A Theory of the Rising Stage of Drift Current in the Ocean

III. The Case of a Finite Bottom-Friction Depending on the Slip Velocity

By

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Abstract

Having found that in a shallow sea there would be too grave a difference between the currents in the two cases of already dealt with, " no bottom-current " and " no bottom-friction ", the writers treat the problem again on the assumption most probable for a real sea, that there can exist a bottom-velocity V_H and a bottom-friction

$$R_s = f\rho V_H^2 \qquad \text{or} \qquad f'\rho V_H$$

according to the strength of V_H.

The results show that:

1) For a deep sea, the current is approximately equal to those in the two cases of " no bottom-current " and " no bottom-friction ".

2) For a shallow sea, the mode of generation and distribution of the current will be very different according to the wind velocity and the turbulence of the sea. For a strong wind the current approaches to that in the case of " no bottom-current ", and for a weak wind to that in the case of " no bottom-friction ".

3) At a given latitude, the greater the wind velocity Q or the smaller the viscosity of water, the greater is the ratio of the bottom velocity V_H to Q; but this ratio can not exceed a limited value, even if the wind velocity becomes infinity.

4) The greater the depth of the sea, the smaller is the ratio V_H/Q, and it becomes zero for an infinitely deep sea.

1. Introduction

One[1] of the writers has already investigated the development of drift current in two cases, (1) on the assumption that there is " no

1. Nomitsu, These Memoirs, A, **16**, 161 and 275 (1933).

current " at the bottom and (2) that " no friction " exists at the bottom. According to the results, for a very deep ocean the current will be almost the same in both cases, but for a very shallow sea the currents in the two cases are entirely different or rather contrary to each other.

For other problems also, such as the fluctuation of mean sea level, the writers[1] used the above two extreme cases of the condition at the bottom, and the phenomena hitherto discussed differed so slightly in the two extreme cases that it was sufficient to say that the result in a real sea would lie intermediate between them.

For the current itself in a shallow sea, however, the two kinds of bottom-conditions assumed above would cause too grave a difference to make it possible to know the result in a real sea from the mere statement that it will be intermediate between the former two. In the case of " no bottom-friction ", the shallower the sea, the stronger is the current and the larger (about 90°) becomes the angle of deviation from the direction of the wind, while in the case of " no bottom-current ", the shallower the sea, the weaker is the current and the nearer its direction approaches that of the wind.

Recently, oceanographers have been laying stress upon the influence of the variation of viscosity with depth and have discussed it very frequently, but almost no one except Jeffreys[2] has touched the far more important effect of the bottom condition, and nearly always the assumption of " no bottom-current " is accepted.

These considerations have led the present writers to treat the drift current again on the supposition that a bottom-velocity may exist and will cause a finite bottom-friction depending on the magnitude of the slip velocity.

In connection with the frictional resistance at a solid surface, there are the numerous experiments[3] made by Froude, Tideman and others to determine the resistance of ships. A part of the resistance of a ship is termed "skin resistance" by ship-building engineers, and this is simply our " surface friction ". The experimental formula for the skin resistance arrived at by Froude and Tideman is

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1. Nomitsu, On the so called Grenzflache etc., These Memoirs, A, **10**, 111 (1927); Nomitsu and Takegami, On the Convection Current and the Surface Level of a Two-layer Ocean. These Memoirs, A, **15**, 131 (1932).

2. Jeffreys, H., The Effect of a Steady Wind on the Sea-level near a Straight Shore. Phil. Mag. **46**, 114 (1923).

3. cf. Taylor, D. W., Resistance of Ships and Screw Propulsion (1910), pp. 16—20.

$$R_s = f\rho V^n, \tag{1}$$

where R_s denotes the skin resistance per unit area, ρ the density of the water, V the relative velocity between the solid and the water, and f and n are constants depending on the nature of the surface.

In the range of the ordinary velocities of ships, the index n is found to be 1.83 for a smooth hard surface such as a well-painted iron surface; but it approaches 2 for a rougher surface and remains at the value 2 for sand or mud, and this value can therefore be used for the sea-bottom.

The value of the coefficient f, recalculated in c. g. s. units from the results of Froude and Tideman in English units, is

$$f = 0.0016 \quad \text{to} \quad 0.002$$

according to the condition of the surface. Jeffreys[1] also made some experiments recently and obtained

$$f = 0.0025.$$

These results are also in accordance with the law[2] of wind-traction T over the sea surface, namely

$$T = 0.0025\,\sigma Q^2, \tag{2}$$

where σ is the density of air and Q the velocity of the wind.

Then, for a very slow velocity also, will the index n take the above value 2? We have no direct experimental data on this subject for sea water. From the numerous data obtained from observations of the sea, however, Thorade[3] found the following experimental formula relating the surface velocity V_0 of the drift current to the velocity Q of its causal wind:

For $Q \leqq 3$ of Beaufort's scale,

$$\left.\begin{aligned} &V_0 = 0.259\sqrt{Q}/\sqrt{\sin\lambda}\,, \\ \text{and for } Q>3 \quad & \\ &V_0 = 0.0126\,Q/\sqrt{\sin\lambda}\,. \end{aligned}\right\} \tag{3}$$

On the other hand, Ekman's theory of the drift current teaches us that

1. Phil. Mag. **49**, 793 (1925).
2. Recalculated from the observations of Colding.
3. Thorade, H., Wiss. Beilage zum Jahresbericht d. Realschule in Eilbeck (1914).

$$V_0 = T/\sqrt{2\mu\rho\omega \sin\lambda}\ ,$$

where the eddy viscosity μ is again a function of T. Hence, if we consider the lower relation in eq. (3) to correspond to law (2), then the upper relation seems to indicate that

$$T \propto Q \qquad \text{for} \quad Q \leqq 3 \text{ of Beaufort's scale,}$$

and analogously for the resistance at a solid surface

$$R_s = f'\rho V \tag{1'}$$

when the velocity is very small.

Thus we will use law (1) or (1′) in the following discussion. Of course a different value must be taken for f' in eq. (1′) from that of f in eq. (1).

2. Steady Current

(a) *When the bottom-friction* $\propto$ (*bottom-velocity*)^2.

