The Octave from Tuning-forks

By

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Abstract

Though the ordinary overtones of a tuning-fork do not make a hramonic series, it is known that the cctave is heard when a fork is vigorously bowed and placed near a suitable resonator. The problem of how the octave is generated has been discussed by Lord Rayleigh, E. H. Barton, and D. C. Miller. In the present paper this problem is studied experimentally and a dynamical theory is proposed to explain it.

1. Introduction

On sounding a tuning-fork without a resonance-box we usually hear many overtones whose frequencies are nearly in the ratios 1 : $6\frac{1}{4}$: $17\frac{1}{2}$: .... Since these ratios well agree with those of the frequencies of the flexural vibration of a clamped-free bar, the tuning-fork is regarded as two clamped-free bars mounted on a heavy base. This way of regarding the mode of vibration is due to Lord Rayleigh.

Chladni, on the other hand, regarded it as a single free-free bar bent in the form of an elongated **U**. This way of regarding the vibration gives a clearer idea of the manner in which the vibration of the fork causes vibration of the attached resonance-box. As the bar deviates more and more from the straight form, the two nodes of the fundamental vibration approach each other more and more closely, and when it is in the form of an elongated **U**, the short piece between the nodes at the base vibrates up and down when the prongs vibrate outwards and inwards, and the up-and-down motion is communicated to the resonance-box.

Although the ordinary normal modes of vibration of a tuning-fork are inharmonic as stated above, it is nevertheless known that the octave is heard with remarkable intensity, if the fork is employed with its stalk pressed against a resonating board.

In his paper 'Octave from Tuning-forks' Lord Rayleigh[1] said: "By the construction of a fork the moving parts are carefully balanced, and the motion is approximately isolated. In the ideal tuning-fork, composed of equal masses moving to and fro in a straight line, the isolation would be complete, and there would be no tendency whatever to communicate motion to surrounding bodies. In an actual fork, however, even if the direction of motion of the masses were as nearly as possible perpendicular to the stalk, the necessary curvature of the paths would give rise to an unbalanced centrifugal force tending to set the sounding board in vibration. The force thus arising is indeed of the second order, and might probably be neglected, were it not that the apparatus is especially suited to bring it into prominence."

After giving a method of generating the octave of the fork, Professor Barton[2] said: "Let us now discuss this result and see where and how the octave may be generated. To this inquiry it is difficult to give any decisive answer. Probably the octave arises or is strengthened at several points in the chain of phenomena." And he pointed out five considerations to explain the octave and then he concluded as follows[3]: "Thus, so far as the present experiment goes, all we can say is, *first*, that the fork does not produce the octave by subdivision of itself, as in the case of a vibrating string or organ-pipe; and, *second*, that the part played by the mechanism of the ear in producing the octave is probably very slight. The octave may be produced at any of the other points in the apparatus concerned in the phenomena, namely, by the fork itself as an asymmetrical vibrator, by the vortical motion of the air immediately round the prongs, or by the resonator as an asymmetrical system under simple harmonic forcing. Probably the relative portions of the effect produced at each point would depend upon the proportions and disposition of the apparatus."

Professor Miller[4] in his book wrote as follows: "When a tuning-fork mounted on a resonance-box is sounded by vigorous bowing, it sometimes produces a strong octave overtone; such a tone is not natural to either the fork or the box, and is probably due to some

1. Lord Rayleigh, Phil. Mag. 3, 456 (1877); Scientific Papers, vol. 1, p. 318.
2. E. H. Barton, Text-Book on Sound, § 310, pp. 395—396 (1919).
3. E. H. Barton, loc. cit. § 314, p. 399 (1919).
4. D. C. Miller, Science of Musical Sound, p. 188 (1926).

peculiar condition of the combination which has not yet been fully explained."

2. Experimental studies on the octaves from tuning-forks

One of the tuning-forks used in the experiments was a fork C' by Maxkohl of which a sketch is given in Fig. 1 (A), and the other was a fork which was constructed for the purpose of the present experiments, a sketch of it being given in Fig. 1 (B). The frequencies of the two forks were almost the same. They were screwed on to various resonance-boxes and were struck by a hammer, and then the sounds emitted were recorded by a Low-Hilger audiometer.

