A Brief Investigation on Zeros of Consecutive Functions defined by the Recurrence Formula

By

Tunezô Satô.

(Received June 4, 1934)

We consider a sequence of functions $\{F_n(x)\}$ which are defined by the following recurrence formula:

$$(1) \qquad F_{n+1}(x)=A_n(x)F_n(x)+B_{n-1}(x)F_{n-1}(x), \quad n=1,2,3,\ldots\ldots\ldots,$$

where two sequences of functions $\{A_n(x)\}$, $\{B_{n-1}(x)\}$, $n=1,2,3,\ldots\ldots$ are supposed to be given by such a way that their elements $A_n(x)$ and $B_n(x)$ are some analytical expressions of x and n.

Actually, however, sometimes $B_n(x)$ does not involve x, but n only, in the recurrence formula (1) which is satisfied by the sequence of functions $\{F_n(x)\}$ e. g. *Legendre's, Tchebyscheff's, Hermite's, Laguerre's* and *Bessel's functions*.[1]

Therefore in this paper, we suppose $B_n(x)\equiv B_n$ independent of x. Now let

$$(2) \qquad \Theta_n(x)=F_n(x)\frac{dF_{n+1}}{dx}-F_{n+1}(x)\frac{dF_n}{dx},$$

then regarding $\frac{dB_n}{dx}=0$, from (1) we obtain

$$(3) \qquad \Theta_n(x)=F_n^2(x)\frac{dA_n(x)}{dx}-B_{n-1}\cdot\Theta_{n-1}(x).$$

Substitute $n-1$, $n-2,\ldots\ldots$, 2, 1 for n successively in the above result, then the Theorem **1** follows:

(1) For example, refer to the following: R. Courant u. Hilbert, *Methoden der Mathematischen Physik* I, 1924. pp. 66–79.

Theorem **1.** *When* $\frac{dA_n(x)}{dx}$ *holds the same sign for any* x *and* n, *and* B_n *for any* n, *if* D_n *be positive, then* $\Theta_n(x)$ *will hold the same sign for all* n *and* x; *where*

$$D_n \equiv \frac{dA_n}{dx} / (-B_{n-1}) \div \Theta_0(x) \text{ and } \Theta_0(x) = F_0(x)\frac{dF_1}{dx} - F_1(x)\frac{dF_0}{dx}.$$

Next when we define a new sequence of functions $\{V_n(x)\}$ by

$$(4) \qquad V_n(x) = \frac{F_{n+1}(x)}{F_n(x)}, \quad n = 0, 1, 2, \ldots\ldots\ldots\ldots,$$

from (2) we have

$$\frac{dV_n}{dx} = \frac{\Theta_n(x)}{[F_n(x)]^2}.$$

Now if all the elements of $\{A_n(x)\}$ are given as regular functions, it is evident by (1) that regularities and singularities of each element of the functional sequence $\{F_n(x)\}$ depend only on those of $F_0(x)$ and $F_1(x)$. Therefore when we take $F_0(x)$, $F_1(x)$ regular in a certain interval, we can make all elements $F_n(x)$, $n = 2, 3, \ldots\ldots\ldots$regular with respect to x.

Moreover these regularities exist in the product (*Durchschnitt*) of the existence-intervals of the functions $A_n(x)$, $n = 1, 2, 3, \ldots\ldots\ldots, F_0(x)$, and $F_1(x)$, which is denoted by (I). If we denote the aggregation of zero-points of $F_n(x)$ in (I) by (E_n) and $(I_n) = (I) - (E_n)$, then from (4) we can at once conclude that $V_n(x)$ *is a regular function of* x *in* (I_n) *and also* $\frac{dV_n}{dx}$. Accordingly, *if* $\frac{dV_n}{dx}$ *holds the same sign throughout the interval* (I_n), *the function* $V_n(x)$ *will become monotone in* (I_n).

Now if the first two Functions $F_0(x)$, $F_1(x)$ have no zero-point in common, no two consecutive functions of the functional sequence $\{F_n(x)\}$ have a common zero-point. For, let $F_{n+1}(a) = F_n(a) = 0$, then, since $B_{n-1} \neq 0$, from (1) we obtain $F_0(a) = F_1(a) = 0$.

