Distorted Outer Layers of the Stars.

By

Shin-ichiro Takeda

(Received June 14, 1934)

Introduction

§1. Recently S. Chandrasekhar,[1] in his investigation of the equilibrium of the slowly rotating and tidally distorted polytropes, made an important extension to Emden's researches on the polytropic gas spheres. His works, however, are confined to rather small distortion so that its higher order terms can be neglected.

From quite different point of view, the same problem has also been discussed by Jeans[2] for the complete sequence of the geometry of configurations for all range of distortion. According to his investigations, a gas sphere rotating as a rigid body can break up in two distinct ways: either by fission into two detached masses as in a liquid star; or by a process of equatorial break-up after a lenticular shape as in the Roche model—according as the polytropic index n is less or greater than $5/6$. A similar result was obtained as for a generalized Roche model consisting of a homogeneous incompressible mass of finite size and of finite density surrounded by an atmosphere of negligible density. It was shown that the critical point occurs when the ratio of the volume of the atmosphere to that of the nucleus is of the value of about $1/3$.

§2. Similar circumstances are found among non-rotating gaseous models. The Roche model corresponds to the polytrope $n=5$ and the liquid star to that of $n=0$. The central condensation is, however, so pronounced for polytropes with larger indices that in the investigation

1. Chandrasekhar: M. N., 93 (1933) 390, 449, 462 and 539
2. *The Problem of Cosmogony and Stellar Dynamics*, p. 186

of the equilibrium of their outer layers, their own gravitational attraction may be well neglected. Or in other words, in such cases, we can determine approximate distribution of the density in the outer regions as an extended Roche model, by taking only the attraction of the nucleus into account and assuming an appropriate polytropic relation between the pressure and the density. At a certain boundary the pressure and the pressure gradient must be continuous with those of the nucleus which is in itself in mechanical equilibrium. For example, the polytrope $n=3$ contains 90 percent of the mass within the sphere of about half the radius and in this case the temperature u for the present model will take the form

$$u=u_1-u_1'\xi_1^2\left(\frac{1}{\xi}-\frac{1}{\xi_1}\right),$$

where $-u_1'\xi_1^2$ is the mass within the sphere of radius ξ_1. The accuracy of this approximation can be seen from the following short table. If we take the boundary of Emden's model as the place of continuation, the divergence from true values will gradually appear as we proceed inwards, but the approximation seems to be most preferable at least for the outermost layers.

Distribution of Temperature u

ξ	$u(\xi_1=4.0)$	$u(\xi_1=6.9)$	u(Emden)
4.0	0.2049	0.2118	0.2049
4.5	0.1561	0.1558	0.1553
5.0	0.1134	0.1110	0.1111
6.0	0.0494	0.0439	0.0441
6.9	0.0079	0.0000	0.0000
7.1	0.0000		

On the other hand, the effect of the rotation or of the tidal force increases rapidly in these regions where the density becomes insignificant. According to Chandrasekhar, for example, the level surface in a slowly rotating polytrope is given by

$$u(\xi)+v\{\psi_0(\xi)-A_2\psi_2(\xi)P_2(\cos\theta)\}=\text{constant}$$

and since the deviation of this surface from a sphere is small, the oblateness σ will be easily found as follows:

$$\sigma\propto-\frac{3}{2}\psi_2(\xi)/\xi u'$$

Taking the value of $\psi_2(\xi)$ from his paper, we find that the oblateness at $\xi=4.0$ is about one fourth of that of the boundary surface.

Consequently we may infer that the deviation of inner leval surfaces from sphere is still small even when the rotation is so rapid that the free surface begins to break up equatrially; that is, in such cases, although the solution of Chandrasekhar gives a good approximation for inner regions, the configurations of outer layers will be represented more preferably by the extended Roche model. Moreover, since in this model the main difficulties met in the calculation of the gravitational potential are eliminated and in actual applications the configurations of outer layers of the polytrope $n=3$ is of most interest, it will be worth while to rediscuss the problem from this point of view and to consider some applications.

§3. The problem is simple. Suppose that throughout the model the total pressure P is related to the density ρ by means of the relation

$$P=K\rho^{\frac{4}{3}} \tag{1}$$

and that it is subject to a finite uniform rotation or is tidally affected. It is required to determine the equilibrium states of the outer layers by taking into account only the gravitational attraction of the nucleus which is in turn in mechanical equilibrium. The level surface forming the boundary of the nucleus is supposed to be ellipsoidal, and the pressure and the pressure gradient in the two regions are made continuous on this surface.

In such a model, as is well known, if we introduce the variable U defined by the equation,

$$\psi=\psi_c U,$$

ψ being the total potential, the temperature, density, and pressure are proportional to U, U^3 and U^4 respectively. It is convenient to measure the radius vector ξ by the standard unit $\sqrt{\dfrac{\psi_c}{4\pi G\rho_c}}$, where the suffix c means their central values.

The Rotational Problem

§4. Assuming that the gravitational potential satisfies Poisson's equation in the nucleus and Laplace's equation in the outer regions, the distribution of the total potential U in the model star rotating with a constant angular velocity ω is governed by the differential equation

$$\nabla^2 U_n=v-U_n^3 \tag{2}$$

or

$$\nabla^2 U_0=v, \tag{3}$$

where v is $\omega^2/2\pi G\rho_c$—according as the nuclear or the outer regions are concerned. In the exact problem, equation (2) should be satisfied throughout the star and formally the approximation by (3) will be adequate when U^3 is small compared with v.

