On the Density Current in the Ocean
III. The Case of a Finite Bottom-Friction Depending on the Slip Velocity

By

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Abstract

The density current in a uniform solenoidal field is here treated again on the assumption most probable for a real sea, that there acts a finite bottom-friction depending on the slip velocity.

The distribution of the steady current and the mode of development of the current are found to be very different according to the magnitude of the density-gradient and the turbulence of the sea.

For a large density-gradient and a little turbulence, the current approaches that seen in the case of "no bottom-current"; and for a small density-gradient and a large viscosity it approaches that seen in the case of "no bottom-friction."

We have already studied the "density current" in the extreme cases of "no bottom-current"[1] and "no bottom-friction".[2] Let us here investigate the current in a sea of uniform solenoidal field on the more probable assumption that the bottom-friction has a finite value depending on the slip velocity there. We use the same notation as in the previous papers.

1. Steady Value of Density Current

§ 1. *When the bottom-friction is proportional to the square of the slip velocity*, we must put the bottom-condition as

$$\left|\frac{dw_1}{dz}\right|_{z=H} = -\frac{f\rho V_H^2}{\mu}e^{i\theta}, \qquad (1)$$

1. Nomitsu, These Memoirs, A. **16**, 261 (1933).
2. Ditto, **16**, 383 (1933).

where f denotes the coefficient of friction, and V_H and θ are the speed and the direction-angle of the bottom-velocity.

If in an open boundless sea we consider the "density current" which is produced by a difference of density alone and not attended by any sloping of the sea-surface, we have already found that the following formula,

$$\left.\begin{aligned} w_1 &= \frac{a}{4k^3}\left[2K\cosh\alpha z + (1-i)e^{-\alpha z} + 2kz\right], \\ a &= -\frac{g}{\mu}\frac{\partial \rho}{\partial y}, \quad \alpha = (1+i)k, \quad k = \sqrt{\frac{\rho\omega\sin\lambda}{\mu}}, \end{aligned}\right\}$$

satisfies the equation of motion and the surface condition.

The arbitrary constant K being determined by (1), the above solution becomes

$$w_1 = \frac{a}{4k^3}\left[(1-i)\frac{\sinh\alpha\left(\frac{H}{2}-z\right)}{\cosh\frac{1}{2}\alpha H} + 2kz\right] - \frac{f\rho V_H^2 e^{i\theta}}{\mu\alpha}\cdot\frac{\cosh\alpha z}{\sinh\alpha H}, \tag{2}$$

of which the former part is nothing but the density current with "no bottom-friction" and the latter part

$$\Delta w_1 = -\frac{f\rho V_H^2 e^{i\theta}}{\mu\alpha}\cdot\frac{\cosh\alpha z}{\sinh\alpha H} \tag{3}$$

denotes the correction term due to the bottom friction $(f\rho V_H^2 e^{i\theta})$ in the form of a drift current, also with "no bottom-friction".

In order to determine V_H and θ, we use the obvious relation

$$V_H e^{i\theta} = |w_1|_{z=H}$$

i. e.,

$$\begin{aligned} V_H &= \frac{a}{4k^3}\left[(i-1)\tanh\tfrac{1}{2}\alpha H + 2kH\right]e^{-i\theta} - \frac{f\rho V_H^2}{\mu\alpha}\coth\alpha H \\ &= \frac{a}{4k^3}\left[(p\cos\theta + q\sin\theta) + i(q\cos\theta - p\sin\theta)\right] - \frac{f\rho V_H^2}{2\mu k}(r - is), \end{aligned} \tag{4}$$

where

$$\left.\begin{aligned} p&=2kH-\frac{\sinh kH+\sin kH}{\cosh kH+\cos kH},\\ q&=\frac{\sinh kH-\sin kH}{\cosh kH+\cos kH},\\ r&=\frac{\sinh 2kH-\sin 2kH}{\cosh 2kH-\cos 2kH},\\ s&=\frac{\sinh 2kH+\sin 2kH}{\cosh 2kH-\cos 2kH}.\end{aligned}\right\} \qquad (5)$$

