A Certain Functional Class

By

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Introduction. We consider the following combinations of functions:

(1) $$F(x, y)=f(x, y)+g(x, y)-\lambda\int_a^b f(x, t)\cdot g(t, y)dt,$$

(2) $$G(x, y)=g(x, y)+f(x, y)-\lambda\int_a^b g(x, t)\cdot f(t, y)dt,$$

(1′) $$F_1(x, y)=f(y, x)+g(x, y)-\lambda\int_a^b f(t, x)\cdot g(t, y)dt,$$

(2′) $$F_2(x, y)=f(x, y)+g(y, x)-\lambda\int_a^b f(x, t)\cdot g(y, t)dt,$$

where λ means a constant parameter, independent of two real variables x and y, and functions $f(x, y)$ and $g(x, y)$ are continuous and finite in the square-domain D, which is defined in $\begin{pmatrix} a\leqq x\leqq b \\ a\leqq y\leqq b\end{pmatrix}$, and moreover it must be so chosen that they belong to two different kinds.[1]

Now we shall define following functional operations:

(3) $$\begin{cases} S_{\lambda A}\varphi\equiv\varphi(x)-\lambda\int_a^b A(x, y)\cdot\varphi(y)dy, \\ S'_{\lambda A}\varphi'\equiv\varphi'(x)-\lambda\int_a^b A(y, x)\cdot\varphi'(y)dy. \end{cases}$$

By substituting one into the other and repeating these operations, we obtain 8 product-operations as follows:

1. G. Kowalewski, Determinantentheorie, 1925. pp. 261—265.

$$\begin{cases} S_{\lambda f}\cdot S_{\lambda g}\equiv S_{\lambda F}, \\ S'_{\lambda g}\cdot S'_{\lambda f}\equiv S'_{\lambda F}, \\ S_{\lambda g}\cdot S_{\lambda f}\equiv S_{\lambda G}, \\ S'_{\lambda f}\cdot S'_{\lambda g}\equiv S'_{\lambda G}, \\ S'_{\lambda f}\cdot S_{\lambda g}\equiv S_{\lambda F_1}, \\ S'_{\lambda g}\cdot S_{\lambda f}\equiv S'_{\lambda F_1}, \\ S_{\lambda f}\cdot S'_{\lambda g}\equiv S_{\lambda F_2}, \\ S_{\lambda g}\cdot S'_{\lambda f}\equiv S'_{\lambda F_2}. \end{cases}$$

But in the present paper, we shall mostly discuss only the combinations (1) and (2).

Before proceeding to the discussion, we must know the following theorem, which will play an important rôle in this paper: *The product of two Fredholm's determinants of $f(x, y)$ and $\varphi(x, y)$ is also another Fredholm's determinant of $F(x, y)$, where*

$$F(x, y)=f(x, y)+\varphi(x, y)-\lambda\int_a^b f(x, t)\cdot\varphi(t, y)dt.$$ [1]

If we denote each Eredholm's determinant of functions f, φ and F by $D_f(\lambda)$, $D_g(\lambda)$ and $D_F(\lambda)$ respectively, the above theorem will become as follows:

(4) $$D_F(\lambda)=D_f(\lambda)\cdot D_g(\lambda).$$

This is called *the multiplication theorem of determinants.*

Now from the above stated product-operations, it is easily known that if $S_{\lambda g}\varphi=0$, then $S_{\lambda F}\varphi=0$,[2] and if $S'_{\lambda f}\varphi'=0$, then $S'_{\lambda F}\varphi'=0$. Hence in a word,

Lemma **1**, *When λ is a characteristic constant of $\varphi(x, y)$, then the F. S.*[3] *of solutions of $\varphi(x, y)$, belonging to λ, becomes independent solutions of $F(x, y)$, belonging to the same constant; and the F. S. of associated solutions of $f(x, y)$ becomes independent associated solutions of $F(x, y)$, belonging to the same constant as that of $f(x, y)$.*

1. When the constant λ_0 is a zero of $D_F(\lambda)$, and at the same

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1. G. Kowalewski, loc. cit., p. 265. and I. Fredholm, Sur une classe d'équations fonctionnelles, pp. 381—383. Acta Math. 1904. Tome 28.
2. Note that $F(x, y)$ depends also on the constant λ of f and φ.
3. An abbreviation of Fundamental System.

time, of $D_f(\lambda)$ and $D_g(\lambda)$, and its multiplicity is respectively m, m_1 and m_2 for each of them, and its rank is respectively p, p_1 and p_2, then it is evident that from (4),

$$m=m_1+m_2,$$

and from Lemma **1**,

$$p\geqq \max.(p_1,\ p_2).$$

Hence let

$$p=p_1+r_1=p_2+r_2\,;$$

and let the F. S. of solutions and associated solutions of $f(x, y)$, belonging to the characteristic constant λ_0, be respectively

$$\psi_1,\ \psi_2,\ \psi_3,\dots\dots,\psi_{p_1},$$

and
$$\overline{\psi}_1,\ \overline{\psi}_2,\ \overline{\psi}_3,\dots\dots\overline{\psi}_{p_1},$$

and each corresponding function of $g(x, y)$ to above those of $f(x, y)$, be respectively

$$\varphi_1,\ \varphi_2,\ \varphi_3,\dots\dots\varphi_{p_2},$$

and
$$\overline{\varphi}_1,\ \overline{\varphi}_2,\ \overline{\varphi}_3,\dots\dots\overline{\varphi}_{p_2}.$$