The solution of equation (2) will be given by

$$U_n(\theta, \xi)=u(\xi)+v\{\psi_0(\xi)+a_2\psi_2(\xi)P_2(\cos\theta)\} \qquad (4)$$

if we neglect, following Chandrasekhar, the higher order terms in v and reserve only the spheroidal terms. Here θ is the colatitude, P_2 the Legendre function of second order, a_2 a constant of integration, and u, ψ_0 and ψ_2 satisfy the following equations:

$$\left.\begin{aligned} &\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{du}{d\xi}\right)+u^3=0\\ &\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)+3u^2\psi_0-1=0\\ &\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right)+\frac{\psi_2}{\xi_2}+3u^2\psi_2=0. \end{aligned}\right\} \qquad (5)$$

The first of these is Emden's differential equation of index 3, and "associated Emden's functions" ψ_0 and ψ_2 are numerically integrated and tabulated by Chandrasekhar (*loc. cit.*, p. 404).

The solution of equation (3) will be of the form,

$$U_0=\frac{M}{\xi}+v\left[\frac{b_2}{\xi^3}P_2(\cos\theta)+\frac{\xi^2}{6}\{1+P_2(\cos\theta)\}\right]+C, \qquad (6)$$

where M, b_2 and C are constants of integration to be determined by the condition on the boundary level surface of the nucleus that U_0 and ∇U_0 should be continuous respectively with U_n and ∇U_n. If we write the equation of the boundary in the form,

$$\xi=\xi_1[1+v\{\varepsilon_0+\varepsilon_2P_2(\cos\theta)\}], \qquad (7)$$

these constants are found to be

$$\left.\begin{aligned} &M=M_1(1+v\Delta M)\\ &C=C_1(1+v\Delta C)\\ &b_2=\frac{\xi_1^3}{6}(\xi_1^2-A)\\ &a_2=\frac{1}{6}A/\psi_2(\xi_1) \end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned} \varepsilon_0 &= -\frac{1}{\xi_1 u_1'}\psi_0(\xi_1) \\ \varepsilon_2 &= \frac{1}{6}\,\frac{1}{\xi_1 u_1'}A, \end{aligned}\right\}$$

where we put

$$\left.\begin{aligned} M_1 &= -\xi_1^2 u_1' \\ C_1 &= u_1 + \xi_1 u_1' \\ M_1 \Delta M &= \left\{\frac{\xi_1}{3} - \psi_0'(\xi_1) - \frac{u_1^3}{u_1'}\psi_0(\xi_1)\right\}\xi_1^2 \\ C_1 \Delta C &= -\frac{\xi_1^2}{2} + \xi_1\psi_0'(\xi_1) + \left(1 + \frac{u_1^3}{u_1'}\xi_1\right)\psi_0(\xi_1) \\ A &= 5\xi_1^2\left\{3 + \xi_1\frac{u_1^3}{u_1'} + \xi_1\psi_2'(\xi_1)/\psi_2'(\xi_1)\right\}^{-1}. \end{aligned}\right\} \quad (9)$$

As already shown by Chandrasekhar, there is mass increase $vM_1\Delta M$ compared with the corresponding spherical star of equal central density. The values of ΔM, a_2, ε_0 and ε_2 reduce to those given by him if we take $\xi_1 = 6.90$ where $u_1 = 0$.

The numerical values of these constants calculated for three cases are summarized in the following table.

Table I

ξ_1	M_1	ΔM	C_1	ΔC	a_2	$b_2\xi_1^2$	ε_0	ε_2
4.50	1.974	5.38	−0.2834	12.68	−0.718	−2.82	4.89	9.13
5.00	2.001	5.77	−0.2890	12.97	−0.721	−2.60	6.80	11.7
6.90	2.015	6.13	−0.2920	13.45	−0.723	−1.42	20.0	27.9

The meaning of the continuation at $\xi_1 = 6.90$ is somewhat ambiguous in its original sense and it would seem that the solution U_0 thus obtained can not be applied to rapidly rotating stars. But it may be taken for a limiting case or that which reduces to the solution of Chandrasekhar when the rotation is slow.

§5. The second term in the equation of U_0 will be ascribed to the deformation of the nucleus but since $|b_2 \div M\xi^2|$ is less than unity near the external boundary indicated by $U_0 = 0$ and v is about 0.004 even at the critical point as shown later, it may be safely neglected compared with the first term so long as the outermost layers are concerned. Thus we find for the equation of the external boundary

$$\frac{M}{\xi}+v\frac{(\xi\sin\theta)^2}{4}+C=0 \tag{10}$$

If we denote the equatorial and the polar radius by ξ_e and ξ_p, we have

$$\left.\begin{aligned}\xi_e&=-\frac{M}{C}\left(1+\frac{v}{4C}\xi_e^2\right)^{-1}\\ \xi_p&=-\frac{M}{C}\end{aligned}\right\} \tag{11}$$

and since the "oblateness" σ may be written

$$\sigma=1-\frac{\xi_p}{\xi_e}=-\frac{v}{4C}\xi_e^2,$$

we have again

$$\xi_e=-\frac{M}{C(1-\sigma)} \tag{11a}$$

and

$$\sigma(1-\sigma)^2=-\frac{v}{4C}\left(\frac{M}{C}\right)^2. \tag{12}$$

Here both M and C are functions of v, and therefore the values of σ, ξ_p and ξ_e can each be calculated by these equations. In particular, when v is small, σ may be written

$$\sigma=-v\frac{M_1^2}{4C_1^3}\equiv v\sigma_0 \tag{13}$$

to the first order and depend on the ordinary Emden function alone. The numerical value of σ_0 is 40.8 for $\xi_1=6.90$, which is a little smaller than that of Chandrasekhar, $\frac{3}{2}\varepsilon_2=41.8$. This is evidently because we neglect the correction due to the deformation of the nucleus.