We use the same notations and the equation of motion and the surface condition as in the previous papers[1], but take the bottom condition different from before. That is, using law (1)

$$-\mu \left| \frac{\partial w_1}{\partial z} \right|_{z=H} = f\rho V_H \boldsymbol{V}_H, \tag{4}$$

where $\boldsymbol{V}_H$ means the velocity-vector at the bottom, and V_H is its scalar value.

Now the general solution of the equation of motion may be written as

$$\left.\begin{aligned} & w_1 = A \sinh az + B \cosh a z, \\ \text{where} \qquad & a = (1+i)k, \quad k = \sqrt{\frac{\bar{\omega}}{\nu}} = \sqrt{\frac{\rho\omega \sin\lambda}{\mu}}, \end{aligned}\right\} \tag{5}$$

and A and B are arbitrary constants.

From the surface condition $iT = -\mu\left(\frac{\partial w_1}{\partial z}\right)_{z=0}$ we get

$$A = -\frac{iT}{\mu a}. \tag{6}$$

1. Nomitsu, These Memoirs, A, **16**, 161 and 275 (1933).

If we denote by θ the angle made by the bottom-current with the x-axis, then the bottom-condition can be written as

$$f\rho V_H^2 e^{i\theta} = -\mu\left(\frac{\partial w_1}{\partial z}\right)_{z=H}, \tag{4'}$$

which gives

$$B = \frac{iT}{\mu a}\coth aH - \frac{f\rho V_H^2 e^{i\theta}}{\mu a \sinh aH}. \tag{7}$$

Introducing (6) and (7) into (5), we obtain

$$w_1 = \frac{iT}{\mu a}(\coth aH \cosh az - \sinh az) - \frac{f\rho V_H^2 e^{i\theta}}{\mu a \sinh aH}\cosh az, \tag{8}$$

of which the part consisting of the first two terms is nothing but the drift current in the case of "no bottom-friction", and the last term may be considered the correction term due to the bottom-friction. Here we must notice that the correction term itself is also a drift current with "no bottom-friction", merely interchanging the surface and the bottom, or putting z instead of $z' = H - z$.

Now, in order to determine the unknown values of V_H and θ corresponding to a given wind, we use the following obvious relation at the bottom

$$V_H e^{i\theta} = |w_1|_{z=H} = \frac{iT}{\mu a}\,\frac{1}{\sinh aH} - \frac{f\rho V_H^2 e^{i\theta}}{\mu a}\coth aH. \tag{9}$$

Separating this equation into two parts, real and imaginary, we get

$$\left.\begin{aligned} f\rho a V_H^2 + \mu k c V_H - T\sin\theta &= 0, \\ f\rho b V_H^2 + \mu k d V_H - T\cos\theta &= 0, \end{aligned}\right\} \tag{10}$$

where

$$\left.\begin{aligned} a &= \cosh kH \cos kH, \\ b &= \sinh kH \sin kH, \\ c &= \sinh kH \cos kH - \cosh kH \sin kH, \\ d &= \sinh kH \cos kH + \cosh kH \sin kH. \end{aligned}\right\} \tag{11}$$

Let Q be the speed of the wind, σ the density of the air and f_a the coefficient of friction for the air, and put by (2)

$$T = f_a \sigma Q^2, \tag{2'}$$

and eliminate θ from (10), then we have the following equation

$$A'\eta^4+B'\eta^3\xi+C'\xi^2\eta^2-1=0, \tag{12}$$

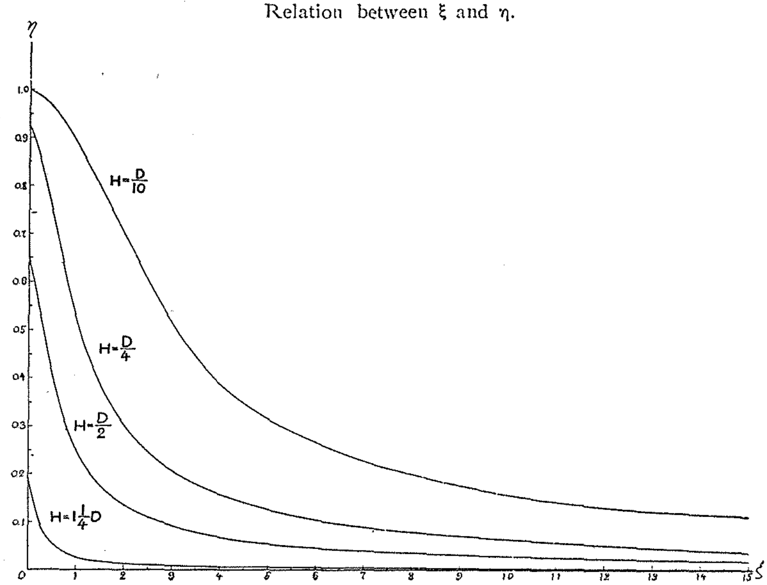
where

$$\left.\begin{aligned}
&\xi=\frac{\mu k}{\sqrt{ff_a\rho\sigma}\ Q},\\
&\eta=\sqrt{\frac{R_s}{T}}=\sqrt{\frac{f\rho}{f_a\sigma}}\cdot\frac{V_H}{Q},\\
&A'=(a^2+b^2)=\frac{1}{2}(\cosh 2kH+\cos 2kH),\\
&B'=2(ac+bd)=\sinh 2kH-\sin 2kH,\\
&C'=(c^2+d^2)=\cosh 2kH-\cos 2kH,
\end{aligned}\right\} \tag{13}$$

Thus, in order to determine the bottom current V_H from a given wind-velocity Q, we must solve the equation (12) which is biquadratic with respect to η. But as the mathematical operation of solving a biquadratic equation is too laborious, we rather take the following graphical method:

Substitute every possible value of η in equation (12) and get the corresponding value of ξ by solving the quadratic equation of ξ, and plot the corresponding values of ξ and η thus obtained in a curve, then by this curve we can find reversely a corresponding value of η

Fig. 1
Relation between ξ and η.



for any given value of ξ. Since ξ must naturally be a real and positive number, equation (12) shows that for $\xi=0$, $\eta=(1/A')^{\frac{1}{4}}$, and for any other value of ξ, η must be smaller than $(1/A')^{\frac{1}{4}}$.