Thus when we find the solutions of the following equation:

$$S_{\lambda_0 F}\cdot\varphi=0$$

i. e.
$$\varphi(x)-\lambda_0\int_a^b F(x,\ y)\cdot\varphi(y)dy=0,$$

these solutions must satisfy either of the two following equations or both:

$$S_{\lambda_0 g}\cdot\varphi=0$$

i. e.
$$\varphi(x)-\lambda_0\int_a^b g(x,\ y)\,\varphi(y)dy=0$$

and
$$S_{\lambda_0 f}\cdot\varphi=0$$

i. e.
$$\psi(x)-\lambda_0\int_a^b f(x,\ y)\cdot\psi(y)dy=0,$$

where
$$\psi(x)=\varphi(x)-\lambda_0\int_a^b g(x,\ y)\cdot\varphi(y)dy.$$

By the conditions[1] for the existence of solutions of the Fredholm's integral equation of the second kind, we see ot once that if

$$\int_a^b \{\overline{\varphi_i(x)}\}\psi_j(x)dx=0^2 \quad \text{for} \quad j=1,\ 2,\dots\dots\dots,\ r_2(\leqq p_1),$$

then any solution of $F(x, y)$, belonging to λ_0, will become a linear function of the following functions:

$$\varphi_1,\ \varphi_2,\ \varphi_3,\dots\dots\dots\dots\dots\dots\dots,\ \varphi_{p_2}$$

and $$\phi_1,\ \phi_2,\ \phi_3,\dots\dots\dots\dots\dots\dots\dots,\ \phi_{r_2},$$

where $$\psi_j(x)=\phi_j(x)-\lambda_0\int_a^b g(x,\ y)\phi_j(y)dy \quad \text{for} \quad j=1,\ 2,\dots\dots\dots,\ r_2.$$

Now we must deal with the independency of these functions

$$\varphi_1,\ \varphi_2,\dots\dots\dots\dots,\ \varphi_{p_2};\ \phi_1,\ \phi_2,\dots\dots\dots\dots,\ \phi_{r_2}.$$

If, for arbitrary constants c's and d's, the equation

$$\sum_{i=1}^{h} c_i\phi_i+\sum_{i=1}^{k} d_i\varphi_i=0,$$

where $h\leqq r_2$, $k\leqq p_2$, is true, then from the construction of ϕ_i and φ_i we obtain

$$\sum_{i=1}^{h} c_i\phi_i+\sum_{i=1}^{k} d_i\varphi_i-\sum_{i=1}^{h} c_i\lambda_0\int_a^b g(x,\ y)\phi_i(y)dy-$$
$$-\sum_{i=1}^{k} d_i\lambda_0\int_a^b g(x,\ y)\varphi_i(y)dy=\sum_{i=1}^{h} c_i\psi_i=0.$$

This contradicts the independency of the ψ's-system.

As the above result can be quite analogously applied to the associated solutions of $F(x, y)$, belonging to the constant λ_0, we can conclude as follows:

Lemma **2.** *When $p=p_2+r_2$, it follows that*

$$(5) \qquad \int_a^b \{\overline{\varphi_i(x)}\}\psi_j(x)dy=0 \quad \text{for} \quad j=1,\ 2,\dots\dots\dots,\ r_2;$$

and the F. S. of the solutions of $F(x, y)$, belonging to λ_0, will be

$$\varphi_1,\ \varphi_2,\dots\dots\dots\dots,\ \varphi_{p_2};\ \phi_1,\ \phi_2,\dots\dots\dots\dots,\ \phi_{r_2},$$

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1. Vivanti-Schwank, Lineare Integralgleichungen, 1928. pp. 107-108.
2. This means that by giving one of j's these equations exist for all i's.

where $\quad \psi_j(x)=\phi_j(x)-\lambda_0\int_a^b g(x,\ y)\phi_j(y)dy, \quad$ *for* $\quad j=1,\ 2,\ \ldots\ldots,\ r_2.$

Quite similarily, when $p=p_1+r_1$, *the equations*

$$(5')\qquad \int_a^b\{\psi_i(x)\}\overline{\varphi_j(x)}dx=0, \quad \text{for} \quad j=1,\ 2,\ 3,\ldots\ldots,\ r_1\ (\leqq p_2),$$

are followed, and the F. S. of the associated solutions of $F(x, y)$ *becomes*

$$\bar{\psi}_1,\ \bar{\psi}_2,\ldots\ldots\ldots,\ \bar{\psi}_{p_1};\ \bar{\phi}_1,\ \bar{\phi}_2,\ldots\ldots\ldots,\ \bar{\phi}_{r_1},$$

where $\quad \bar{\varphi}_j(x)=\bar{\phi}_j(x)-\lambda_0\int_a^b f(y,\ x)\bar{\phi}_j(y)dy \quad$ *for* $\quad j=1,\ 2,\ldots\ldots,\ r_1.$

The above results can also be applied to $G(x,\ y)$, $F_1(x,\ y)$ and $F_2(x,\ y)$.