The radius of the non-rotating star is given by

$$\xi_2=-\frac{M_1}{C_1} \tag{14}$$

and since apparently

$$\xi_p\leqslant\xi_2\leqslant\xi_e,$$

there must be somewhere the radius $\xi=\xi_2$, for which

$$\sin^2\theta_2=\frac{\Delta C-\Delta M}{\sigma_0} \tag{15}$$

indifferently to v.

Finally the critical stage occurs when the rotation becomes rapid and $\frac{dU_0}{d\xi}$ vanishes at the equator. This condition follows

$$\xi_e^3 = 2\frac{M}{v}$$

or substituting this in (11), we find

$$\xi_e = \frac{3}{2}\xi_p = \frac{3}{2}\frac{M}{C},$$

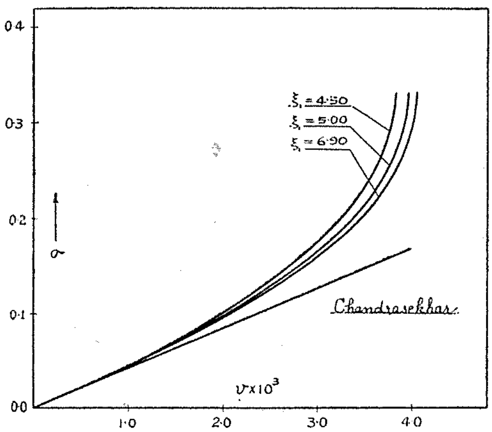
showing that the critical oblateness is one third irrespective of the value of ξ_1 as in the case of the ordinary Roche model.

The numerical values of these characteristics are calculated for three values of ξ_1 (Table II). Their mutual divergence is not so great that the ambiguity owing to the arbitrary selection of ξ_1 will make much difference in the actual applications.

Table II

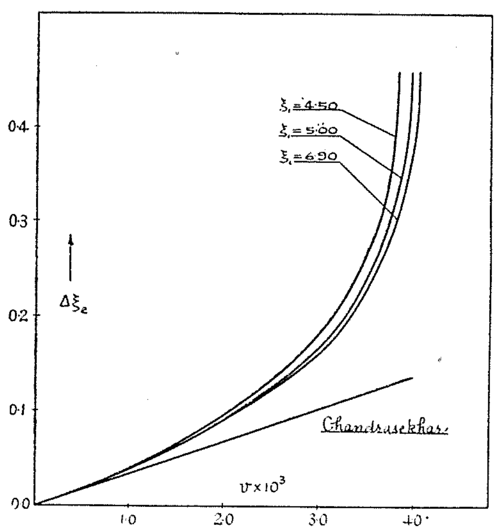
ξ_1	ξ_2	θ_2	σ_0	$v_{critical}$
4.50	6.97	24°.3	42.8	0.00383
5.00	6.92	24°.6	41.5	0.00397
6.90	6.90	25°.0	40.8	0.00405

Fig. 1. Oblateness σ



In order to illustrate the contrast with the solution of Chandrasekhar, I plot in Fig. 1 and 2 the oblateness σ and the equatorial elongation $\Delta\xi_e = \frac{\xi_e}{\xi_2} - 1$ against v, where the lower straight lines correspond to his solutions. The divergence comes into appearance rapidly for v greater than 0.001, the oblateness being about double of his

Fig. 2. Equatorial Elongation $\Delta\xi_e$



value for the critical rotation. The variation ξ_p with v is nearly constant so that the critical equatorial elongation is practically independent of ξ_1 and is found to be about 0.46.

§6. The arguments made in the proceeding article can be applied to inner level surfaces if we replace C by $C-U_0$ but here the effect of the deformation of the nucleus gradually becomes more significant as we proceed inwards and it is safe to solve equation (6) directly.

The oblateness at $\xi_1=4.50$ and 5.00 is estimated by (7) to be about 0.05 and 0.07 for critical rotation respectively, showing that the solution of Chandrasekhar, as expected, gives a good approximation for inner regions even in the critical stage.

Above calculations all refer to the standard radius ξ, and since the actual radius is quite arbitrary for the spherical polytrope $n=3$, we can say nothing about the variation of the actual radius with the rotation. It will be of some interest to convert ρ_c into the mean density $\bar{\rho}$ and to express the critical rotation by using the latter. The volume V of the Roche model at the critical point, as well known, is equal to

$$4\pi\xi_e^3\times 0.180373 \doteqdot \frac{4\pi}{3}\xi_2^3\times 1.68$$

so that

$$\bar{\rho}=\frac{M}{V}=\rho_m\frac{1+\Delta M\cdot v_{\text{critical}}}{1.68}$$

Here ρ_m is the mean density of the corresponding spherical polytrope and is equal to $\rho_c/54.36$. Hence we have for the critical rotation

$$\frac{\omega^2}{2\pi G\bar{\rho}}=89.1\times v_{\text{critical}},$$

which becomes 0.341, 0.354 and 0.361 respectively for the above three values of ξ_1 and their mean value is practically the same as that estimated by Jeans (*op. cit.*, p. 206).

The Binary Problem

§7. We consider in this section the steady configurations of a binary whose orbit is circular and for which the period of the rotation is equal to that of the revolution.