In that region of η we constructed Table 1 and Fig. 1 for a sea of depth $H=D/10$, $D/4$, $D/2$, $1.25D$.

Table 1

Relation between ξ and η.

$H=\frac{D}{10}$		$H=\frac{D}{4}$		$H=\frac{D}{2}$		$H=1\frac{1}{4}D$	
ξ	η	ξ	η	ξ	η	ξ	η
0	1.000	0	0.945	0	0.653	0	0.194
0.618	0.95	0.148	0.9	0.103	0.6	0.095	0.15
0.984	0.9	0.381	0.8	0.283	0.5	0.227	0.1
1.456	0.8	0.595	0.7	0.494	0.4	0.258	0.09
1.908	0.7	0.821	0.6	0.789	0.3	0.305	0.08
2.411	0.6	1.089	0.5	1.311	0.2	0.361	0.07
3.029	0.5	1.448	0.4	2.763	0.1	0.433	0.06
3.885	0.4	2.016	0.3	3.082	0.09	0.532	0.05
5.251	0.3	3.102	0.2	3.476	0.08	0.676	0.04
7.928	0.2	6.288	0.1	3.983	0.07	0.915	0.03
15.901	0.1	6.990	0.09	4.657	0.06	1.383	0.02

For values of ξ greater than in Fig. 1, η becomes very small and its higher powers may be neglected, so that eq. (12) can be written

$$C'\xi^2\eta^2-1\doteq 0, \quad \text{or} \quad \eta\doteq\frac{1}{\sqrt{C'}\,\xi}, \tag{12'}$$

which will give the value of η corresponding to a larger ξ.

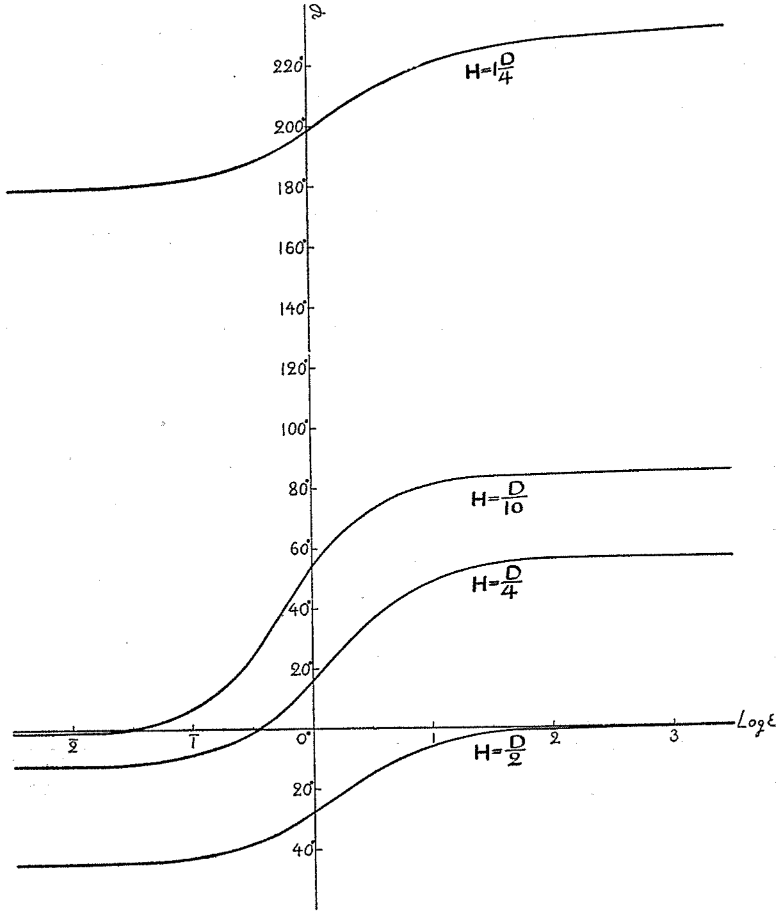
Having determined the value of V_H, we can easily calculate the direction of current, θ, with the following equation derived from (10):

$$\left.\begin{aligned} &\tan\theta=\frac{\sinh kH\cos kH-\cosh kH\sin kH+\varepsilon\cosh kH\cos kH}{\sinh kH\cos kH+\cosh kH\sin kH+\varepsilon\sinh kH\sin kH} \\ &\text{where} \\ &\varepsilon=\frac{f\rho V_H}{\mu k}=\frac{\eta}{\xi}. \end{aligned}\right\} \tag{14}$$

By means of these formulae we get Table 2 and Fig. 2.

Fig. 2

Direction of bottom-current according to $\epsilon=\eta/\xi$.



Using the values of V_H and θ obtained above, we can calculate the correction term for the drift current

$$\Delta w_1 = -\frac{f\rho V_H^2 e^{i\theta}}{\mu a \sinh aH} \cosh az, \tag{15}$$

or in the x and the y direction separately

$$\left.\begin{aligned} \Delta u_1 &= -\frac{T}{\mu k}(E \cosh kz \cos kz + F \sinh kz \sin kz), \\ \Delta v_1 &= \frac{T}{\mu k}(F \cosh kz \cos kz - E \sinh kz \sin kz), \end{aligned}\right\} \tag{15'}$$

Table 2.

Direction of bottom-current according to $\epsilon=\eta/\xi$.