2. Now denote each reciprocal kernel of $f(x,\ y)$, $g(x,\ y)$ and $F(x,\ y)$ respectively by $f(x,\ y;\ \lambda)$, $g(x,\ y;\ \lambda)$ and $F(x,\ y;\ \lambda)$, then these functions of λ are meromorphic everywhere on the λ-plane under the exclusion of the infinity point, and have poles to be the characteristic constants of $f(x,\ y)$, $g(x,\ y)$ and $F(x,\ y)$. Hence we expand these meromorphic functions of λ into the Laurent's series in the neighbourhood of $\lambda=\lambda_0$, one of the poles; namely

$$\begin{aligned} F(x,\ y;\ \lambda) &= \frac{a_t(x,\ y)}{(\lambda_0-\lambda)^t}+\ldots\ldots\ldots+\frac{a_1(x,\ y)}{\lambda_0-\lambda}+ \\ &\quad +b_0(x,\ y)+(\lambda_0-\lambda)b_1(x,\ y)+\ldots\ldots\ldots \\ &= P_F(x,\ y;\ \lambda)+R_F(x,\ y;\ \lambda), \end{aligned}$$

where $P_F(x,\ y;\ \lambda)$ is the principal part and $R_F(x,\ y;\ \lambda)$ the regular part of the Laurent's expansion of the reciprocal kernel $F(x,\ y;\ \lambda)$, corresponding to the characterisric constant λ_0.

From the above equation, we have

$$\begin{aligned} F(x,\ y;\ 0) &\equiv \bar{F}(x,\ y) \\ &= P_F(x,\ y)+R_F(x,\ y) \\ &= f(x,\ y)+g(x,\ y), \end{aligned}$$

where $\quad P_F(x,\ y)\equiv P_F(x,\ y;\ 0) \quad$ and $\quad R_F(x,\ y)\equiv R_F(x,\ y;\ 0).$

Now we wish to prove the following

Theorem **1.**

$$\frac{1}{2\pi i}\int F(x,\ y\ ;\ \lambda)dy=\frac{1}{2\pi i}\int f(x,\ y\ ;\ \lambda)d\lambda+\frac{1}{2\pi i}\int g(x,\ y\ ;\ \lambda)d\lambda,$$

where these integrals are taken along the closed curve (C) *involving the pole* $\lambda=\lambda_0$.

Before entering into the proof of the above theorem, we must know the following: The residue of the reciprocal kernel of any kernel is represented in the bilinear form by fundamental functions with respect to the characteristic constant.[1]

By taking the residues of reciprocal kernels $F(x,\ y\ ;\ \lambda)$, $f(x,\ y\ ;\ \lambda)$ and $g(x,\ y\ ;\ \lambda)$, we obtain the equality:

$$(\phi)\qquad a_1(x,\ y)=\overset{(f)}{a_1}(x,\ y)+\overset{(g)}{a_1}(x,\ y)+a_1'(x,\ y),$$

where functions $\overset{(f)}{a_1}$, $\overset{(g)}{a_1}$ are the coefficients of $1/(\lambda_0-\lambda)$ in the Laurent's expansions of reciprocal kernels $f(x,\ y\ ;\ \lambda)$ and $g(x,\ y\ ;\ \lambda)$ respectively, but a_1' is an unknown term.

And still a_1, $\overset{(f)}{a_1}$ and $\overset{(g)}{a_1}$ can be together written by the biorthogonal and normalized systems in the following bilinear forms:

$$a_1(x,\ y)=\sum_{h=1}^{m}a_h(x)\cdot\beta_h(y)$$

$$\overset{(f)}{a_1}(x,\ y)=\sum_{h=1}^{m_1}\overline{a_h}(x)\cdot\overline{\beta_h}(y)$$

$$\overset{(g)}{a_1}(x,\ y)=\sum_{h=1}^{m_2}\overline{\overline{a_h}}(x)\cdot\overline{\overline{\beta_h}}(y),$$

where

$$(a_h,\ \beta_i)=\begin{cases}0 & \text{for}\quad h\neq i\\ 1 & \phantom{\text{for}}\quad h=i\end{cases}$$

$$(\overline{a_h},\ \overline{\beta_i})=\begin{cases}0 & \text{for}\quad h\neq i\\ 1 & \phantom{\text{for}}\quad h=i\end{cases}$$

$$(\overline{\overline{a_h}},\ \overline{\overline{\beta_i}})=\begin{cases}0 & \text{for}\quad h\neq i\\ 1 & \phantom{\text{for}}\quad h=i;\end{cases}$$

1. Bryon Heywood, Sur l'équation fonctionnelle de Fredholm, Jordan Journ., 1908. pp. 300—307.

and m, m_1 and m_2 mean respectively the multiplicity of the characteristic constant λ_0 of each kernel F, f and g.