Take the centre of gravity of the primary, M as the origin of the coordinate system, x-axis in the direction of the secondary, M' and

z-axis parallel to that of the revolution. Then the direction cosines of a radius vector are given by

$$\lambda=\cos\varphi\sin\theta,\quad \mu=\sin\varphi\sin\theta\quad \text{and}\quad \nu=\cos\theta, \tag{16}$$

where θ is the colatitude and φ the longitude measured from x-axis.

The fundamental differential equations (2) and (3) to be satisfied by the potential will remain unchanged in the present problem, but here the rotation should be replaced by the revolution around the centre of gravity of the system and it follows

$$v=\frac{\omega^2}{2\pi G\rho_c}=2\frac{M'+M}{a^3}, \tag{17}$$

where a is the distance between the two bodies measured in the standard unit.

Now their solutions will be expressed in the form

$$U_n=u+v\{\psi_0(\xi)+a_2\psi_2(\xi)P_2(\nu)\}+a_2'\psi_2(\xi)P_2(\lambda) \tag{18}$$

$$U_0=\frac{M_n}{\xi}+v\Big[\frac{b_2}{\xi^3}P_2(\nu)+\frac{\xi^2}{6}\{1-P_2(\nu)\}\Big]+C$$
$$+\frac{b_2'}{\xi^3}P_2(\lambda)+M'\Big\{\frac{1}{\xi'}-\frac{1}{a}-\frac{\xi}{a^2}P_2(\lambda)\Big\}, \tag{19}$$

so long as the effect of the deformation of the secondary can be neglected and the boundary surface of the nucleus can be taken for an ellipsoid. Here M_n is the mass of the nucleus and ξ' means the distance from the centre of gravity of the secondary, viz.

$$\frac{1}{\eta'}\equiv\frac{a}{\xi'}=(1-2\lambda\cdot\eta+\eta^2)^{-\frac{1}{2}}=1+\eta P_1(\lambda)+\eta^2P_2(\lambda)+\ldots, \tag{20}$$

η being ξ/a.

Constants of integration in (18) and (19) can be determined, as before, by the conditions on the boundary surface, which may be now written

$$\xi=\xi_1\{1+v\varepsilon_0+v\varepsilon_2P_2(\nu)+\varepsilon'_2P_2(\lambda)\}. \tag{21}$$

But since the terms independent of λ in these equations are of the same form as in the rotational problem, the values of M_n, b_2, C, a_2, ε_0 and ε_2 will be given by (8). The same process gives the other three constants as follows:

$$-b_2'=\frac{M'}{a^3}\xi_1^3(\xi_1^2-A)\qquad\Big)$$

$$\varepsilon_2' = -\frac{M'}{a^3}\,\frac{A}{\xi_1 u_1'} \qquad a_2' = \frac{M'}{a^3}\,\frac{A}{\psi_2(\xi_1)}, \qquad (22)$$

where A is the same as given by (9).

It is convenient for the present problem to adopt $\xi_1 = 6.90$ as the place of continuation and to introduce the following two parameters:

$$\left.\begin{aligned} v'' &= \eta_1^3 \frac{M'}{M} \\ v' &= \eta_1^3\left(1 + \frac{M'}{M}\right) = v\frac{\xi_1^3}{2M}. \end{aligned}\right\} \qquad (23)$$

The constants appearing in U_0 will now become after some reductions

$$\left.\begin{aligned} M &= \frac{M'}{1 - v'\Delta M'} \\ \frac{C}{M} &= -\frac{1}{\xi_1}(1 + v'\Delta C') \\ \frac{1}{3}\,\frac{v'}{v''}\,\frac{b_2'}{M} &= -v\frac{b_2}{M} = -\frac{2b_2}{\xi_1^3}v' = 0.412v', \end{aligned}\right\} \qquad (24)$$

where

$$\left.\begin{aligned} \Delta M' &= \frac{2}{3} - 2\frac{\psi_0'(\xi_1)}{\xi_1} = 0.075 \\ \Delta C' &= \frac{1}{3} - 2\frac{\psi_0(\xi_1)}{\xi_1^2} = 0.088. \end{aligned}\right\}$$

Substituting these values in (19), we have

$$\frac{U_0}{M} = \frac{1}{\xi} - \frac{0.412}{\xi^3}\{v'P_2(\nu) - 3v''P_2(\lambda)\} + v'\frac{\xi^2}{3\xi_1^3}\{1 - P_2(\nu)\} + \frac{M'}{aM}\left\{\frac{1}{\eta'} - (1 + \lambda\eta)\right\} - \frac{1}{\xi_1}(1 + 0.088v'). \quad (25)$$

§8. The terms containing ξ^{-3} in the right of the above equation may be ascribed to the effect of the deformation of the nucleus. However, since, near the external boundary indicated by $U_0 = 0$, they are insignificant as compared with the first term, we neglect them in the following discussions as in the rotational problem. Thus we obtain for the free boundary

$$\frac{1}{\eta} + \frac{1}{3}\left(1 + \frac{M'}{M}\right)\eta^2\{1 - P_2(\nu)\} + \frac{M'}{M}\left\{\frac{1}{\eta'} - (1 + \lambda'\eta)\right\}$$

$$=(1+0.088v')\frac{1}{\eta_1}. \tag{26}$$

Now if we take φ and θ for the independent variables, the extreme radii of the surface will be given by the conditions:

$$\frac{\partial\eta}{\partial\varphi}=0 \quad \text{and} \quad \frac{\partial\eta}{\partial\theta}=0,$$