$H=\frac{D}{10}$		$H=\frac{D}{4}$		$H=\frac{D}{2}$		$H=1\frac{D}{4}$	
$\epsilon=\eta/\xi$	θ	$\epsilon=\eta/\xi$	θ	$\epsilon=\eta/\xi$	θ	$\epsilon=\eta/\xi$	θ
1.536	62.7°	6.102	45.5°	5.825	−9.0°	1.579	203.8°
0.915	51.8°	2.104	29.9°	1.767	−20.9°	0.441	190.2°
0.549	37.7°	1.176	18.7°	0.810	−29.8°	0.349	188.5°
0.367	27.5°	0.731	10.0°	0.380	−36.5°	0.262	186.5°
0.249	19.2°	0.495	4.0°	0.153	−41.3°	0.194	185.0°
0.165	12.6°	0.269	−2.7°	0.036	−44.1°	0.189	184.8°
0.103	7.3°	0.149	−6.7°	0.029	−44.3°	0.094	182.5°
0.057	3.3°	0.064	−9.7°	0.023	−44.4°	0.059	181.7°
0.025	0.4°	0.016	−11.5°	0.018	−44.6°	0.033	180.8°
0.006	−1.3°	0.013	−11.7°	0.013	−44.7°	0.014	180.5°

where

$$\left.\begin{aligned} E=&\frac{\eta^2}{\cosh 2kH-\cos 2kH}\Big[(\cosh kH\sin kH\\ &+\sinh kH\cos kH)\sin\theta-(\cosh kH\sin kH\\ &-\sinh kH\cos kH)\cos\theta\Big],\\ F=&\frac{\eta^2}{\cosh 2kH-\cos 2kH}\Big[(\cosh kH\sin kH\\ &-\sinh kH\cos kH)\sin\theta+(\cosh kH\sin kH\\ &+\sinh kH\cos kH)\cos\theta\Big]. \end{aligned}\right\} \qquad (16)$$

For seas of several depths we already know the value of the drift current in the case of "no bottom-friction" from our previous work[1] on it, and the correction term for the present case can be calculated by the above formulae for several values of ξ, and then adding the results we obtain Table 3 and the curves in Figs. 3, 4, 5, 6. In all these, the coefficient $T/\mu k$ was assumed as unity; hence a point to be attended to in looking at the figures is that the assumed unit $(T/\mu k)$ is different for different values of wind or viscosity and consequently of ξ.

1. Nomitsu, These Memoirs, A, **16**, 275 (1933).

Table 3

Vertical distribution of steady current when bottom friction $\propto$ (bottom velocity)^2.

	$H=\frac{D}{10}$									
	$\xi=0.618$		$\xi=0.984$		$\xi=1.456$		$\xi=1.908$		$\xi=2.411$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	0.341	0.803	0.608	0.934	1.012	0.931	1.258	0.785	1.424	0.639
0.1	0.340	0.773	0.607	0.903	1.011	0.900	1.257	0.755	1.423	0.609
0.2	0.337	0.742	0.604	0.873	1.008	0.872	1.254	0.727	1.421	0.581
0.3	0.333	0.709	0.599	0.841	1.003	0.842	1.250	0.698	1.417	0.553
0.4	0.327	0.679	0.592	0.813	0.996	0.817	1.244	0.675	1.412	0.531
0.5	0.321	0.655	0.584	0.791	0.988	0.798	1.238	0.659	1.408	0.517
0.6	0.313	0.629	0.575	0.767	0.979	0.779	1.230	0.641	1.401	0.501
0.7	0.305	0.603	0.566	0.746	0.970	0.761	1.223	0.626	1.395	0.488
0.8	0.293	0.569	0.553	0.720	0.957	0.742	1.211	0.608	1.385	0.472
0.9	0.281	0.548	0.538	0.699	0.942	0.728	1.199	0.602	1.379	0.469
1.0	0.269	0.519	0.523	0.681	0.926	0.715	1.186	0.593	1.367	0.465

Table 3

continued

	$H=\frac{D}{4}$						$H=\frac{D}{2}$				$H=1\frac{1}{4}D$	
	$\xi=0.148$		$\xi=0.381$		$\xi=0.595$		$\xi=0.103$		$\xi=0.494$		$\xi=0.095$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	0.374	0.684	0.597	0.642	0.620	0.566	0.541	0.519	0.503	0.471	0.501	0.500
0.1	0.370	0.608	0.594	0.568	0.617	0.492	0.529	0.375	0.493	0.326	0.442	0.182
0.2	0.358	0.528	0.582	0.491	0.607	0.416	0.500	0.258	0.467	0.206	0.325	0.000
0.3	0.341	0.469	0.567	0.440	0.594	0.365	0.456	0.164	0.429	0.108	0.203	−0.083
0.4	0.316	0.407	0.543	0.387	0.574	0.312	0.405	0.092	0.387	0.031	0.106	−0.104
0.5	0.288	0.346	0.514	0.341	0.551	0.267	0.350	0.043	0.344	−0.026	0.041	−0.092
0.6	0.255	0.292	0.484	0.300	0.523	0.228	0.291	0.009	0.300	−0.067	0.001	−0.067
0.7	0.217	0.238	0.450	0.266	0.496	0.197	0.231	−0.011	0.261	−0.093	−0.019	−0.043
0.8	0.178	0.189	0.412	0.237	0.465	0.170	0.174	−0.018	0.226	−0.104	−0.023	−0.024
0.9	0.135	0.140	0.368	0.213	0.431	0.150	0.117	−0.017	0.196	−0.106	−0.020	−0.012
1.0	0.091	0.096	0.326	0.193	0.394	0.137	0.061	−0.010	0.171	−0.098	−0.012	−0.006

Fig. 3

Vertical distribution of steady current when bottom-friction ∞ (bottom-current)².

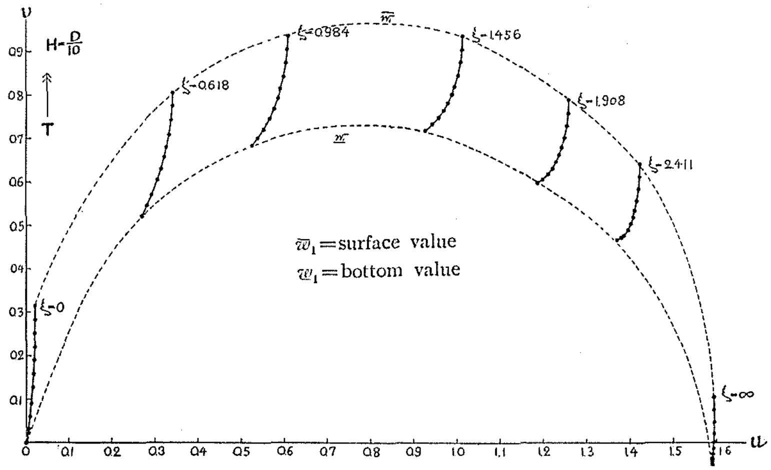


Fig. 4

Vertical distribution of steady current when bottom-friction ∞ (bottom-current)².