Fig. 1

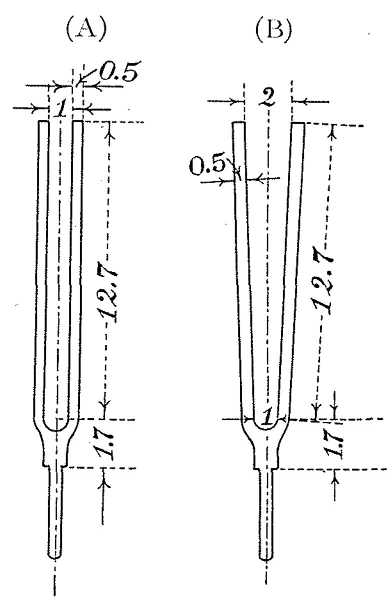


Fig. 1, A in Plate I shows the sound-waves from the fork A screwed to a resonance-box in tune with the fork, and Fig. 1, B in Plate I shows the waves from the fork B screwed to the same box. In these two figures we see that the fundamental tones only are present and the octave overtones are scarcely found. In Fig. 3, A in Plate I, i. e. in the waves from fork A screwed to a box tuned to G'^{z}, the presence of the octave is marked, while in the waves from fork B with the same box the octave is not yet noticeable.

When the fork was screwed on to a box whose fundamental frequency was the octave of the fork, we found that the octave tone was predominant and the fundamental tone was scarcely heard. Their wave-forms are given in Fig. 3 in Plate II.

From the figures in Plate I and Plate II we see that the tuning-forks with resonance-boxes, whether the prongs are parallel or not, produce octave overtones, and that the fundamental tones are always more intense in fork B than in fork A.

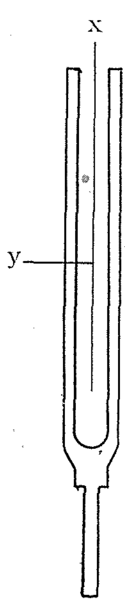
Fig. 4, A in Plate II shows the sound-waves from fork A placed near a box in tune with the octave. In this figure we see that the fundamental frequency predominates and the octave is of small intensity. Comparing this figure with the other figures in Plate II, we see that the octave vibration is predominant especially when the stalk of the fork is screwed on to a resonance-box and consequently it may be supposed that at the stalk the amplitude of the octave vibration is comparable to that of the fundamental, although the latter is more prominent at the vibrating prongs.

The curves in the plates were analyzed by the usual method of harmonic analysis, and the results are shown in the diagrams. The abscissae in the diagrams represent the ratios of the frequencies of the partial tones and the ordinates represent the ratios of their amplitudes. In one of the diagrams, which corresponds to Fig. 3, A in Plate II, we find a partial tone with a frequency 6.2 times that of the fundamental. This tone is the ordinary second partial tone of the tuning-fork.

3. A theory explanatory of the octaves from tuning-forks with parallel prongs

Now let us consider dynamically how the octave may be generated. If the curvature at the base of the fork is taken into account, the analysis naturally becomes very difficult, so we assume that the fork consists of two straight parallel bars mounted on a block of the same material.

Fig. 2



Let the axis of x be taken along the bars and let (ξ, η) represent the displacement of any point, at time t. It is assumed that the motion is restricted to flexural vibrations in the plane of the fork. Then, the rotatory inertia being neglected, the equation of motion in the direction of y is given by

$$\frac{\partial^2 \eta}{\partial t^2} + \frac{IE}{\omega\rho} \frac{\partial^4 \eta}{\partial x^4} = 0, \qquad (1)$$

where I is the moment of inertia of the cross-section, E the Young's modulus, ω the area of a cross-section, and ρ the volume density of the material. When the vibrations of the two bars are symmetrical with respect to the x-axis, the displacement of the base in the direction of the y-axis vanishes. Accordingly the boundary conditions at the base of the bars are given by

$$\eta=0, \quad \frac{\partial\eta}{\partial x}=0 \quad \left[\text{at } x=-\frac{l}{2}\right], \tag{2}$$

where l is the length of the bars. The conditions at the free ends of the prongs are

$$\frac{\partial^2\eta}{\partial x^2}=0, \quad \frac{\partial^3\eta}{\partial x^3}=0 \quad \left[\text{at } x=\frac{l}{2}\right]. \tag{3}$$

The fundamental vibration which satisfies differential equation (1) and the boundary conditions (2) and (3) is given by

$$\eta=a\left\{\sin m\frac{l}{2}\cosh mx+\cosh m\frac{l}{2}\sin mx\right\}\cos 2\pi\nu t, \tag{4}$$

where

$$m^4=\frac{4\pi^2\omega\rho}{IE}\nu^2, \tag{5}$$

m being the smallest positive root of the equation

$$\coth m\frac{l}{2}=\tan m\frac{l}{2},$$

whose numerical value is $0.59686\ \pi/l$.