or differentiating

$$\left.\begin{aligned} &\sin\theta\sin\varphi(\eta'^{-3}-1)=0 \\ &\left[\left(1+\frac{M'}{M}\right)\eta\sin\theta+\frac{M'}{M}(\eta'^{-3}-1)\cos\varphi\right]\cos\theta=0, \end{aligned}\right\} \tag{27}$$

which are satisfied by the following four sets of equations

$$\left.\begin{aligned} &\sin\varphi=0,\ \cos\theta=0; \\ &\sin\varphi=0,\ (1-\eta'^{-3})=\left(1+\frac{M}{M'}\right)\eta\sin\theta; \\ &\cos\varphi=0,\ \sin\theta=0; \\ &\cos\theta=0,\ 1-\eta'^{-3}=0. \end{aligned}\right\} \tag{28}$$

The first set defines the maximum axis, lying in x-direction and the other three the minimum. In the diametrical plane $\left(\varphi=\frac{\pi}{2}\right)$, the minimum radius coincides with z-axis; but in the principal meridian $(\varphi=0)$, such is not the case. In fact neglecting the higher order terms in η, we find for its direction

$$\left(4+\frac{M}{M'}\right)\sin\theta=\frac{3}{2}(1-5\sin^2\theta)\eta+\ldots\ldots, \tag{29}$$

Finally in the equatorial planes $\left(\theta=\frac{\pi}{2}\right)$, it gives simply

$$\cos\varphi=\frac{1}{2}\eta, \tag{30}$$

or combining this with the boundary equation (26),

$$\frac{1}{\eta}+\frac{1}{2}\eta^2=(1+0.088)\frac{1}{\eta_1}. \tag{31}$$

The boundary surface at the critical point is determined by the condition,

$$\left(\frac{\partial U_0}{\partial\xi}\right)_{\lambda=1}=0,$$

which follows indifferently to η_1

$$\frac{M'}{M}=\frac{1-\eta_x^3}{1-(1-\eta_x)^3}\cdot\frac{(1-\eta_x)^2}{\eta_x^2}. \qquad (32)$$

By substituting (32) in (26), we can find the radius η_1 of the corresponding spherical star for different values of M'/M. The results are collected in the following table.

Table III

Characteristics of the Equator at the Critical Point

$\frac{M'}{M}$	η_1	η_x	η_{-x}	η_{min}	φ_{min}
0.0	1.000	1.000	1.000	1.000	
0.5	0.425	0.571	0.469	0.438	77°.4
1.0	0.367	0.500	0.406	0.374	79°.2
1.5	0.334	0.458	0.372	0.337	80°.3
2.0	0.310	0.429	0.347	0.312	81°.0
2.5	0.293	0.407	0.328	0.295	81°.5
3.0	0.279	0.389	0.314	0.280	82°.0
4.0	0.258	0.362	0.292	0.258	82°.6
5.0	0.242	0.341	0.274	0.242	83°.1
6.0	0.230	0.324	0.262	0.229	83°.4

The Phase Effect

§9. The stellar luminocity observed to the earth depends on the shape of the external boundary and the distribution of the surface brightness. In a binary star, the surface receives the radiation from the secondary, superimposed on the ordinary radiation streaming out from the interior. The reflection effect is of the order of magnitude of η^2, and is maximum at $\varphi=0$ and minimum at $\varphi=\pi$. The proper radiation is proportional to the temperature gradient on the surface; and since the latter is larger as the radius is smaller, it tends to increase the phase effect because of the deformation of the surface. Therefore it gives the maximum luminocity when observed in quarature, the contrast being greater if the limb-darkening is taken into account. The order of magnitude of the deformation effect is η^3; but as its coefficient is large, it becomes significant for close binaries. In the following we shall discuss the question more closely.

Let I_1 and I_2 be the flow per unit cross section per sec. of the ordinary and reflected radiation travelling in the direction α to the surface normal. Then the total luminocity observed in the direction (θ_0, φ_0) is

$$L(\theta_0, \varphi_0)=\int(I_1+I_2)\cos\alpha\, dF, \tag{33}$$

where dF is the surface element, and the integration should be extended over the whole visible hemisphere. If we denote by β the angle between the surface normal and the radius vector, dF will be given by

$$\cos\beta dF=\eta^2\sin\theta d\theta d\varphi \tag{34}$$

and we have

$$\cos\alpha=ll_0+mm_0+nn_0 \tag{35}$$

$$\cos\beta=\lambda l+\mu m+\nu n, \tag{36}$$

where (λ, μ, ν), (l, m, n) and (l_0, m_0, n_0) are the direction consines respectively of the radius vector, the surface normal, and the line of sight.

Now represent the net flux through the boundary surface by πH_1. Then the ordinary outward flow I_1 for a darkened star will be

$$I_1=\frac{1}{1+s}H_1\left(1+\frac{3}{2}s\cos\alpha\right), \tag{37}$$

s being the coefficient of darkening. s can assume any value between 0 (no darkening) and ∞ (complete darkening) but usually we may take it to be unity. Naturally H_1 is not constant but can be calculated if we suppose additionally that the star is in radiative equilibrium so that the net flux is proportional to $-\nabla U_0$.[1] Consequently we find from (25) for its components