On the other hand, the solutions and associated solutions, belonging to the constant λ_0, of each kernel F, f and g are always respectively represented in the linear function by the two following systems:

$$\{\alpha_h\},\ \{\overline{\alpha}_h\},\ \{\overline{\overline{\alpha}}_h\}\quad (h=1,\ 2,\ 3,\dots\dots\dots)$$

and

$$\{\beta_h\},\ \{\overline{\beta}_h\},\ \{\overline{\overline{\beta}}_h\}\quad (h=1,\ 2,\ 3,\dots\dots\dots).$$

Hence, when we write each F. S. of the solutions and associated solutions belonging to the constant λ_0 as follows:

$$\chi_i(x)=\sum_{h=1}^{m} C_{ih}\alpha_h(x)\quad \text{for}\quad i=1,\ 2,\ 3,\dots\dots\dots,p \tag{6}$$

and

$$\overline{\chi}_i(x)=\sum_{h=1}^{m} \overline{C}_{ih}\beta_h(x)\quad \text{for}\quad i=1,\ 2,\ 3,\dots\dots\dots,p, \tag{6'}$$

then from Lemma **1** we can obtain the equalities:

$$\begin{cases}\chi_i(x)\equiv\varphi_i(x) & \text{for}\quad i=1,\ 2,\ 3,\dots\dots\dots,p_2\\ \overline{\chi}_i(x)\equiv\psi_i(x) & \text{for}\quad i=1,\ 2,\ 3,\dots\dots\dots,p_1.\end{cases} \tag{7}$$

On the other hand, just as in the case of the above functions $\chi_i(x)$, $\overline{\chi}_i(x)$ we have

$$\varphi_i(x)=\text{linear function of } \overline{\overline{\alpha}}\text{'s}$$

$$\overline{\psi}_i(x)=\text{linear function of } \overline{\beta}\text{'s},$$

and by applying orthogonal relations (5) and (5′) in Lemma **2** to (6) and (6′),

$$C_{ih}=0\quad\text{for}\quad\begin{cases}h=1,\ 2,\ 3,\dots\dots\dots,m_1\\ i=1,\ 2,\ 3,\dots\dots\dots,p_2\end{cases}$$

$$\overline{C}_{ih}=0\quad\text{for}\quad\begin{cases}h=m_1+1,\ m_1+2,\dots\dots\dots,m\\ i=1,\ 2,\ 3,\dots\dots\dots\dots\dots\dots,p_1.\end{cases}$$

Thus from the above results and the equation $m=m_1+m_2$, we can find the two following equations:

$$\overset{(f)}{a_1}(x,\ y)=\sum_{h=1}^{m_1}\overline{\alpha}_h(x)\cdot\beta_h(y)$$

and $$\overset{(g)}{a_1}(x,\ y)=\sum_{h=m_1+1}^{m}a_h(x)\cdot\overline{\overline{\beta_h}}(y).$$

Consequently, in view of the left-hand side of the equality (ϕ)

$$\overline{a_h}(x)=\sum_i e_{hi}a_i(x) \quad \text{and} \quad \overline{\overline{\beta_h}}(x)=\sum_i e'_{hi}\beta_i(x).$$

Therefore we have

$$(\overline{a_h},\ \beta_i)\equiv\sum_i e_{hi}(a_i,\ \beta_j)=e_{hj}=\begin{cases} 0 & \text{for } h\neq j \\ 1 & \phantom{\text{for }} h=j. \end{cases}$$

Accordingly $\overline{a_h}(x)\equiv a_h(x)$.

In a like manner we have

$$\overline{\overline{\beta_h}}(x)\equiv\beta_h(x).$$

After all we can rewrite $\overset{(f)}{a_1}$ and $\overset{(g)}{a_1}$ in the following form:

$$(8)\qquad \begin{cases} \overset{(f)}{a_1}(x,\ y)=\sum_{h=1}^{m_1}a_h(x)\cdot\beta_h(y) \\ \overset{(g)}{a_1}(x,\ y)=\sum_{h=m_1+1}^{m}a_h(x)\cdot\beta_h(y). \end{cases}$$

Now let us replace these results in the equality (ϕ), then by observing $m=m_1+m_2$ we obtain

$$a_1'(x,\ y)\equiv 0. \qquad \text{Q. E. D.}$$

From the preceding Theorem **1**, we have easily the following

Corollary **1**. *Let λ_0 be the characteristic constant of $F(x, y)$ and p be its rank for $F(x, y)$, p_1 for $f(x, y)$, p_2 for $g(x, y)$, then the equation*

$$p=p_1+p_2$$

will necessarily follow.

For, from the equality (8) any associated solution of $f(x, y)$, belonging to the characteristic constant, is arranged for the linear function by the fundamental functions as follows:

$$\beta_1,\ \beta_2,\dots\dots\dots\dots,\ \beta_{m_1},$$

and in the same way any solution of $g(x,\ y)$ is represented in the linear function by the following fundamental functions:

$$\alpha_{m_1+1},\ \alpha_{m_1+2},\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \alpha_m,$$[1]

while $(\alpha_i,\beta_i)=0$ for $i \neq j$; therefore the equations (5) and (5′) in Lemma **2** exist for all the functions ψ's and $\overline{\varphi}$'s. Hence we have

$$r_2=p_1 \qquad r_1=p_2,$$

accordingly $\quad p=p_1+p_2.$

This corollary is true.