Let x, y and x', y' be the coordinates of a point before and after the strain. Then the condition that the displacement may be inextensional is given by

$$(dx')^2+(dy')^2=(dx)^2+(dy)^2.$$

If we assume that the bars vibrate by pure bending, the displacement (ξ, η) must satisfy the above condition. In the present case, as the axis of x is taken along the bars, $dy=0$ and dx', dy' are given by

$$dx'=\left(1+\frac{\partial\xi}{\partial x}\right)dx, \qquad dy'=\frac{\partial\eta}{\partial x}dx;$$

and therefore the condition of no extension becomes

$$\left(1+\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2=1,$$

or

$$2\frac{\partial\xi}{\partial x}+\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2=0.$$

Since the second term is of a higher order of small quantity than the first term, it may be neglected, and we get

$$\frac{\partial\xi}{\partial x} = -\frac{1}{2}\left(\frac{\partial\eta}{\partial x}\right)^2 \qquad (6)$$

as the condition of no extension.

Substituting the value of η given by (4) in (6), we obtain

$$\frac{\partial\xi}{\partial x} = -\frac{m^2a^2}{4}\left\{\sin m\frac{l}{2}\sinh mx + \cosh m\frac{l}{2}\cos mx\right\}^2$$
$$\times(1+\cos 4\pi\nu t). \qquad (7)$$

It follows from this that ξ is also a simple harmonic function of the time, but its frequency is 2ν, namely twice the frequency of the fundamental vibration. From (7) we see that ξ is of the order a^2 and is small compared with η which is of the order a, and therefore at the free ends of the prongs the fundamental tone may be prominent and the octave will scarcely be heard there. But the motion which is communicated to the sounding board is the motion of the base and ξ does not vanish at the base, while η vanishes there.

This result well explains how the octave may be generated from the tuning-fork, especially when it is employed with its stalk pressed against a resonating board.

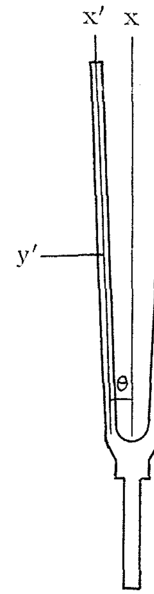
Although the ordinary octaves produced by the vibrations of a string or an air column in an open pipe are the normal modes of vibration which are perfectly independent of the fundamental vibration, the octaves from tuning-forks are due to the fundamental vibrations. If the prongs are in a state of fundamental vibration, the stalk vibrates necessarily with a frequency twice that of the fundamental. The ratio of the amplitude of the octave to that of the fundamental is of the order a and it follows from this that the octave becomes predominant when the fork vibrates vigorously.

From (4) and (7) we see that in our approximate results the fundamental vibration vanishes at the base and the octave alone remains there. But if the curvature at the base is taken into account, it is easily seen that the fundamental vibration does not entirely vanish at the base although its amplitude is very small.

4. Case where the prongs are not parallel

Next let us consider a tuning-fork of which the prongs are not parallel and study how the octave is affected. Let the fork consist of two straight bars mounted on a base. Take the axis of x' along one of the bars, and denote by θ the angle between the axis of x' and the axis of the fork which is taken as the x-axis.

Fig. 3



In the case where the prongs of the fork are parallel the condition at the base is very briefly given by (2), but in the present case such a simple condition does not hold for the displacement (ξ', η') referred to the (x', y') coordinates and the analysis becomes complicated, and therefore instead of obtaining the vibrations of the base we shall calculate the forces at the base due to the vibrations of the prongs, when the base is fixed so that the stalk may be unable to vibrate up and down.

In this state the condition at the base is

$$\left.\begin{aligned} &\eta'=0, \quad \frac{\partial \eta'}{\partial x'}=0 \quad \left[\text{at } x'=-\frac{l}{2}\right], \\ &\xi'=0 \qquad\qquad\quad\; \left[\text{at } x'=-\frac{l}{2}\right], \end{aligned}\right\} \quad (2')$$

and the displacement (ξ', η') is given by

$$\eta'=a\left\{\sin m\frac{l}{2}\cosh mx'+\cosh m\frac{l}{2}\sin mx'\right\}\cos 2\pi\nu t, \qquad (4')$$

$$\xi'=-\frac{m^2a^2}{4}\int_{-\frac{l}{2}}^{x'}\left\{\sin m\frac{l}{2}\sinh mx'+\cosh m\frac{l}{2}\cos mx'\right\}^2dx'$$
$$\times(1+\cos 4\pi\nu t)\,; \qquad (7')$$

the lower limit of integration in (7′) being $-\frac{l}{2}$ by condition (2′).