$$\left.\begin{aligned}\eta^2\frac{H_{1x}}{H_0}&=\lambda-\left(1+\frac{M'}{M}\right)\eta^3\lambda-\frac{M'}{M}\left\{\frac{1-\lambda\eta}{\eta'^3}-1\right\}\eta^2\\ \eta^2\frac{H_{1y}}{H_0}&=\mu\left\{1-\left(1+\frac{M'}{M}\right)\eta^3+\frac{M'}{M}\left(\frac{\eta}{\eta'}\right)^3\right\}\\ \eta^2\frac{H_{1z}}{H_0}&=\nu\left\{1+\frac{M'}{M}\left(\frac{\eta}{\eta'}\right)^3\right\}\end{aligned}\right\} \tag{38}$$

where H_0 is a constant and can be determined by the following relation giving the total radiation $4L$ emitted from the whole surface:

$$4L=\int H_1 dF. \tag{39}$$

Further, since $-\nabla U_0$ is of the direction of the surface normal, it gives

1. von Zeipel: M. N., 84 (1924) 665

$$l=\frac{H_{1x}}{H_1}, \quad m=\frac{H_{1y}}{H_1} \quad \text{and} \quad n=\frac{H_{1z}}{H_1}. \tag{40}$$

Next we consider the reflected radiation. The subject has been investigated by Eddington[1], Milne[2] and others. In a steady state, the whole of incident radiation is necessarily re-emitted from the surface but the reflected radiation will not be distributed uniformly in direction. Milne derives the following formula for the law of reflection:

$$I_2=H_2\cos\gamma\frac{\left((\cos\gamma+\frac{1}{2}\right)\left(\cos\alpha+\frac{1}{2}\right)}{\cos\gamma+\cos\alpha}, \tag{41}$$

where πH_2 is the flux per unit cross section of radiation incident at angle γ to the surface normal and may be approximately given by

$$\pi H_2=\frac{L'}{\eta'^2}, \tag{42}$$

$4L'$ being the total radiation emitted by the secondary whose deformation and limb-darkening are neglected. The angle γ will be given by

$$\cos\gamma=\frac{1}{\eta'}(l-\eta\cos\beta). \tag{43}$$

Equation (42) is valid while η' is within the common inner tangent cone to the two stars. Nevertheless, it can be used in further calculations with sufficient approximation since, outside the cone, the incident radiation is small and decreasing rapidly in this region.

§10. As it is impossible to evaluate the integral (33) rigorously, to proceed further we have to confine ourselves to some simplifications. Accordingly we neglect here the higher order terms of distortion and consider only the case, $\theta_0=\frac{\pi}{2}$, that is the line of sight being in the equatorial plane.

Expanding η in powers of η_1 we obtain from (26)

$$\eta=\eta_1[1-0.088v'+\frac{1}{2}(1-\nu^2)v'+v''\{P_2(\lambda)+\eta_1P_3(\lambda)$$
$$+\eta_1^2P_4(\lambda)\}] \tag{44}$$

to the order of approximation. (44) is the same as the expression given by Chandrasekhar (*loc. cit.*, p. 467) except for a small difference

1. Eddington: M. N., 86 (1926) 320
2. Milne: M. N., 87, (1926) 43

in the coefficients of variable terms. His exact values are 1.0289, 1.00736 and 1.00281 instead of our unity.

On the other hand, since we have from (38)

$$H_1 = \sqrt{H_{1x}^2 + H_{1y}^2 + H_{1z}^2} = \lambda H_{1x} + \mu H_{1y} + \nu H_{1z} + O(\eta^6),$$

it follows

$$\cos\beta = 1 + O(\eta^6) \tag{45}$$

and

$$\eta^2 \frac{H_1}{H_0} = 1 - v'(1-\nu^2) + v''\left\{\frac{1}{\eta'^3} - \frac{\lambda}{\eta_1}\left(\frac{1}{\eta'^3} - 1\right)\right\} + O(\eta)$$
$$= 1 - v'(1-\nu^2) - v''\{2P_2(\lambda) + 3\eta_1 P_3(\lambda) + 4\eta_1^2 P_4(\lambda)\} \tag{46}$$

by inserting the value of η' from (20).

Similarly we find from (38), (40) and (46)

$$\left.\begin{aligned} l &= \lambda(1 - v'\nu^2) + v''(\lambda^2 - 1)\{P_2'(\lambda) + \eta_1 P_3'(\lambda) + \eta_1 P_4'(\lambda)\} \\ m &= \mu[1 - v'\nu^2 + v''\lambda\{P_2'(\lambda) + \eta_1 P_3'(\lambda) + \eta_1^2 P_4'(\lambda)\}] \\ n &= \nu[1 + v'(1-\nu^2) + v''\lambda\{P_2'(\lambda) + \eta_1 P_3'(\lambda) + \eta_1^2 P_4'(\lambda)\}], \end{aligned}\right\} \tag{47}$$

where $P_n'(\lambda)$ means $\dfrac{dP_n(\lambda)}{d\lambda}$.

Now we can evaluate our integrals. The total emission becomes

$$4L = \int_0^\pi\int_0^{2\pi} H_1\eta^2 \sin\theta d\theta d\varphi = 4\pi H_0\left(1 - \frac{2}{3}v'\right). \tag{48}$$

Here the term in v' gives the diminution of emission owing to the rotation (von Zeipel, *loc. cit.*), since by definition

$$\frac{2}{3}v' = \frac{1}{3}\frac{M}{\xi_1^3}\frac{\omega^2}{2\pi G\rho_c} = \frac{\omega^2}{2\pi G\bar{\rho}} \tag{49}$$

to the order of accuracy we are working.