Now from the equalities (8) we obtain at once

$$\int_a^b \overset{(f)}{a_1}(x,\ t)\cdot\overset{(g)}{a_1}(t,\ y)dt=\int_a^b \overset{(g)}{a_1}(x,\ t)\cdot\overset{(f)}{a_1}(t,\ y)dt=0.$$

By recalling the preceding studies we discover that the existence of the above equations is independent of the function $F(x,\ y)$, and dependent only on having the characteristic constant in common; hence we can conclude as a corollary as follows:

Corollary **2**. *When both kernels $f(x,\ y)$ and $g(x,\ y)$ have the same characteristic constant λ_0, the residues of their reciprocal kernels at the pole $\lambda=\lambda_0$ are orthogonal to each other in the interval $(a,\ b)$.*

As all the coefficients $a_2(x,\ y),\ a_3(x,\ y),\ldots\ldots\ldots$ of every term $(\lambda_0-\lambda)^{-2},\ (\lambda_0-\lambda)^{-3},\ldots\ldots\ldots$ in the Laurent's expansion of $F(x,\ y;\ \lambda)$ are formed by bilinear functions[2] of the fundamental functions, corresponding to the constant λ_0,

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \alpha_m,$$

$$\beta_1,\ \beta_2,\ \beta_3,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \beta_m;$$

moreover the results obtained for $F(x,\ y;\ \lambda)$ hold good also for $f(x,\ y;\ \lambda)$ and $g(x,\ y;\ \lambda)$, so their corresponding principal parts $P_f(x,\ y;\ \lambda)$ and $P_g(x,\ y;\ \lambda)$ can be represented by the following fundamental functions:

$$\alpha_1,\ \alpha_2,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \alpha_{m_1},$$

$$\beta_1,\ \beta_2,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \beta_{m_1},$$

and

$$\alpha_{m_1+1},\ \alpha_{m_1+2},\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \alpha_m,$$

1. Heywood, loc. cit., p. 300.
   T. Lalesco, Introduction à la théorie des équations intégrales, 1912. p. 50.
2. Vivanti-Schwank, loc. cit., p. 134.

$$\beta_{m_1+1},\ \beta_{m_1+2},\ldots\ldots\ldots\ldots\ldots\ldots,\ \beta_m$$

in the bilinear forms respectively.

Consequently the following relations:

$$\int_a^b P_f(x,\ t;\ \lambda)P_g(t,\ y;\ \lambda)dt=\int_a^b P_g(x,\ t;\ \lambda)P_f(t,\ y;\ \lambda)dt=0 \tag{9}$$

will exist; therefore we have

Corollary **3**. *When the characteristic constant λ_0 is held in common by both kernels $f(x, y)$ and $g(x, y)$, the relative parts*[1] *of their reciprocal kernels $f(x, y; \lambda)$ and $g(x, y; \lambda)$, corresponding to λ_0, are orthogonal to each other in the interval (a, b).*

Now let us consider the principal part of the Laurent's expansion of $F(x, y; \lambda)$ in the neighbourhood of the λ_0, then we obtain at once the following:

$$P_F(x,\ y;\ \lambda)=P_f(x,\ y;\ \lambda)+P_g(x,\ y;\ \lambda).$$

Substitute $\lambda=0$ in the above equality, then

$$P_F(x,\ y)=P_f(x,\ y)+P_g(x,\ y); \tag{10}$$

while let $\lambda=0$ in equations (9), then we have

$$\int_a^b P_f(x,\ t)P_g(t,\ y)dt=\int_a^b P_g(x,\ t)P_f(t,\ y)dt=0.$$

Thus corresponding to Corollary **3**, we have the following

Corollary **4**. *When both kernels $f(x, y)$ and $g(x, y)$ have the same characteristic constant λ_0 in common, the relative parts*[2] *of these kernels, corresponding to the characteristic constant λ_0, are orthogonal to each other in the interval (a, b).*

Remark: we must recall that functions $f(x, y)$ and $g(x,y)$ ought to be of different kinds as shown by the preceding foot-note. For instance the following functions will be of the same kind,

$$f(x,\ y)=\sin x\cdot\sin y$$
$$g(x,\ y)=\sin x\cdot\sin y+\sin 2x\cdot\sin 2y$$
$$\text{interval } (a,\ b)=\text{interval } (-\pi,\ +\pi),$$

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1. We call the principal part of any reciprocal kernel at the pole λ_0 *the relative part of the reciprocal kernel,* corresponding to the characteristic constant λ_0 after Heywood.

2. The function, produced by substituting $\lambda=0$ in the relative part of any reciprocal kernel is called *the relative part* of the original kernel corresponding to the same characteristic constant λ_0.

while on the other hand, the following are of different kinds,

$$f(x, y) = \sin x \cdot \sin y$$

$$g(x, y) = \cos x \cdot \cos y.$$

Now we see immediately from (10) that the function $P_F(x, y)$ is represented by the sum of two functions orthogonal to each other, and hence under some hypothesis *the chracteristic function of* $P_F(x, y)$ *may be a linear function of the characteristic functions of* $P_f(x, y)$ *and* $P_g(x, y)$.
Also, we are familiar with the following: *the solutions of the equations*

$$\varphi(x) - \lambda_0 \int_a^b F(x, y) \cdot \varphi(y) dy = 0$$

and

$$\psi(x) - \lambda_0 \int_a^b F(y, x) \cdot \psi(y) dy = 0$$

become those of the equations

$$\varphi(x) - \lambda_0 \int_a^b P_F(x, y) \cdot \varphi(y) dy = 0$$

and

$$\psi(x) - \lambda_0 \int_a^b P_F(y, x) \cdot \psi(y) dy = 0 \qquad \textit{respectively.}$$ [1]