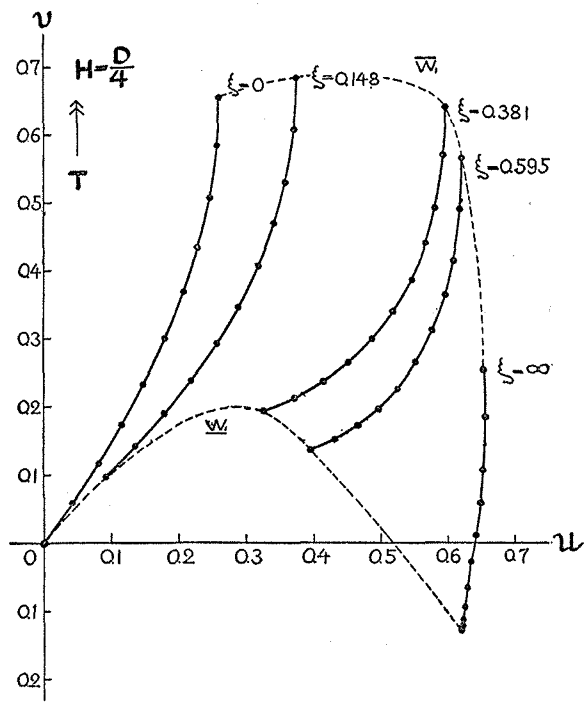


Fig. 5

Vertical distribution of steady current when bottom-friction ∞ (bottom-current)².

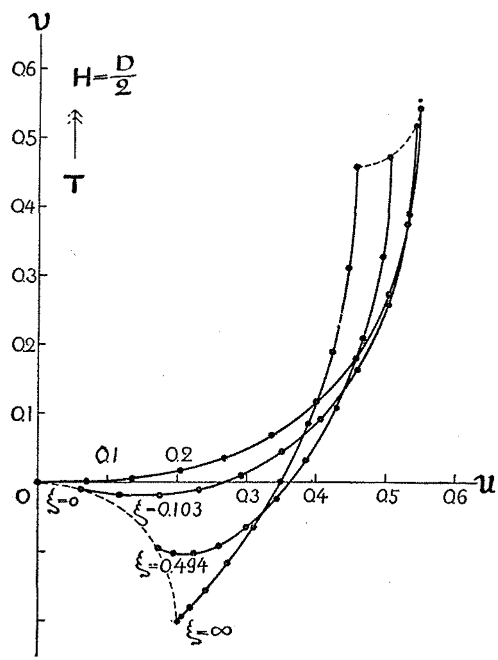
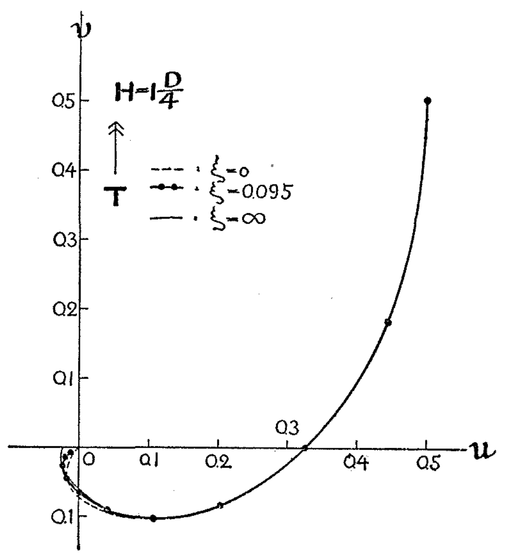


Fig. 6

Vertical distribution of steady current when bottom-friction ∞ (bottom-current)².



Glancing at the results we see that:

1) At a given latitude, the greater the wind velocity Q or the smaller the viscosity of the water, the greater is the ratio of bottom velocity V_H to Q; but this ratio can not exceed a limited value, even if the wind velocity becomes infinity.

2) The greater the depth of the sea, the smaller is the ratio V_H/Q and it becomes zero for an infinitely deep sea.

3) The mode of vertical distribution of the current is intermediate between those in the case of "no bottom-current" and "no bottom-friction." The mode of distribution approaches that seen in the case of "no bottom-current" for the smaller values of ξ, i. e., for the greater values of Q or the smaller viscosity μ, and that seen in the case of "no bottom-friction" for the greater values of ξ, i. e., for the smaller values of Q or the larger viscosity.

4) In shallow water, the mode of the current-distribution differs greatly according to the value of ξ, but in very deep water it is almost the same.

(b) *When the bottom-friction ∞ the bottom-velocity.*

If the skin resistance obeys law (1′), bottom-condition (4) must be replaced by

$$-\mu\frac{dw_1}{dz}=f'\rho w_1=f'\rho V_H e^{i\theta} \qquad \text{at} \quad z=H. \qquad (17)$$

Then the correction term for the drift current will become

$$\left.\begin{aligned} \varDelta w_1 &= -\frac{f'\rho V_H e^{i\theta}}{\mu a \sinh aH}\cosh az \\ &= -\frac{iT}{\mu a}\cdot\frac{2f'\rho\cosh az}{f'\rho\sinh 2aH+\mu a(\cosh 2aH-1)}. \end{aligned}\right\} \qquad (18)$$

Thus we see that if $T/\mu k$ is taken as unit velocity, the mode of the current-distribution for a sea of given depth is no more dependent of the wind velocity, but depends only on the viscosity of water and the frictional coefficient of the bottom.

To calculate this correction term numerically, we may of course use the lower formula of (18), but it is rather convenient to follow a similar method as in the foregoing paragraph, because it is not only desirable to know the bottom-velocity itself, but for $\cosh az/\sinh aH$

we already have the numerical data of drift current with "no bottom-friction".

From equations analogous to (10) we have

$$\left.\begin{aligned} \eta' &\equiv \frac{f'\rho V_H}{T} = \frac{1}{\sqrt{(a+\xi' c)^2+(b+\xi' d)^2}}, \\ &\text{and} \\ \tan\theta &= \frac{a+\xi' c}{b+\xi' d}, \end{aligned}\right\} \tag{19}$$

where $\xi' \equiv \mu k / f'\rho,$ (20)

and a, b, c, d are given by eq. (11).

