steady value w_1, with a period of 12 pendulum hours as before, but it does not attain the value w_1 itself even when $t=\infty$, except for an infinitely deep ocean.

ii) The hodograph has an asymptotic circle of radius

$$\frac{1}{2}\frac{D}{\pi H}\cdot\frac{T}{\mu k} = \frac{T}{2\rho H\omega \sin \lambda},$$

with center at the arrow-head of w_1, so that water particles will finally describe *cycloidal paths*. The circle of the hodograph is very large for a shallow sea, and becomes indefinitely small for an infinitely deep sea.

iii) The time required for the current to attain the quasi-steady state is considerably shorter (about one-third) than the time required to become steady in the previous case.

Depth of sea	D/10	D/4	D/2	D
Time required (pendulum hours)	1/4	2	6	15

iv) The shallower the sea, the stronger is the current and the larger becomes the angle of deviation from the direction of the wind.

v) The deeper the sea, the nearer the current approaches to that in the case of "no bottom-current"; and if the sea is infinitely deep, the mode of development of the current is exactly the same in both cases, as is shown below.

Infinitely deep ocean. The mathematical formula of w may be written in the form:

$$w=\sum_{n=0}^{\infty}C_n \cos\beta_n z\left\{1-e^{-(\nu\beta_n^2+i2\varpi)t}\right\}$$

$$=\frac{1}{2}\,\frac{T}{\rho H\varpi}\left(1-e^{-i2\varpi t}\right)+i\frac{2T}{\rho H}\sum_{n=1}^{\infty}\cos\beta_n z\int_0^t dt\cdot e^{-(\nu\beta_n^2+i2\varpi)t}. \quad (12)$$

Now, when the depth is infinite, the first term corresponding to $n=0$ is of course zero; and for the remaining terms we put

$$\beta_n=n\pi/H=\beta, \quad d\beta=\pi/H,$$

then we get

$$w=i\frac{2T}{\rho\pi}\int_0^{\infty}d\beta\cos\beta z\int_0^t dt\cdot e^{-(\nu\beta^2+i2\varpi)t}=\frac{iT}{\sqrt{\pi\mu\rho}}\int_0^t dt\frac{e^{-\frac{z^2}{4\nu t}}}{\sqrt{t}}e^{-i2\varpi t}, \quad (13)$$

which is nothing but Fredholm's formula.

3. Development of Current by a Varying Wind

When the wind varies with the time and T is a function of t such that $T=T(t)$, we get from eq. (12) in a similar way as in the previous case

$$w=\frac{2i}{\rho H}\left\{\frac{1}{2}\int_0^t d\tau\, T(\tau)e^{-i2\varpi(t-\tau)}\right.$$
$$\left.+\sum_{n=1}^{\infty}\cos\beta_n z\int_0^t d\tau\, T(\tau)e^{-(\nu\beta_n^2+i2\varpi)(t-\tau)}\right\}.$$

If $T(t)$ is linearly increasing, or exponentially or harmonically varying, the above integration (14) can be easily executed just as in the previous case[1].

1. loc. cit. p. 172.