The force at the base due to the two bars is

$$F=\int_{-\frac{l}{2}}^{\frac{l}{2}}\omega\rho\left(-\frac{\partial^2\eta'}{\partial t^2}\sin\theta+\frac{\partial^2\xi'}{\partial t^2}\cos\theta\right)dx'.$$

Substituting the values of (4′), (7′), and (5) in the above equation, we obtain

$$F=IEPm^4a\sin\theta\cos 2\pi\nu t+IEQm^6a^2\cos\theta\cos 4\pi\nu t, \qquad (8')$$

where

$$\left.\begin{aligned} P &\equiv \int_{-\frac{l}{2}}^{\frac{l}{2}} \left\{ \sin m\frac{l}{2} \cosh mx' + \cosh m\frac{l}{2} \sin mx' \right\} dx', \\ Q &\equiv \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{l}{2}}^{x''} \left\{ \sin m\frac{l}{2} \sinh mx' + \cosh m\frac{l}{2} \cos mx' \right\}^2 dx' dx'' \end{aligned}\right\} \tag{9'}$$

From equation (8′) we see that the force due to the fundamental vibration is proportional to $\sin\theta$ and it is therefore much affected by the magnitude of θ, while the force due to the octave vibration is almost independent of θ so long as the value of θ is small since the force is proportional to $\cos\theta$.

In conclusion the writers wish to express their cordinal thanks to Professor K. Tamaki on whose suggestion this investigation was undertaken.

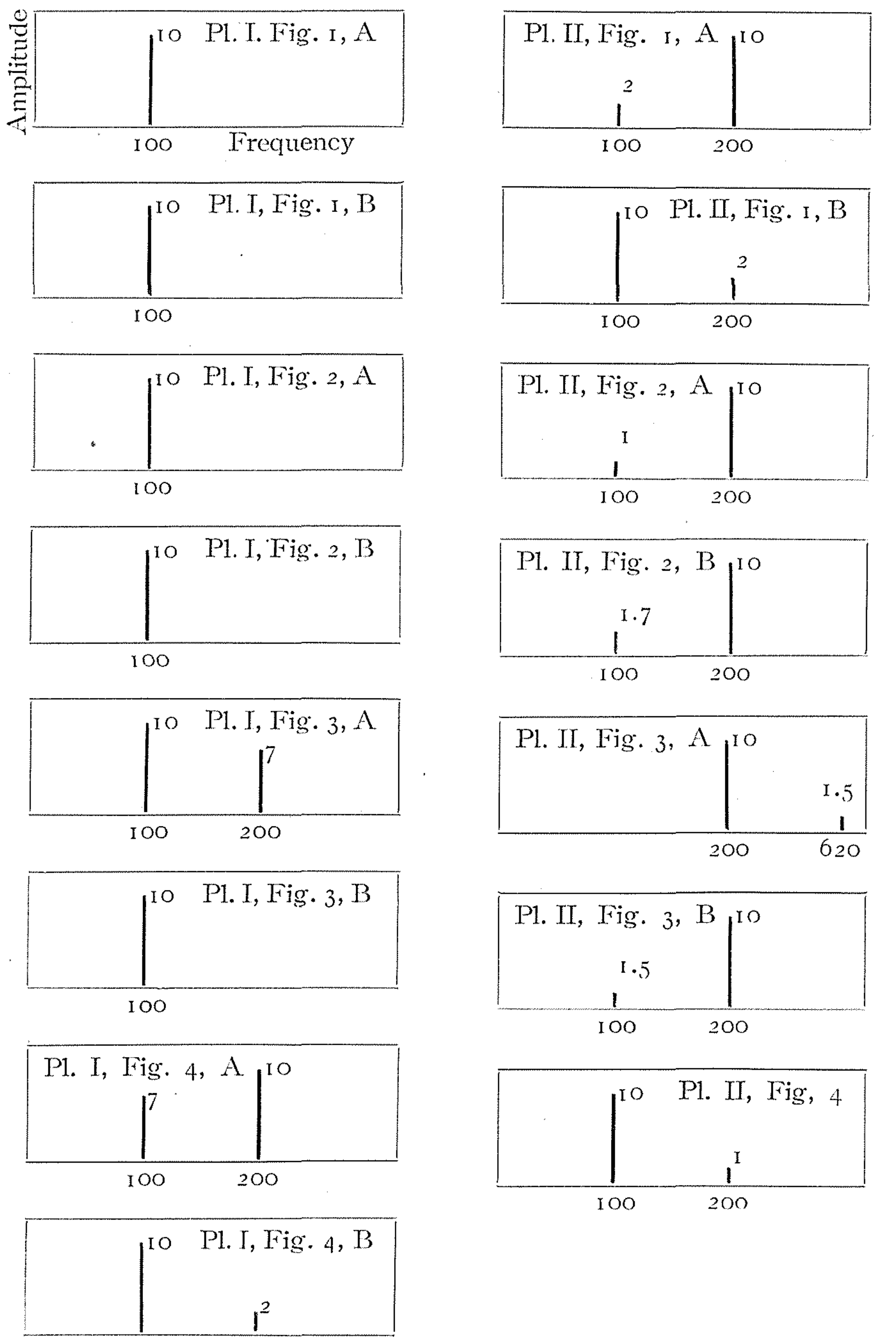
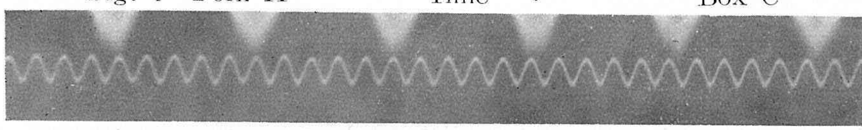