Therefore from these results, if we denote the first zero[1] of $F_n(x)$ in (I) by $\overset{(n)}{a_i}$, from (4) we have

$$V_n\left(\overset{(n+1)}{a_1}\right) = 0 \quad \text{and} \quad \lim_{x \to \overset{(n)}{a_1} - 0} V_n(x) = \pm\infty.$$

---

1. We enumerate the zeros E_n, giving the order successively from the left.

Hence let

$(I)=(a, b)$ and

$V_n(a)\dfrac{dV_n}{dx}<0$[1]

then clearly $\overset{(n+1)}{\alpha_1}<\overset{(n)}{\alpha_1}$. (see fig.)

In general we have

$\overset{(n+1)}{\alpha_p}<\overset{(n)}{\alpha_p}$,

$p=1, 2, 3\ldots\ldots\ldots$

Consequently if $V_n(a)$ holds the same sign for any n, the inequality

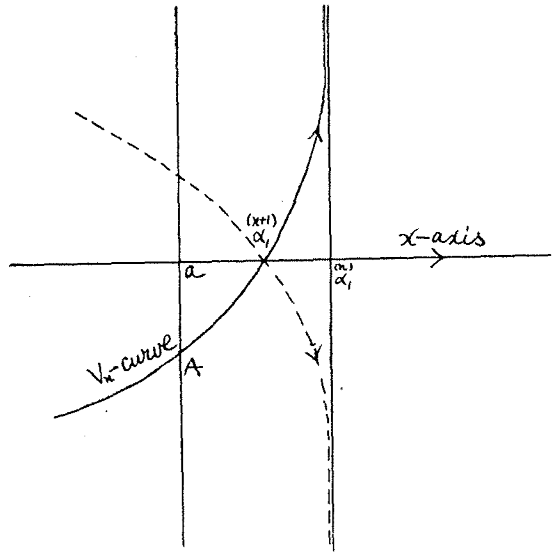
$V_n(a)\dfrac{dV_n}{dx}<0$

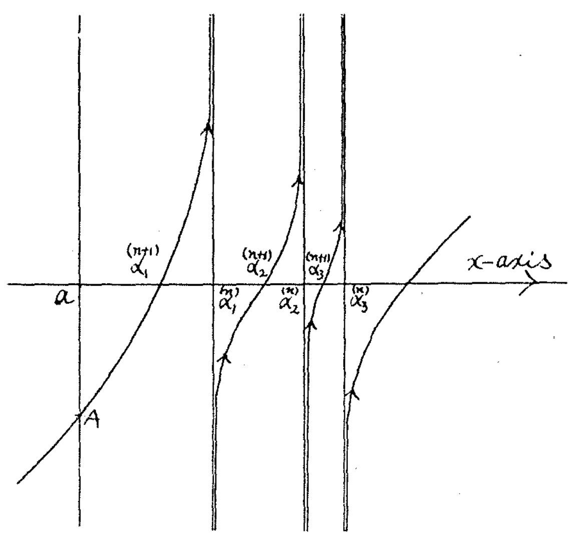
will mean that for any x and n:

$\varepsilon\Theta_n(x)<0$,

where

$\varepsilon=|V_n(a)|\div V_n(a)$.





By combining these results with Theorem **1**, *we can add the following to Theorem* **1**:

Theorem **2.** (i) *when* $\varepsilon<0$, *if* $\dfrac{dA_n}{dx}$ *is positive, the first zero of* $F_n(x)$ *will decrease as* n *increases to* $+\infty$;

(ii) *when* $\varepsilon>0$, *if* $\dfrac{dA_n}{dx}$ *is negative, the same result exists also.*

Application.