Separating (4) into the real and the imaginary part we have

$$\left.\begin{aligned} V_H&=\frac{a}{4k^3}(p\cos\theta+q\sin\theta)-r\frac{f\rho V_H^2}{2\mu k},\\ 0&=\frac{a}{4k^3}(q\cos\theta-p\sin\theta)+s\frac{f\rho V_H^2}{2\mu k}.\end{aligned}\right\} \qquad (6)$$

The sum of the above two equations multiplied by p and q respectively, and the difference between the equations multiplied by q and p respectively, are

$$\left.\begin{aligned} pV_H+(pr-qs)\frac{f\rho V_H^2}{2\mu k}&=\frac{a}{4k^3}(p^2+q^2)\cos\theta,\\ qV_H+(qr+ps)\frac{f\rho V_H^2}{2\mu k}&=\frac{a}{4k^3}(p^2+q^2)\sin\theta.\end{aligned}\right\} \qquad (7)$$

Eliminating θ from the above equations, we get

$$\frac{r^2+s^2}{4}\cdot\frac{f^2\rho^2}{\mu^2k^2}V_H^4+r\frac{f\rho}{\mu k}V_H^3+V_H^2=(p^2+q^2)\left(\frac{a}{4k^3}\right)^2,$$

which we write in the form

$$A\eta^4+B\eta^3\xi+\eta^2\xi^2-C=0, \qquad (8)$$

where

$$\left.\begin{aligned} &\xi=2k^2\sqrt{\frac{\mu}{f\rho a}}=2\omega\sin\lambda\sqrt{\frac{\rho}{fg\left(-\frac{\partial\rho}{\partial y}\right)}},\\ &\eta=2k\sqrt{\frac{f\rho}{\mu a}}V_H=\frac{V_H}{\xi V_a}, \qquad V_a=\frac{a}{4k^3},\\ &A=\frac{r^2+s^2}{4}, \quad B=r, \quad C=p^2+q^2,\end{aligned}\right\} \qquad (9)$$

and these are all dimensionless quantities.

Since it is very laborious to find directly V_H or η for a given value of $\frac{\partial\rho}{\partial y}$ or ξ, we rather construct a curve of ξ, η by reverse calculation of ξ corresponding to every given value of η. Table 1 and Fig. 1 show some concrete examples.