Amplitude
Frequency
Pl. I. Fig. 1, A
10
100
Pl. I, Fig. 1, B
10
100
Pl. I, Fig. 2, A
10
100
Pl. I, Fig. 2, B
10
100
Pl. I, Fig. 3, A
10
7
100
200
Pl. I, Fig. 3, B
10
100
Pl. I, Fig. 4, A
10
7
100
200
Pl. I, Fig. 4, B
10
2
100
200
Pl. II, Fig. 1, A
10
2
100
200
Pl. II, Fig. 1, B
10
2
100
200
Pl. II, Fig. 2, A
10
1
100
200
Pl. II, Fig. 2, B
10
1·7
100
200
Pl. II, Fig. 3, A
10
1·5
200
620
Pl. II, Fig. 3, B
10
1·5
100
200
Pl. II, Fig, 4
10
1
100
200

Plate I

Fig. 1 Fork A Time⟶ Box C^1



Fork B $|\leftarrow {}^1/_{50}S \rightarrow|$ Box C^1

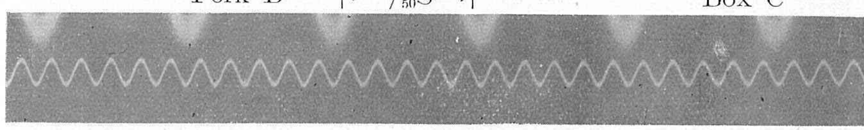
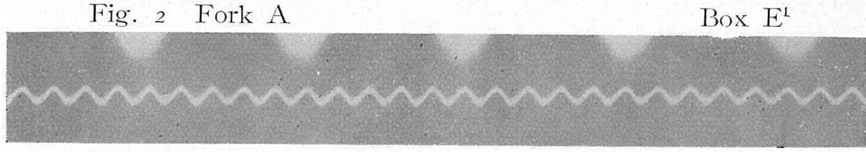


Fig. 2 Fork A Box E^1



Fork B Box E^1

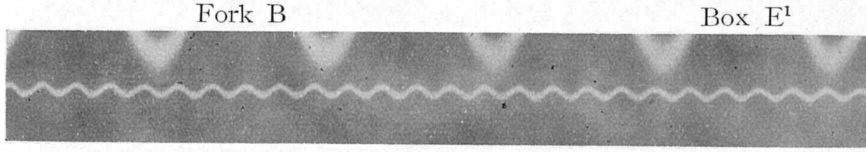
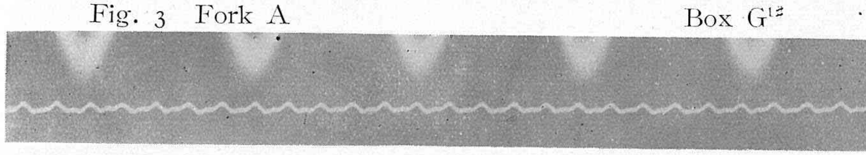