1. *Legendre's functions* $\{P_n(x)\}$:

In this case the recurrence formula is

---

1. If $V_n(a)=0$, the point a should be excluded from (I).

$$P_{n+1}(x)=\frac{2n+1}{n+1}x.P_n(x)-\frac{n}{n+1}P_{n-1}(x),\ \text{where}\ P_0(x)=1,$$
$$P_1(x)=x.$$

$$(I)=(-1,+1),\ A_n(x)=\frac{2n+1}{n+1}x,\ B_{n-1}=-\frac{n}{n+1}.$$

$$V_n(-1)=\frac{P_{n+1}(-1)}{P_n(-1)}=-1,\ \text{i. e.}\ \varepsilon<0.\quad \frac{dA_n}{dx}=\frac{2n+1}{n+1}>0.$$

$$-B_{n-1}=\frac{n}{n+1}>0.\quad \Theta_0(x)=1>0.$$

Hence for all x and n, $D_n>0$, then from the Theorem **2**, (i) we have the conclusion that *the first zero of* $P_n(x)$ *decreases as* $n\to+\infty$.

2. *Tschebyscheff's functions* $\{T_n(x)\}$:

The recurrence formula is

$$T_{n+1}(x)=x.T_n(x)-\frac{1}{4}T_{n-1}(x),\quad \text{where}\quad T_0(x)=1,$$
$$T_1(x)=\text{Cos}(\arccos x)\quad \text{i. e.}\ x.$$

$$(I)=(-1,+1).\quad A_n(x)=x.\quad B_{n-1}=-\frac{1}{4}.\quad \Theta_0(x)=1>0.$$

$$V(-1)=\frac{T_{n+1}(-1)}{T_n(-1)}=\frac{\frac{1}{2^n}(-1)^{n+1}}{\frac{1}{2^{n-1}}(-1)^n}=-\frac{1}{2}\ \text{i. e.}\ \varepsilon<0.$$

Hence $D_n>0$,

namely Theorem **2**, (i) is applicable in the same way as above.

3. *Hermite's functions* $\{H_n(x)\}$:

The recurrence formula is

$$H_{n+1}(x)=2x.H_n(x)-2n.H_{n-1}(x),\ \text{where}\ H_0(x)=1,\ \ H_1(x)=2x.$$
$$(I)=(-\infty,+\infty).\ A_n(x)=2x.\ B_{n-1}=-2n.\ \lim_{x\to-\infty}V_n(x)\to-\infty,$$

hence $\varepsilon<0$.

Therefore in this case also Theorem **2**, (i) is applicable.

4. *Laguerre's functions* $\{L_n(x)\}$:

The recurrence formula is

$$L_{n+1}(x)=(2n+1-x)L_n(x)-n^2.L_{n-1}(x),\ \text{where}\ L_0(x)=1,$$
$$L_1(x)=-x+1.$$

$$(I)=(0,+\infty).\quad A_n(x)=2n+1-x.\quad B_{n-1}=-n^2.\quad \Theta_0(x)=-1<0.$$

$$F_n(0)=\frac{(n+1)!}{n!}\ \text{i. g. } \varepsilon>0.\quad \frac{dA_n}{dx}=-1<0.\quad -B_{n-1}=n^2>0.$$

Hence $D_n>0$. Hence, although this case is a little different from the preceding three cases, Theorem **2**, (ii) is applicable.

Now on the other hand it is well-known that all these functions as above stated are also the regular functions of such linear homogeneous differential equations of the second order as

$$\frac{d^2F_n}{dx^2}+\alpha(x)\frac{dF_n}{dx}+\beta_n(x)F_n(x)=0.$$

From this stand-point, Professor Toshizô Matumoto[1] proves that *if x_0 be a zero point of $F_n(x)$, when $\frac{\partial\beta_n}{\partial n}>0$, we may conclude that x_0 decreases as $n\to+\infty$.* In practice since β_n of each of $P_n(x)$, $T_n(x)$, $H_n(x)$ and $L_n(x)$ is respectively

$$-\frac{(n+1)(n+2)}{x^2-1}\quad(|x|<1);\quad \frac{(n+1)^2}{1-x^2}\quad(|x|<1);$$

$$2n\quad(-\infty<x<+\infty);\quad \frac{n}{x}\quad(0<x<+\infty),\quad \text{and clearly}$$

$$\frac{\partial\beta_n}{\partial n}>0\quad\text{in } (I),$$

this conclusion coincides well with our result.

In conclusion the author wishes to express his thanks to Professor Toshizô Matsumoto for his interest in this paper.

---

1. T. Matsumoto, On the definition of functions by the recurrence formula.........etc., These Memoirs A, **14**, (1931) pp. 317–325.