Next we calculate the observed luminocity. For the observer in the equatorial plane

$$l_0 = \cos\varphi_0, \quad m_0 = \sin\varphi_0 \quad \text{and} \quad n_0 = 0. \tag{50}$$

Hence the contribution from the proper radiation will be

$$\frac{L_1\left(\frac{\pi}{2}, \varphi_0\right)}{\pi H_0} = \frac{1}{\pi H_0}\int_0^\pi\int_{\varphi_0 - \frac{\pi}{2}}^{\varphi_0 + \frac{\pi}{2}} I_1\eta^2 \sin\theta d\theta d\varphi$$

$$=1-\frac{1}{1+s}[v'+v''\{2P_2(l_0)-\eta_1^2P_4(l_0)\}]$$
$$-\frac{s}{1+s}\left[\frac{6}{5}v'+v''\left\{\frac{16}{5}P_2(l_0)+\frac{15}{8}\eta_1P_3(l_0)\right\}\right] \quad (51)^1$$

by (37), (46) and (47). Here as expected s, the coefficient of limb-darkening, tends to increase the deformation effect. It is of interest to note that leaving out the limb-darkening and regarding only the elliptical terms in the above equation, the variation in the surface brightness behaves as though the excentricity of the external surface is just twice as large as its proper value.

In the calculation of the reflection effect, we neglect the deformation of the surface and reserve only the terms to η_1^4. Accordingly it is sufficient for the integrand in (33) to write λ and η_1 for l and η and we have by (42) and (43)

$$\pi H_2\cos\gamma=\frac{L'}{4}\{\lambda+2\eta_1P_2(\lambda)+3\eta_1^2P_3(\lambda)\} \quad (52)$$

It may be noted[2] that here the darkening does not play an important rôle since the coefficient of cos γ in Milne's formula (41) of reflection is equal to unity for $\gamma=60^\circ$ and varies from 9/8 to 3/4 for $\gamma=0^\circ$ while α varies from 0° to 90°, the latter corresponding to the case when $s=1/3$ in the ordinary law of darkening (37). Indeed near $\gamma=90^\circ$, the amount of its variation becomes greater in the opposite direction, but in this region cos γ is so insignificant as not to concern us. Consequently here we may use cos γ instead of Milne's formula without much error.

Now we find for the reflection effect

$$L_2\left(\frac{\pi}{2},\ \varphi_0\right)=\int_0^\pi\int_{\varphi_0-\frac{\pi}{2}}^{\frac{\pi}{2}} H_2\cos\gamma\cos\alpha\,\eta_1^2\sin\theta d\theta d\varphi$$
$$=L'\eta_1^2\left[\frac{2}{3\pi}\left\{(\pi-\varphi_0)l_0+m_0\right\}\right.$$
$$\left.+\frac{1}{4}\eta_1\left\{l_0+P_2(l_0)\right\}-\frac{1}{\pi}\eta_1^2m_0^3\right] \quad (53)$$

from (33), (35) and (52). To the first order, L_2 is the same as given by Eddington (*loc. cit.*).

1. If we add $\frac{1}{1+s}(1+\frac{8}{5}s)\ v'n_0^2$ to this expression, we have the general formula for the observer not in the equatorial plane.

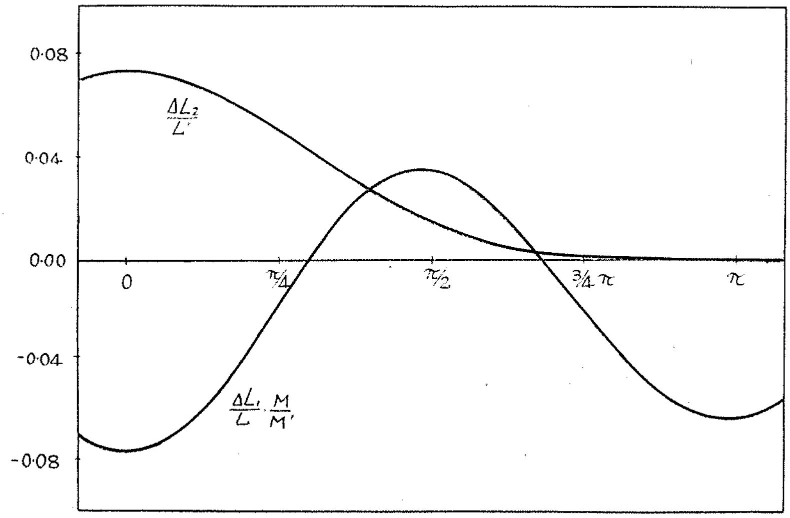
2. Pike: Ap. J., 73 (1931) 205

L_1 and L_2 combined give the observed luminocity

$$L\left(\frac{\pi}{2},\ \varphi_0\right)=L_1+L_2, \tag{54}$$

showing its variation with the phase. In conventional scales the contributions from these two effects are illustrated in Fig. 3 separately.