Fig. 1

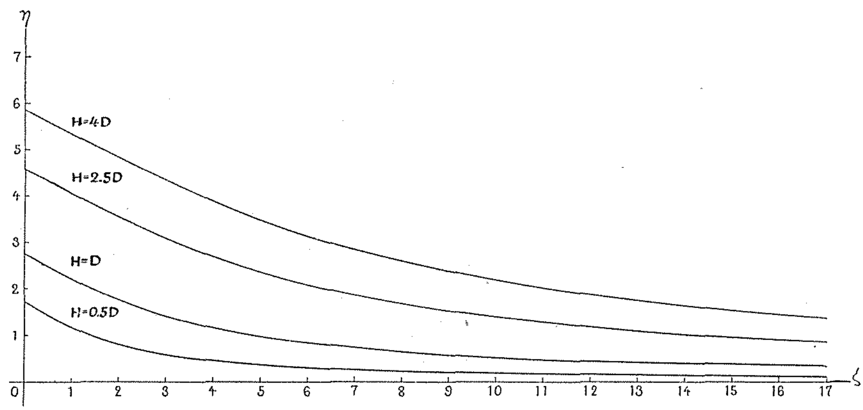


Fig. 2

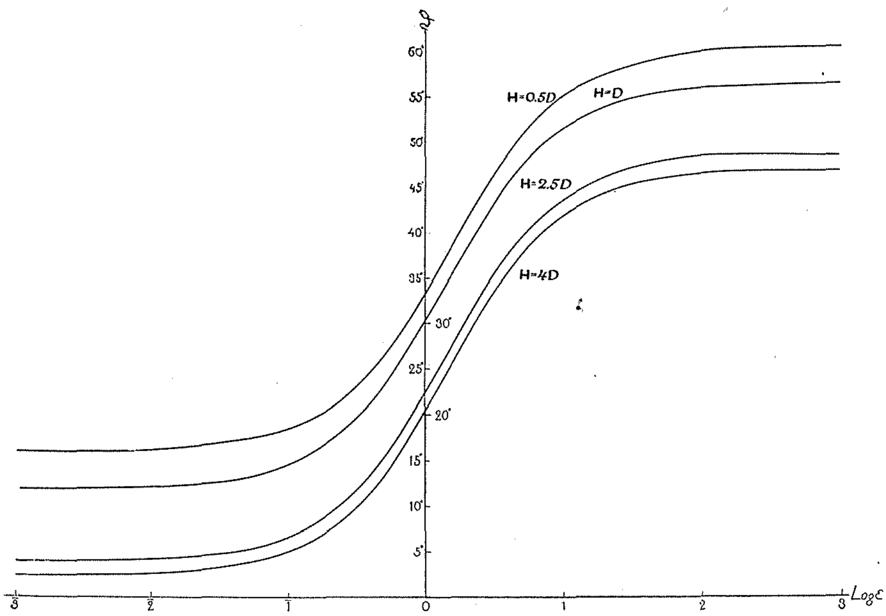


Table 1

$H=\frac{1}{2}D$		$H=D$		$H=2.5D$		$H=4D$	
ξ	η	ξ	η	ξ	η	ξ	η
0	1.712	0	2.470	0	4.567	0	5.845
0.382	1.5	0.660	2.4	0.343	4.4	0.116	5.8
0.758	1.3	1.049	2.2	0.726	4.2	0.491	5.6
0.932	1.2	1.458	2.0	1.114	4.0	0.842	5.4
1.149	1.1	1.909	1.8	1.475	3.8	1.249	5.2
1.383	1.0	2.416	1.6	1.868	3.6	1.633	5.0
1.655	0.9	3.025	1.4	2.303	3.4	2.008	4.8
1.977	0.8	3.779	1.2	2.727	3.2	2.404	4.6
2.370	0.7	4.783	1.0	3.177	3.0	2.850	4.4
2.876	0.6	6.222	0.8	3.686	2.8	3.210	4.2
3.560	0.5	8.539	0.6	4.221	2.6	3.724	4.0
4.557	0.4	13.065	0.4	4.831	2.4	4.154	3.8
6.188	0.3			5.518	2.2	4.655	3.6
				6.305	2.0	5.139	3.4
				7.247	1.8	5.792	3.2
				8.386	1.6	6.405	3.0
				9.827	1.4	7.127	2.8
				11.669	1.2	7.896	2.6
				14.237	1.0	8.804	2.4
						9.835	2.2
						11.037	2.0
						12.497	1.8
						14.263	1.6
						16.535	1.4

Having found the value of η, we can calculate the direction angle θ by the formula

$$\tan\theta=\frac{2q+(qr+ps)\varepsilon}{2p+(pr-qs)\varepsilon}, \qquad \varepsilon=\frac{\eta}{\xi}, \qquad (9)$$

which is derived from eq. (7). Table 2 and Fig. 2 show the values of θ corresponding to Table 1 and Fig. 1.