Thus we can conclude that *by a certain hypothesis the solutions of the kernel* $F(x, y)$ *may be written linearly of the solutions of two kernels* $P_f(x, y)$ *and* $P_g(x, y)$, *belonging to the same constant as that of* $F(x, y)$, *where* $P_f(x, y)$ *and* $P_g(x, y)$ *are the relative parts of the kernels* $f(x, y)$ *and* $g(x, y)$ *respectively, corresponding to the same constant.*

Now from the independency of the system χ's in (6), the rank of the following matrix of coefficients is p:

$$\begin{pmatrix} 0, & 0, \ldots\ldots, & 0, & C_{1, m_1+1}, & C_{1, m_1+2}, \ldots\ldots, & C_{1, m} \\ 0, & 0, \ldots\ldots, & 0, & C_{2, m_1+1}, & C_{2, m_1+2}, \ldots\ldots, & C_{2, m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, \ldots\ldots, & 0, & C_{p_2, m_1+1}, & C_{p_2, m_1+2}, \ldots\ldots, & C_{p_2, m} \\ C_{p_2+1, 1}, & C_{p_2+1, 2} \ldots, & C_{p_2+1, m_1}, & C_{p_2+1, m_1+1}, & C_{p_2+1, m_1+2}, \ldots\ldots, & C_{p_2+1, m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{p,1}, & C_{p,2}, \ldots\ldots & & & \ldots\ldots, & C_{p, m} \end{pmatrix}$$

1. B. Heywood, loc. cit. pp. 290—291.

On the other hand, from the independency of φ's the rank of the matrix:

$$\begin{pmatrix} C_{1,m_1+1}, & C_{1,m_1+2}, \ldots\ldots\ldots\ldots, & C_{1,m} \\ C_{2,m_1+1}, & C_{2,m_1+2}, \ldots\ldots\ldots\ldots, & C_{2,m} \\ \ldots\ldots\ldots\ldots & & \cdot \\ C_{p_2,m_1+1}, & C_{p_2,m_1+2}, \ldots\ldots\ldots\ldots, & C_{p_2,m} \end{pmatrix}$$

is p_2. Therefore if we suppose the determinant:

$$|C_{ij}| \neq 0 \quad \text{for} \quad \begin{cases} i=1, \ 2, \ 3, \ldots\ldots\ldots, p_2 \\ j=m_1+1, m_1+2, \ldots\ldots, m_1+p_2, \end{cases}$$

we will obtain

$$a_{m_1+h} = \sum_{i=1}^{p_2} \xi_{hi}\varphi_i + \sum_{i=p_2+1}^{m_2} \eta_{hi} a_{m_1+i} \quad \text{for} \quad h=1, \ 2, \ 3, \ldots\ldots, p_2.$$

Quite similarily we have

$$\psi_i = \text{linear function of } (a_1, a_2, \ldots\ldots, a_{m_1})$$

$$\text{for} \quad i=1, 2, 3, \ldots\ldots, p_1,$$

where the rank of the matrix of coefficients is p_1, therefore

$$a_i = \text{linear function of } (\psi_1, \psi_2, \ldots\ldots, \psi_{p_1}; \ a_{p_1+1}, \ldots\ldots, a_{m_1})$$

$$\text{for} \quad i=1, \ 2, \ 3, \ldots\ldots\ldots, p_1.$$

Substitute these above results in the expression of the function $\chi_i(x)$, then

(II) $\qquad \chi_i(x) = \text{linear function of}$

$$(\psi_1, \psi_2, \ldots\ldots, \psi_{p_1}; \ \varphi_1, \ldots\ldots, \varphi_{p_2}; \ a_{p_1+1}, \ a_{p_1+2}, \ldots\ldots$$

$$\ldots\ldots, a_{m_1}; \ a_{m_1+p_2+1}, \ a_{m_1+p_2+2}, \ \ldots\ldots, a_m)$$

$$\text{for} \quad i=1, \ 2, \ 3, \ldots\ldots, p.$$

We can obtain the same result for the functions $\overline{\chi_i}(x)$ as for

$$\chi_i(x) \quad i=1, \ 2, \ 3, \ldots\ldots\ldots, p.$$

Therefore if two reciprocal kernels $f(x, y; \lambda)$ and $g(x, y; \lambda)$ have the simple pole at $\lambda=\lambda_0$ at the same time, the multiplicity of the characteristic constant λ_0 will be coincident with its rank; namely

$$m_1 = p_1 \quad \text{and} \quad m_2 = p_2.$$

In conclusion we have the following

Theorem **2.** *If the reciprocal kernels of $f(x, y)$ and $\varphi(x, y)$ have the simple pole at $\lambda=\lambda_0$ at the same time, the eigen-function of $F(x, y)$ which belongs to the characteristic constant λ_0 will be written linearly by the independent eigen-functions. corresponding to the same characteristic constant λ_0 of the kernels $f(x, y)$ and $\varphi(x, y)$, where*

$$F(x, y)=f(x, y)+\varphi(x, y)-\lambda_0\int_a^b f(x, t)\cdot\varphi(t, y)dt.$$

The same result is applicable also to the associated eigen-function of the kernel $F(x, y)$.