Fig. 3 Fork A Box $G^{1\sharp}$



Fork B Box $G^{1\sharp}$

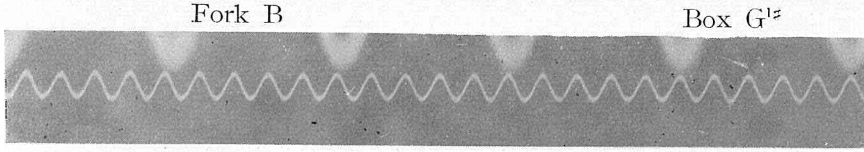
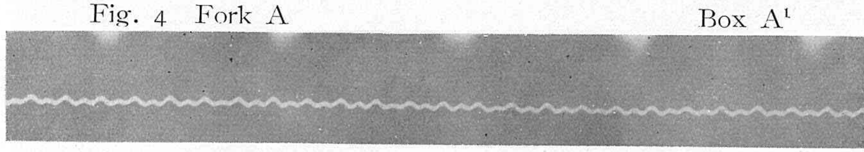


Fig. 4 Fork A Box A^1



Fork B Box A^1

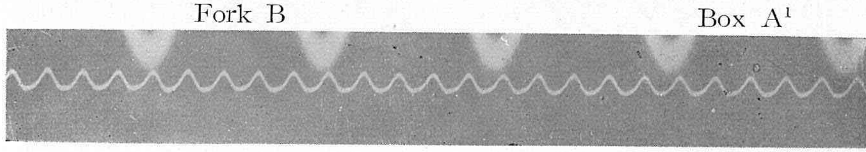
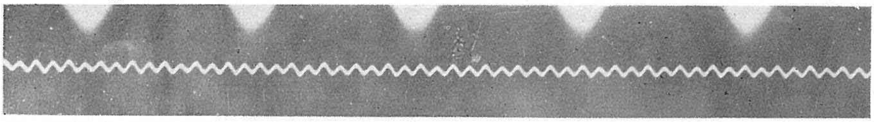


Plate II

Fig. 1 Fork A Box $A^{1\sharp}$



Fork B Box $A^{1\sharp}$

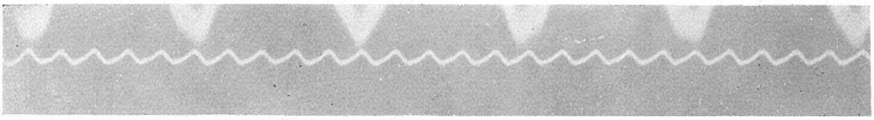
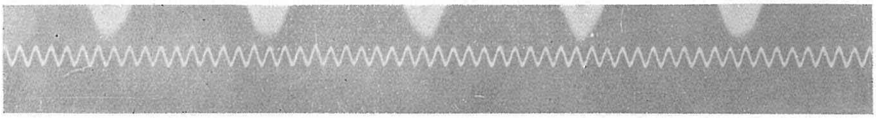


Fig. 2 Fork A Box B^1



Fork B Box B^1

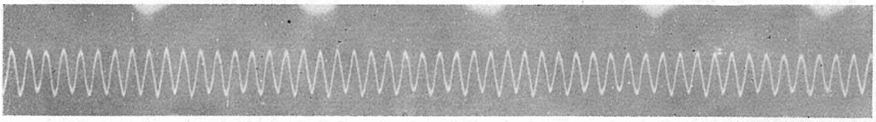
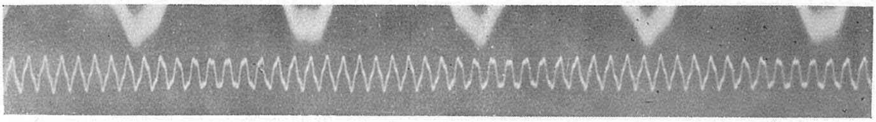


Fig. 3 Fork A Box C^2



Fork B Box C^2

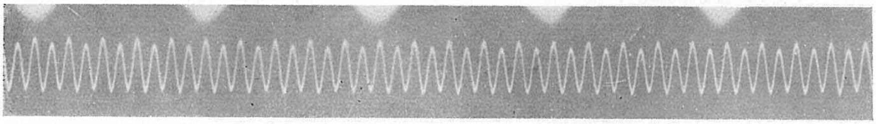


Fig. 4 Fork A Box C^2 (not fixed)