The values of η and θ thus being known, the correction term (3) may be easily evaluated by

$$\Delta w_1=-\frac{a}{4k^3}\cdot\frac{(1-i)\eta^2e^{i\theta}}{2\sinh aH}\cosh az. \qquad (3')$$

Table 2

$H=\frac{1}{2}D$		$H=D$		$H=2.5D$		$H=4D$	
ϵ	θ	ϵ	θ	ϵ	θ	ϵ	θ
∞	60.°8	∞	56.°8	∞	48.°9	∞	47.°3
3.926	48.°5	3.636	44.°7	12.828	44.°8	50.000	46.°2
1.715	39.°7	2.097	39.°0	5.785	40.°5	11.405	42.°6
1.288	36.°2	1.372	34.°0	3.591	36.°7	6.413	39.°4
0.957	32.°8	0.945	29.°7	2.576	33.°3	4.163	36.°0
0.723	29.°8	0.662	25.°8	1.927	30.°0	3.062	33.°2
0.544	27.°2	0.463	22.°5	1.476	26.°8	2.390	30.°5
0.405	24.°8	0.318	19.°7	1.173	24.°2	1.689	26.°5
0.295	22.°7	0.209	17.°2	0.944	21.°7	1.544	25.°4
0.209	20.°8	0.129	15.°3	0.760	19.°3	1.308	23.°4
0.140	19.°3	0.070	13.°8	0.616	17.°2	1.072	21.°2
0.088	18.°2	0.030	12.°7	0.497	15.°2	0.915	19.°3
0.048	17.°1	0	11.°8	0.399	13.°3	0.773	17.°5
0	15.°8			0.317	11.°7	0.662	15.°8
				0.248	10.°2	0.552	14.°2
				0.190	8.°8	0.468	12.°7
				0.142	7.°8	0.393	11.°4
				0.103	6.°7	0.329	10.°1
				0.070	5.°8	0.273	8.°9
				0	3.°8	0.224	7.°8
						0.181	7.°1
						0.144	6.°0
						0.112	5.°2
						0.085	4.°6
						0	2.°3

Finally, the density current in the present case will be obtained by adding this correction to the density current with "no bottom-friction" already given in our second paper[1] on the density current. Table 3 and Figs. 3—6 show the results thus obtained.

§ 2. *When the bottom-friction varies with the bottom-velocity*, the bottom condition may be written as

$$\mu\left|\frac{dw_1}{dz}\right|_{z=H} = -f'\rho V_H e^{i\theta}, \qquad (10)$$

1. Nomitsu, These Memoirs, A, **16**, 383 (1933).

Table 3

	$H=0.5D$						$H=D$					
	$\xi=0.758$		$\xi=1.383$		$\xi=3.560$		$\xi=1.049$		$\xi=1.909$		$\xi=3.779$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	1.360	−0.046	1.390	−0.247	1.347	−0.455	1.337	−1.121	1.282	−1.142	1.171	−1.128
0.1	1.361	−0.015	1.396	−0.215	1.358	−0.424	1.446	−1.009	1.393	−1.035	1.280	−1.032
0.2	1.363	0.063	1.413	−0.130	1.391	−0.342	1.751	−0.731	1.705	−0.774	1.592	−0.804
0.3	1.361	0.177	1.435	−0.006	1.439	−0.221	2.198	−0.353	2.168	−0.419	2.059	−0.506
0.4	1.351	0.309	1.458	0.142	1.499	−0.078	2 715	0.089	2 715	−0.008	2.628	−0.171
0.5	1.320	0.443	1.472	0.301	1.560	0.079	3.213	0.566	3.262	0.439	3.229	0.184
0.6	1.271	0.563	1.473	0.453	1.620	0.235	3.600	1.057	3.721	0.910	3.794	0.556
0.7	1.190	0.702	1.453	0.586	1.667	0.375	3.777	1.513	4.004	1.366	4.213	0.955
0.8	1.073	0.755	1.400	0.683	1.697	0.495	3.659	1.841	4.013	1.747	4.519	1.276
0.9	0.924	0.734	1.318	0.723	1.702	0.570	3.184	1.892	3.682	1 918	4.541	1.516
1.0	0.737	0.620	1.200	0.684	1.679	0.590	1.786	1.449	2.980	1.693	4.267	1.530