Having determined the bottom-velocity, we can easily calculate the correction term with the same equations as (15′) and (16), only replacing η^2 with $\eta' = f'\rho V_H / T$. Adding this result to the current in the case of "no bottom-friction", we obtain the total current w_1. Some examples are shown by Table 4 and the curves between two lines $\overline{w}_1$ and $\underline{w}_1$ in Figs. 7, 8, 9.

Table 4

Vertical distribution of steady current.
(When bottom friction ∝ bottom velocity.)

	$H=\frac{D}{10}$					
	$\xi'=0.662,\ \eta'=0.9$		$\xi'=2.016,\ \eta'=0.6$		$\xi'=3.540,\ \eta'=0.4$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1
0	0.348	0.810	1.073	0.905	1.379	0.704
0.1	0.347	0.778	1.072	0.874	1.378	0.674
0.2	0.344	0.748	1.069	0.845	1.376	0.646
0.3	0.339	0.714	1.064	0.816	1.372	0.618
0.4	0.333	0.685	1.057	0.791	1.366	0.595
0.5	0.327	0.661	1.050	0.774	1.361	0.580
0.6	0.319	0.633	1.041	0.754	1.354	0.564
0.7	0.312	0.609	1.032	0.737	1.347	0.550
0.8	0.300	0.575	1.019	0.716	1.338	0.535
0.9	0.288	0.554	1.004	0.707	1.326	0.531
1.0	0.275	0.528	0.970	0.695	1.315	0.526

Table 4

(continued)

	$H=\frac{D}{4}$					
	$\xi'=0.177,\ \eta'=0.8$		$\xi'=0.562,\ \eta'=0.6$		$\xi'=1.175,\ \eta'=0.4$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1
0	0.383	0.693	0.554	0.625	0.658	0.511
0.1	0.379	0.607	0.555	0.550	0.650	0.437
0.2	0.368	0.527	0.545	0.473	0.642	0.362
0.3	0.351	0.468	0.529	0.420	0.630	0.312
0.4	0.326	0.406	0.507	0.365	0.612	0.262
0.5	0.297	0.346	0.481	0.316	0.592	0.218
0.6	0.267	0.292	0.449	0.273	0.568	0.182
0.7	0.226	0.240	0.419	0.236	0.544	0.153
0.8	0.186	0.191	0.380	0.204	0.516	0.131
0.9	0.144	0.144	0.341	0.176	0.488	0.114
1.0	0.100	0.100	0.300	0.153	0.459	0.105

(continued)

	$H=\frac{D}{2}$					
	$\xi'=0.076,\ \eta'=0.4$		$\xi'=0.874,\ \eta'=0.2$		$\xi'=2.322,\ \eta'=0.1$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1
0	0.544	0.533	0.512	0.477	0.487	0.463
0.1	0.532	0.389	0.502	0.333	0.476	0.318
0.2	0.502	0.272	0.475	0.214	0.450	0.197
0.3	0.456	0.178	0.437	0.116	0.414	0.096
0.4	0.403	0.108	0.394	0.039	0.374	0.017
0.5	0.344	0.058	0.349	−0.014	0.333	−0.044
0.6	0.282	0.022	0.303	−0.053	0.293	−0.088
0.7	0.218	0.003	0.261	−0.077	0.258	−0.118
0.8	0.155	−0.004	0.222	−0.086	0.229	−0.134
0.9	0.092	−0.006	0.187	−0.086	0.205	−0.140
1.0	0.029	−0.003	0.157	−0.076	0.189	−0.136

3. Rising stage

In a deep sea, the modes of the rising stage in the case of " no bottom-current " and of " no bottom-friction " are the same, and in the present case of bottom-friction also we may allow the same result without any further discussion.

In shallow water, the development of current differs greatly according to the bottom-condition. Having already determined the final current, however, the most important point for the rising stage will only be the rapidity of becoming steady, and the rapidity will not be of different order either when the bottom-friction is proportional to the bottom-velocity or when proportional to its square.

Now, while it is very complicated to solve the case when the bottom-friction is proportional to the square of the slip velocity, we can easily solve the rising stage if the friction varies with the bottom velocity and we shall be contented of it.

Let the bottom-condition be

$$\frac{\partial w}{\partial z} = -\frac{f'\rho}{\mu} w \qquad \text{at} \quad z = H. \tag{21}$$

Now putting $w = w_1 - w_2$, we separate the above condition into two parts :

$$-\mu \partial w_1/\partial z = f'\rho w_1 \qquad \text{at} \quad z = H,$$
$$-\mu \partial w_2/\partial z = f'\rho w_2 \qquad \text{at} \quad z = H. \tag{21'}$$

Then the decaying part w_2 will be given by

$$w_2 = \sum_{n=1}^{\infty} A_n \cos \beta_n z \, e^{-(\nu\beta_n^2 + i2\varpi)t}, \tag{22}$$

where β_n must be a real root of

$$\beta \tan \beta H = \frac{f'\rho}{\mu} \equiv h \text{ (say)} \tag{23}$$

by condition (21).

Since with such β_n

$$\int_0^H \cos \beta_m z \cos \beta_n z dz = 0 \qquad \text{for} \quad m \neq n$$
$$= \frac{H(\beta_n^2 + h^2) + h}{2(\beta_n^2 + h^2)} \qquad \text{for} \quad m = n,$$

the initial condition ($w_2 = w_1$ when $t = 0$) gives

$$A_n = \frac{2(\beta_n^2 + h^2)}{H(\beta_n^2 + h^2) + h} \int_0^H w_1 \cos \beta_n z dz.$$

Noting that

$$\int_0^H \cosh a(H-z) \cos \beta_n z dz = \frac{h \cos \beta_n H + a \sinh aH}{a^2 + \beta_n^2},$$

$$\int_0^H \cosh az \cos \beta_n z dz = \frac{\cos \beta_n H}{a^2 + \beta_n^2}(a \sinh aH + h \cosh aH),$$

and substituting w_1 in the expression of A_n by the previously obtained value, we have

$$A_n = \frac{T}{\mu a} \frac{2(\beta_n^2 + h^2)}{H(\beta_n^2 + h^2) + h} \cdot \frac{1}{(a^2 + \beta_n^2) \sinh aH}$$
$$\times \Big[i(h \cos \beta_n H + a \sinh aH) - \eta' e^{i\theta} \cos \beta_n H (a \sinh aH$$
$$+ h \cosh aH) \Big]. \tag{24}$$