But it is evident that a pole of the reciprocal kernel of any symmetric kernel is simple[1], so that there exists the following

Corollary **1.** *If both kernels $f(x, y)$ and $\varphi(x, y)$ are symmetric, the eigen-functions of $F(x, y)$ which belong to the characteristic constant λ_0 will represent linear functions with respect to the eigen-functions of the kernels $f(x, y)$ and $\varphi(x, y)$, belonging to the same constant λ_0. And the same can be said also for the associated eigen-functions of $F(x, y)$.*

On the other hand, in order that the reciprocal kernel may have the simple pole at $\lambda=\lambda_0$,[2] it is necessary and sufficient that no associated eigen-function of the original kernel becomes orthogonal to any eigen-function, belonging to the same charateristic constant λ_0. Hence we may rewrite the above corollary **1** as the following

Corollary **2.** *When no associated solution is orthogonal to any eigen-function of the kernel $f(x, y)$, belonging to the same characteristic constant λ_0, and similarly for the kernel $\varphi(x, y)$, then the eigen-function of the kernel $F(x, y)$ which belongs to the constant can be expressed linearly by eigen-functions of kernels $f(x, y)$ and $\varphi(x, y)$, belonging to the same constant λ_0.*

Now if the pole of the relative part of the reciprocal kernel $F(x, y; \lambda)$, corresponding to the constant λ_0, be only simple, we shall have

$$\overset{(f)}{a_i}(x, y)=-\overset{(\varphi)}{a_i}(x, y) \quad \text{for} \quad i=2, 3,\ldots, t;$$

accordingly the coefficients of $(\lambda_0-\lambda)^{-i}$, $i=2, 3,\ldots\ldots, t$ must cancel out each other.

1. Vivanti-Schwank, loc. cit., p. 148.
2. Vivanti-Schwank, loc. cit., p. 144.

Again, as those coefficients $\overset{(f)}{a_i}$, $\overset{(g)}{a_i}$ are always the linear functions of α's, β's, so we have

$$\overset{(f)}{a_i}(x,\ y)=\sum_{j,\,h=1}^{m_1}E_{jh}\alpha_j(x)\cdot\beta_h(y)$$

$$\overset{(g)}{a_i}(x,\ y)=\sum_{j,\,h=m_1+1}^{m}E'_{jh}\alpha_j(x)\cdot\beta_h(y)$$

$$i=2,\ 3,\ldots\ldots\ldots\ldots,\ t.$$

Therefore

$$\sum_{j,\,h=1}^{m_1}E_{jh}\cdot\alpha_j(x)\cdot\beta_h(y)+\sum_{j,\,h=m_1+1}^{m}E'_{jh}\alpha_j(x)\cdot\beta_h(y)=0.$$

Multiply both sides of the above equality by $\beta_h(y)$ $(h\leqq m_1)$ and integrate with respect to y from a to b:

$$E_{jh}\cdot\alpha_j(x)=0.$$

since

$$\int_a^b\beta_j(y)\cdot\beta_h(y)dy=\begin{cases}1, & j=h\\ 0, & j\neq h.\end{cases}$$

Consequently $\quad E_{jh}=0$,

quite in the same way, $E'_{jh}=0$, for all the values of j, h.
Therefore we have

$$\overset{(f)}{a_i}(x,\ y)\equiv 0 \quad\text{and}\quad \overset{(g)}{a_i}(x,\ y)\equiv 0$$

$$i=2,\ 3,\ldots\ldots\ldots\ldots,\ t.$$

This shows that if the reciprocal kernel $F(x,\ y\ ;\ \lambda)$ has the simple pole at $\lambda=\lambda_0$, both reciprocal kernels $f(x,\ y\ ;\ \lambda)$ and $\varphi(x,\ y\ ;\ \lambda)$ necessarily have also simple poles at the same point λ_0. Thus we can rewrite the above theorem **2** and corollary as follows:

Theorem **3**. *When the reciprocal kernel of $F(x,\ y)$ has the simple pole at $\lambda=\lambda_0$, the eigen-function and associated function of $F(x,\ y)$ can be written in linear functions of the corresponding eigen- and associated eigen-functions of kernels $f(x,\ y)$ and $\varphi(x,\ y)$ respectively, belonging to the same characteristic constant λ_0.*

Corollary **1**. *If a kernel $F(x,\ y)$ be symmetric, the eigen- and associated eigen-function will become a linear function of those of* *kernels $f(x, y)$ and $g(x, y)$ respectively, corresponding to the same characteristic constant λ_0.*

Corollary **2**. *If no associated solution is orthogonal to any eigen-function of the kernel $F(x, y)$, the eigen- and associated eigen-functions of $F(x, y)$ will be linear functions of the corresponding eigen- and associated eigen-functions of kernels $f(x, y)$ and $g(x, y)$ respectively, belonging to the same constant λ_0.*