Table 3 (*Continued*)

	$H=2.5D$						$H=4D$					
	$\xi=1.868$		$\xi=3.686$		$\xi=7.247$		$\xi=2.008$		$\xi=3.724$		$\xi=7.896$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	1.002	−0.992	1.002	−0.995	1.001	−0.998	1.000	−1.000	1.000	−1.000	1.000	−1.000
0.1	1.569	−0.637	1.571	−0.639	1.571	−0.642	2.330	−0.359	2.330	−0.359	2.330	−0.359
0.2	2.916	−0.203	2.923	−0.203	2.929	−0.205	4.914	0.018	4.913	0.018	4.913	0.018
0.3	4.542	−0.021	4.553	−0.006	4.565	−0.003	7.536	0.032	7.535	0.032	7.534	0.032
0.4	6.216	−0.046	6.215	−0.010	6.225	−0.017	10.059	0.014	10.060	0.011	10.061	0.007
0.5	7.944	−0.156	7.888	−0.098	7.862	−0.044	12.537	0.008	12.547	0.008	12.559	0.005
0.6	9.853	−0.136	9.683	−0.141	9.554	−0.090	15.014	−0.097	15.022	−0.061	15.047	−0.025
0.7	11.886	0.436	11.630	0.166	11.348	0.040	17.839	−0.322	17.731	−0.259	17.634	−0.136
0.8	13.269	2.046	13.292	1.227	13.047	0.598	21.449	0.466	21.084	0.175	20.606	−0.011
0.9	12.039	4.259	13.350	3.027	13.950	1.672	22.522	4.993	23.133	3.568	23.208	1.459
1.0	5.861	3.369	9.801	3.484	12.826	2.305	8.342	5.111	13.763	5.588	20.216	3.733

where f' denotes the new coefficient of friction. Then the correction term will become

$$\Delta w_1 = -\frac{f'\rho V_H e^{i\theta}}{\mu a \sinh aH} \cosh az, \qquad (11)$$

Fig. 3

Vertical distribution of steady current when bottom-friction ∞ (bottom-current)[2]