Writing $h = f'\rho/\mu = k/\xi'$, $a = (1+i)k$, and separating A_n into the real part (A_{nx}) and the imaginary part (A_{ny}), we finally obtain

$$\left.\begin{aligned}
A_{nx} = C\Bigg\{ & \Big[\frac{\beta_n^2}{k^2} \cdot a' \cos \beta_n H + 2(\xi' + b' \cos \beta_n H) \Big] \\
& - \eta' \cos \beta_n H \Big[\Big\{ \frac{\beta_n^2}{k^2} \Big(\xi' + \frac{a}{2} \Big) - b \Big\} \cos \theta \\
& + \Big\{ 2\xi' + a + \frac{b}{2} \frac{\beta_n^2}{k^2} \Big\} \sin \theta \Big] \Bigg\}, \\
A_{ny} = C\Bigg\{ & \Big[\frac{\beta_n^2}{k^2} (\xi' + b' \cos \beta_n H) - 2a' \cos \beta_n H \Big] \\
& - \eta' \cos \beta_n H \Big[\Big\{ \frac{\beta_n^2}{k^2} \Big(\xi' + \frac{a}{2} \Big) - b \Big\} \sin \theta \\
& - \Big\{ 2\xi' + a + \frac{b}{2} \frac{\beta_n^2}{k^2} \Big\} \cos \theta \Big] \Bigg\},
\end{aligned}\right\} \tag{25}$$

where

$$C = \frac{T}{\mu k} \cdot \frac{2\left(\frac{\beta_n^2}{k^2} + \frac{1}{\xi'^2} \right)}{kH\left(\frac{\beta_n^2}{k^2} + \frac{1}{\xi'^2} \right) + \frac{1}{\xi'}} \cdot \frac{1}{\left(4 + \frac{\beta_n^4}{k^4} \right) \xi'},$$

$$\left.\begin{aligned} a&=\frac{\sinh 2kH-\sin 2kH}{\cosh 2kH-\cos 2kH}, \\ b&=\frac{\sinh 2kH+\sin 2kH}{\cosh 2kH-\cos 2kH}, \\ a'&=\frac{\sinh kH\cos kH+\cosh kH\sin kH}{\cosh 2kH-\cos 2kH}, \\ b'&=\frac{\sinh kH\cos kH-\cosh kH\sin kH}{\cosh 2kH-\cos 2kH}, \end{aligned}\right\} \qquad (26)$$

and ξ', η' and θ have the same meaning as in the preceding paragraph.

We can now evaluate the decaying part w_2 and consequently the developing current w by the above equations; and Table 5 and the spiral curves in Figs. 7, 8, and 9 give some examples, in which $T/\mu k$ is taken as unit velocity and the time is reckoned in pendulum hours as usual.

Table 5

Decaying part w_2.

(when bottom-friction ∝ bottom-velocity)

	$H=\frac{D}{10}$					
	$\xi'=0.662$		$\xi'=2.016$		$\xi'=3.540$	
t	u_2	v_2	u_2	v_2	u_2	v_2
0^h	+0.348	+0.810	+1.073	+0.905	+1.376	+0.704
1	+0.227	+0.149	+0.902	+0.106	+1.192	−0.135
2	+0.093	+0.007	+0.564	−0.242	+0.768	−0.566
3	+0.028	−0.014	+0.248	−0.332	+0.304	−0.696
6	−0.001	−0.001	−0.102	−0.076	−0.351	−0.153
9	−0.000	+0.000	−0.023	+0.031	−0.077	+0.177
12			+0.010	+0.007	+0.089	+0.039
18			−0.001	−0.001	−0.023	−0.010
24			+0.000	+0.000	+0.006	+0.003
30					−0.000	−0.000

Table 5

(continued)

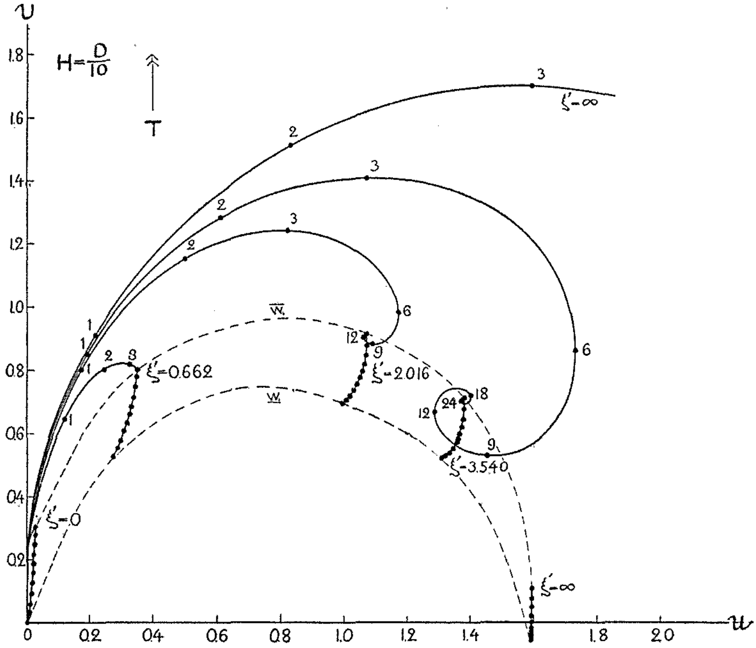
	$H=\frac{D}{4}$					
	$\xi'=0.177$		$\xi'=0.562$		$\xi'=1.175$	
t	u_2	v_2	u_2	v_2	u_2	v_2
0^h	+0.383	+0.693	+0.554	+0.625	+0.658	+0.511
1	+0.286	+0.123	+0.456	+0.058	+0.554	−0.081
2	+0.153	−0.018	+0.285	−0.119	+0.345	−0.272
3	+0.061	−0.045	+0.126	−0.166	−0.129	−0.323
6	−0.006	−0.007	−0.051	−0.039	−0.160	−0.064
9	−0.001	+0.001	−0.012	+0.016	−0.031	+0.078
12	+0.000	+0.000	+0.005	+0.004	+0.040	+0.016
18			−0.000	−0.000	−0.009	−0.004
24					+0.003	+0.001
30					−0.000	−0.000