3. From Lemma **2** and corollary **1** to Theorem **1** in the preceding sections, we know that two F. S.'s of the characteristic functions and associated functions of the kernel $F(x, y)$, belonging to the characteristic constant λ_0, are as follows:

$$\varphi_1, \varphi_2, \ldots\ldots, \varphi_{p_2}; \ \phi_1, \phi_2, \ldots\ldots, \phi_{p_1},$$

and

$$\overline{\psi}_1, \overline{\psi}_2, \ldots\ldots, \overline{\psi}_{p_1}; \ \overline{\phi}_1, \overline{\phi}_2, \ldots\ldots, \overline{\phi}_{p_2},$$

where

$$\psi_j(x) = \phi_j(x) - \lambda_0 \int_a^b g(x, y) \cdot \phi_j(y) dy,$$

$$\overline{\varphi}_j(x) = \overline{\phi}_j(x) - \lambda_0 \int_a^b f(y, x) \cdot \overline{\phi}_j(y) dy \quad \text{respectively.}$$

In the present instance, when we repeat the same process for $G(x, y)$ as for $F(x, y)$, then we obtain the two following F. S.'s of eigen-functions and associated eigen-functions of the kernel $G(x, y)$:

$$\psi_1, \psi_2, \ldots\ldots, \psi_{p_1}; \ \Psi_1, \Psi_2, \ldots\ldots, \Psi_{p_2},$$

$$\overline{\varphi}_1, \overline{\varphi}_2, \ldots\ldots, \overline{\varphi}_{p_2}; \ \overline{\Psi}_1, \overline{\Psi}_2, \ldots\ldots, \overline{\Psi}_{p_1},$$

where

$$\varphi_j(x) = \Psi_j(x) - \lambda_0 \int_a^b f(x, y) \cdot \Psi_j(y) dy$$

$$\overline{\psi}_j(x) = \overline{\Psi}_j(x) - \lambda_0 \int_a^b g(y, x) \cdot \overline{\Psi}_j(y) dy \qquad \text{respectively.}$$

Now let us suppose the following permutability between two kernels $f(x, y)$ and $g(x, y)$:

$$\int_a^b f(x, t) \cdot g(t, y) dt = \int_a^b g(x, t) \cdot f(t, y) dt,$$

then from (1) and (2) we have

$$F(x, y) \equiv G(x, y).$$

Consequently, the above two corresponding systems of eigen-functions must be coincident with each other, while the necessary and sufficient conditions are

$$\phi_j = \sum_{h=1}^{p_1} A_{jh} \psi_h + \sum_{h=1}^{p_2} B_{jh} \Psi_h; \quad j = 1, 2, 3, \ldots\ldots, p_1$$

and $$\Psi_j=\sum_{h=1}^{p_2}C_{jh}\psi_h+\sum_{h=1}^{p_1}D_{jh}\phi_h;\quad j=1,\ 2,\ 3,\ldots\ldots\ldots\ldots,p_2.$$

Substitute the second equation in the first equation, then we obtain

$$\phi_j=\sum_{h=1}^{p_1}A_{jh}\psi_h+\sum_{h=1}^{p_2}\xi_{jh}\varphi_h+\sum_{h=1}^{p_1}\eta_{jh}\cdot\phi_h,$$

$$\text{for}\quad j=1,2,3,\ldots\ldots\ldots\ldots,p_1,$$

where $$\xi_{ji}=\sum_{h=1}^{p_2}B_{jh}\cdot C_{hi},$$

$$\eta_{ji}=\sum_{h=1}^{p_1}B_{jh}\cdot D_{hi},$$

and moreover $B_{jh}=D_{hi}=0$ for $h\geqq p_2+1$.

Now we consider the following determinant:

$$\begin{vmatrix} 1-\eta_{11}, & -\eta_{12}, & \cdots\cdots\cdots\cdots, & -\eta_{1p_1} \\ -\eta_{21}, & 1-\eta_{22}, & \cdots\cdots\cdots\cdots, & -\eta_{2p_1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ -\eta_{p_11}, & -\eta_{p_12}, & \cdots\cdots\cdots\cdots, & 1-\eta_{p_1p_1} \end{vmatrix}\equiv\Delta.$$

By applying the above relations, the determinant can be written as follows:

$$\Delta=\text{determi. of }|\eta_{ij}|=\text{determi. }|B_{jh}|\times\text{determi. }|D_{hi}|.$$

Therefore let the rank of the determinant Δ be r, then there will exist (p_1-r)-functional relations among the following functions:

$$\phi_1,\ \phi_2,\ldots\ldots\ldots\ldots\ldots\ldots,\phi_{p_1};$$

namely one of $(\phi_1,\ \phi_2,\ldots\ldots,\ \phi_{p_1-r})$ is expressed linearly by the other r-functions $\phi_{p_1-r+1},\ \phi_{p_1-r+2},\ \ldots\ldots\ldots\ldots,\phi_{p_1}$.

Hence we shall have

Theorem **4.** *When two kernels $f(x,\ y)$ and $g(x,\ y)$ are permutable to each other, and have the same characteristic constant, in order that the eigen-function of the kernel $F(x,\ y)$ may be represented by a linear function of eigen-functions of kernels $f(x,\ y)$ and $g(x,\ y)$, it is necessary and sufficient that the above stated determinant Δ does not vanish.*

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