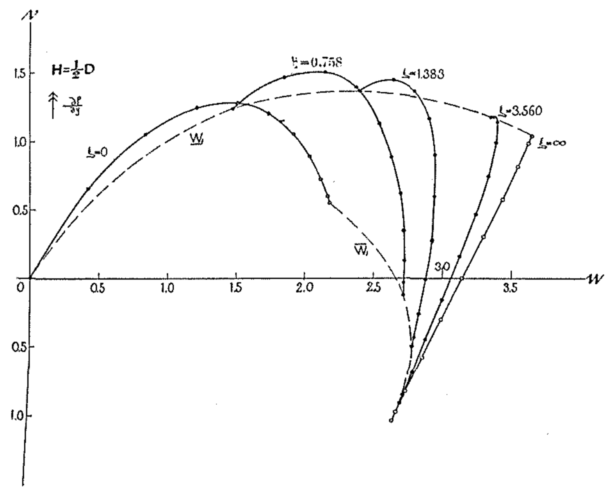


Fig. 4

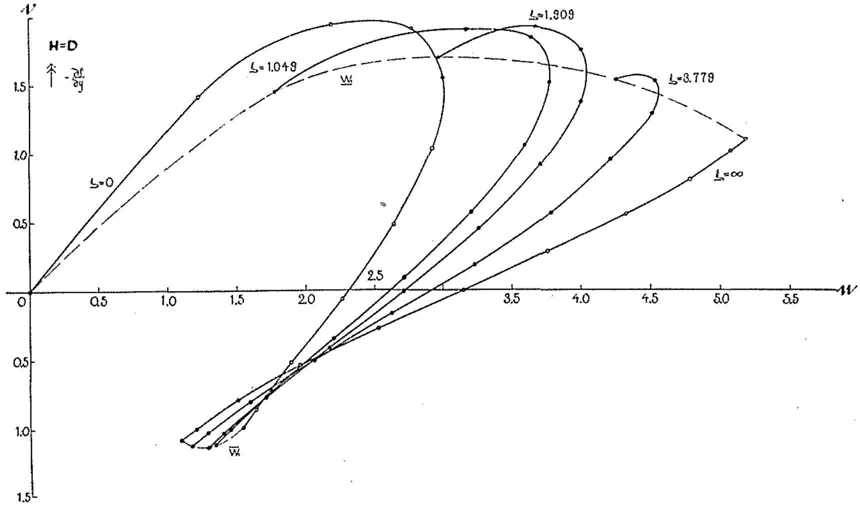


Fig. 5

Vertical distribution of steady current when bottom-friction ∝ (bottom-current)[2]

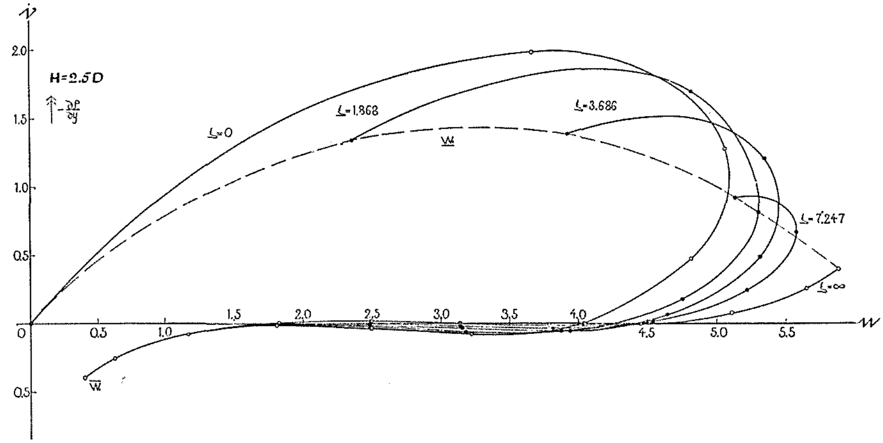
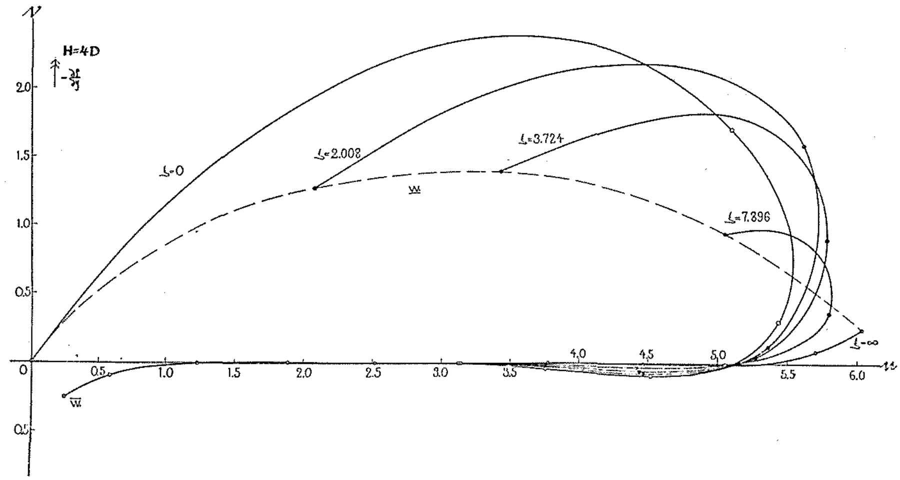


Fig. 6



and the equations corresponding to (7) will be

$$\left.\begin{aligned}\left[p+(pr-qs)\frac{f'\rho}{2\mu k}\right]V_H&=\frac{a}{4k^3}(p^2+q^2)\cos\theta,\\ \left[q+(qr+ps)\frac{f'\rho}{2\mu k}\right]V_H&=\frac{a}{4k^3}(p^2+q^2)\sin\theta.\end{aligned}\right\}\quad(12)$$