(continued)

	$H=\frac{D}{2}$					
	$\xi'=0.076$		$\xi'=0.874$		$\xi'=2.32$	
t	u_2	v_2	u_2	v_2	u_2	v_2
0^h	+0.544	+0.533	+0.512	+0.477	+0.487	+0.463
1	+0.435	−0.044	+0.374	−0.119	+0.331	−0.146
2	+0.281	−0.201	+0.235	−0.256	+0.197	−0.269
3	+0.111	−0.246	+0.067	−0.303	+0.034	−0.312
6	−0.119	−0.054	+0.212	−0.045	−0.259	−0.030
9	−0.026	+0.059	−0.034	+0.155	−0.023	+0.211
12	+0.032	+0.014	+0.107	+0.024	+0.173	+0.019
18	−0.007	−0.003	−0.065	−0.014	−0.117	−0.021
24	+0.002	+0.001	+0.026	+0.005	+0.079	+0.008
48	+0.000	+0.000	+0 002	+0.000	+0.016	+0.001
72					+0.003	+0.000
96					+0.000	+0.000

Fig. 7 & Fig. 8

Vertical distribution of steady current, and development of surface current.
(When bottom-friction ∝ bottom-current.)



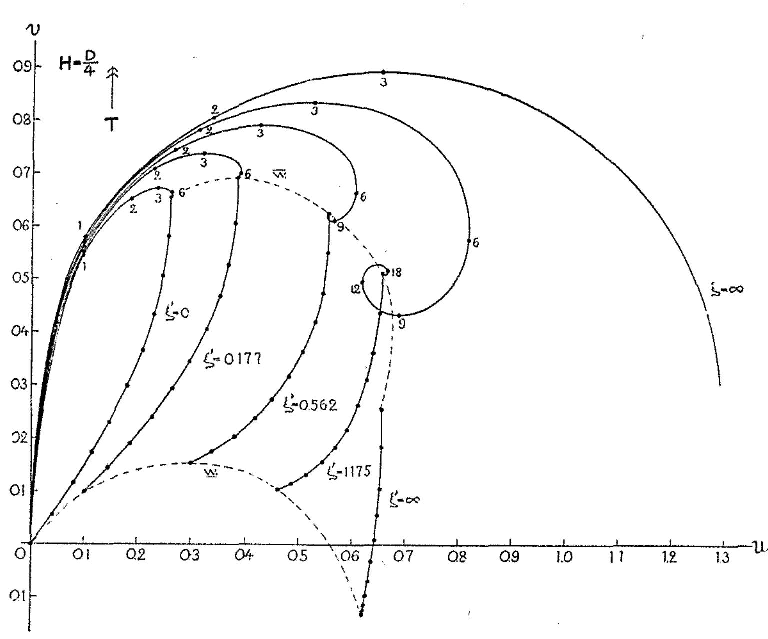
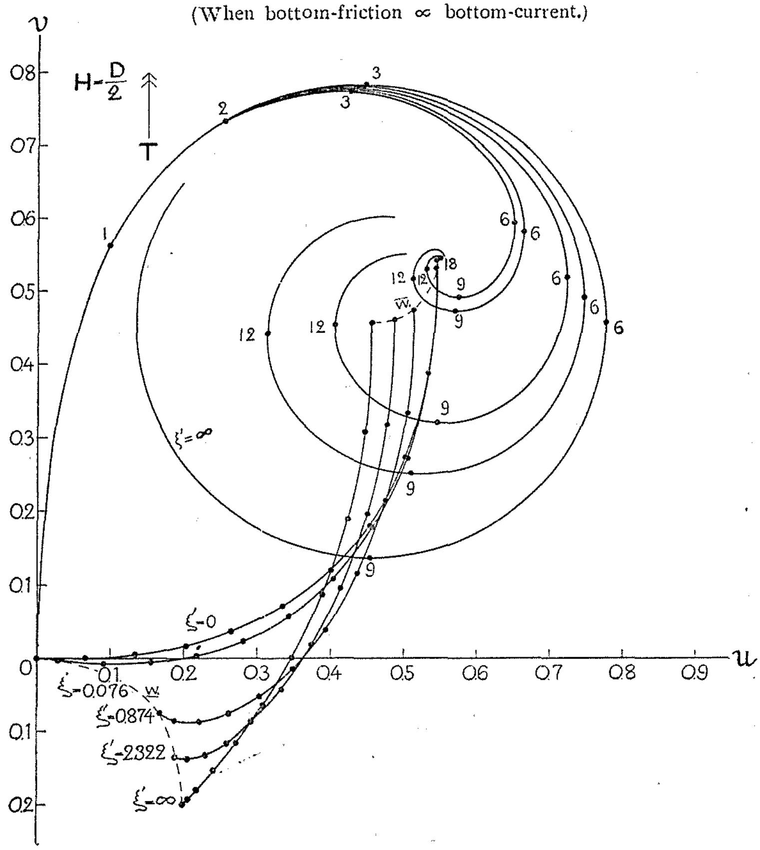


Fig. 9

Vertical distribution of steady current, and development of surface current.

(When bottom-friction ∝ bottom-current.)



The above results show that:

1) The current oscillates with a period of 12 pendulum hours as usual.

2) The convergency factor $e^{-\nu\beta_n^2 t}$ is quite effective for a shallow sea ($H \leqq D/4$), especially for $n \geqq 2$, so that all terms except the first in eq. (22) become negligible in one pendulum hour only.

3) The smaller the value of ξ' and accordingly the larger the value of viscosity μ, the more and more the current approaches that seen in the case of "no bottom-current".

Reversely the greater the value of ξ' and accordingly the smaller the value of μ, the nearer the current approaches that found in the case of "no bottom-friction"; but the damping is very good beyond expectation, even for a very slight friction at the bottom and accordingly for when the steady current itself is very near to that with " no bottom-friction ".

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