From these equations we easily find the magnitude and the direction of the slip velocity :

$$V_H = \frac{a}{4k^3}\sqrt{\frac{p^2+q^2}{1+r\xi'+\frac{r^2+s^2}{4}\xi'^2}} = V_a\eta', \tag{13}$$

$$\tan\theta = \frac{2q+(qr+ps)\xi'}{2p+(pr-qs)\xi'}, \tag{14}$$

where
$$\left.\begin{aligned} &\xi' = \frac{f'\rho}{\mu k}, \qquad V_a = \frac{a}{4k^3}, \\ &\eta' = \sqrt{\frac{p^2+q^2}{1+r\xi'+\frac{r^2+s^2}{4}\xi'^2}}. \end{aligned}\right\} \tag{15}$$

Then the correction term is calculated by

$$\Delta w_1 = -\frac{a}{4k^4}\frac{(1-i)\xi'\eta' e^{i\theta}}{2\sinh aH}\cosh az, \tag{11'}$$

and adding it to the density current with "no bottom-friction", we finally obtain the density current in the required case.

2. Development of the Density Current

§ 3. *Decaying of the steady current.*

If the local difference in density suddenly disappears, the mode of decaying of the steady current will be given by the current w_2 such that

$$\frac{\partial w_2}{\partial t} = \nu\frac{\partial^2 w_2}{\partial z^2} - 2i\bar{\omega}\, w, \qquad \left(\nu = \frac{\mu}{\rho},\ \bar{\omega} = \omega\sin\lambda\right),$$

with the surface and the initial conditions in the usual forms. But the bottom condition will be

$$-\frac{\partial w_2}{\partial z} = \frac{f'\rho}{\mu} w_2 \qquad \text{at} \quad z = H,$$

if the bottom-friction is proportional to the slip velocity.

The current w_2 which satisfies the eq. of motion and all the conditions will be

$$w_2 = \sum_{n=1}^{\infty} C_n \cos\beta_n z\, e^{-(\nu\beta_n^2+2i\bar{\omega})t}, \tag{16}$$

where β_n denotes the roots of

$$\beta \tan \beta H = \frac{f'\rho}{\mu} \equiv h \text{ (say)}, \tag{17}$$

and

$$C_n = \frac{2(\beta_n^2 + h^2)}{h + (\beta_n^2 + h^2)H} \int_0^H w_1 \cos \beta_n z \, dz. \tag{18}$$

§ 4. *Development of the density current*

If the difference in density is suddenly generated in a motionless sea, the development of current in an invariable solenoidal field will be represented by

$$w = w_1 - w_2 = \sum_{n=1}^{\infty} C_n \cos \beta_n z \left\{ 1 - e^{-(\nu\beta_n^2 + 2i\omega)t} \right\}, \tag{19}$$

provided that the bottom-friction varies with the slip velocity.

If we put $f' = 0$, we get the development of the current in the sea with " no bottom-friction ", which we had left untreated in the second paper on the subject.

At any rate, we shall be satisfied merely to note here that the formula contains the same damping factor $e^{-\nu\beta_n^2 t}$ as usual, so that the steady state will be generally attained in a few hours or days. In a real sea, however, the density of the water shows only a very slow variation and never a sudden change of appreciable amount, and accordingly the density current may be considered as always stationary. Hence we will not calculate numerically the current in (19).

In conclusion, the writers' thanks are due to the Hattori Hōkō-kwai for the subvention given to one of the writers (T. N.) with which he was able to obtain an assistant to help him in the laborious numerical calculations.

Errata for our paper on the development of the slope current etc., II.

p. 338, Table 1, first line of numbers. For 0.710 read 0.790.
p. 338. Replace Fig. 2 by the following diagram.