Fig. 3 Effects of the Deformation and Reflection. $\eta_1=0.3$



§11 In order to compare their relative amounts in the same scale, we must know the ratio L'/L. According to Jeans[1],

$$L \propto M^3 \tag{55}$$

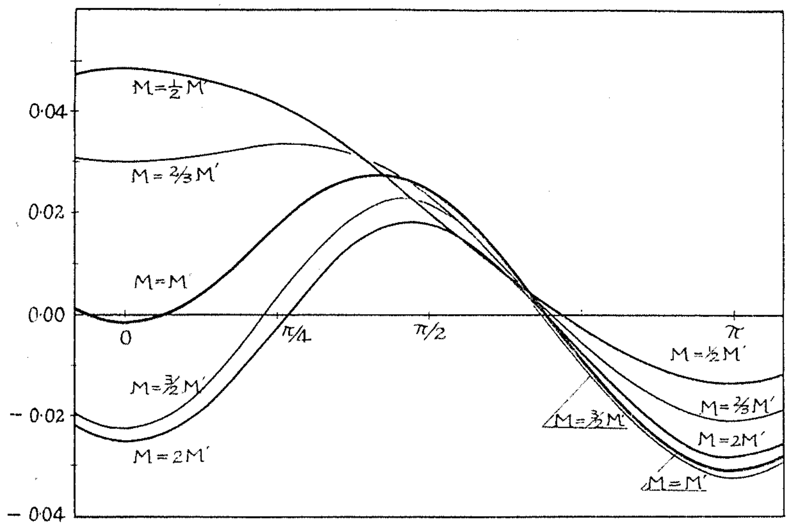
approximately. By means of this relation, the total phase effect for a component can be calculated for different values of η_1 and M'/M, a few examples for $\eta_1=0.3$ being illustrated in Fig. 4, where the resultant luminocity $(L+L')$ is taken to be unity. As apparent in the diagrams, its dependency on the mass ratio is remarkable. The reflection effect becomes predominate for the component whose mass is a little smaller than the other, and the oscillating light variation will then disappear.

The transition point can be estimated by (54) approximately, for it will occur first when $\dfrac{dL}{d\varphi_0}=0$ at $\varphi_0=0$. Differentiating and equating to 0, we have for $s=1$

---

1. *Astronomy and Cosmogony*, p. 130

Fig. 4 Phase Effect for the Component, M. $\eta_1=0.3$



$$\frac{2}{3\pi}(\pi-\varphi_0)+\frac{1}{4}\eta_1(1+3l_0)-\frac{3}{\pi}m_0l_0\eta_1^2$$
$$=\eta_1\left(\frac{M}{M'}\right)^2\left\{\frac{39}{5}l_0+\frac{15}{32}\eta_1(5l_0^2-1)\right.$$
$$\left.+\frac{5}{4}\eta_1^2(7l_0^2-3)l_0\right\} \tag{56}$$

whence for $\varphi_0=0$

$$\left(\frac{M'}{M}\right)^2=\eta_1\frac{\frac{39}{5}+\frac{15}{8}\eta_1+5\eta_1^2}{\frac{2}{3}+\eta_1}. \tag{57}$$

Combining this relation with that defining the critical boundary given in Table III, we find for the required transition point nearly

$M'/M=1.7$.

The deduction depends on Jeans' relation (55) of mass and luminocity but this estimate will not differ much if another plausible law is assumed.

Beyond this point, the deformation effect becomes appreciable and increases its relative importance for close binaries, although ultimately its absolute amount begins to decrease proportionally to M'/M. It is hardly necessary to say that the observed phase effect is the resultant of those of the two components and is to be found for each case.

§12 Now we proceed to compare our result for the deformation effect with observed data. It has been found by Shapley[1] that of 90 eclipsing variables whose orbits had been computed, 25 show measurable ellipticity. The ratio b/a of their equatorial axes is collected in the following table directly taken from his paper.

Mean separation $1-(a_b+a_f)$	No Stars	Uniform b/a	Darkened b/a	Darwin b/a
0.501	5	0.971	0.983	0.944
0.399	5	0.900	0.939	0.902
0.320	6	0.838	0.900	0.858
0.196	4	0.809	0.883	0.772
0.106	4	0.700	0.782	0.692

It will be seen as noted by him that while the uniform value of b/a agrees remarkably well with Darwin's theoretical value for a hypothetical homogeneous fluid, the darkened stars show throughout considerably less ellipticity than the homogeneous bodies for given separation, in spite of general adaptability of the darkened solution.

In order to compare our model star with Shapley's data it is convenient to reduce it to the uniform brightening as theoretically the variation in the surface brightness which has the same tendency as the limb-darkening should also be taken into account.

Consider a binary of equal ellipsoidal components. If they were uniformly brightened, the observed luminocity would be proportional to the area of the cross section perpendicular to the line of sight. Consequently apparent b/a will be equal to the ratio of the maximum and the minimum light, leaving out the variation due to the eclipse. Thus we find from (51) for $s=1$

$$\frac{b}{a}=\frac{L_{1\,min}}{L_{1\,max}}=\frac{1-0.9\eta_1^3}{1-4.8\eta_1^3} \tag{58}$$

to the first order.

On the other hand the ratio of the true axes is found from (44) to be

$$\frac{\eta_y}{\eta_x}=\frac{1+0.324\eta_1^3}{1+1.824\eta_1^3}. \tag{59}$$

Similarly the mean separation is given by

$$1-2\eta_x=1-2\eta_1(1+1.824\eta_1^3). \tag{60}$$

1. Shapley: Contr. Princeton Univ. Obs., No. 3, p. 115

Numerical results are shown in the following table.

Table IV

η_x	η_1	$\frac{L_{1\,min}}{L_{1\,max}}$	O−C Polytrope	O−C Darwin	
0.250	0.244	0.943	+0.028	+0.027	0.989
0.300	0.288	0.910	−0.010	−0.002	0.967
0.340	0.321	0.866	−0.028	−0.020	0.953
0.402	0.369	0.795	+0.014	+0.037	0.931
0.447	0.400	0.735	−0.035	+0.008	0.919

The general agreement between the theory and the observation is not worse than in the case of a uniform liquid star. But as expected, the ratio of the true axes is much closer to unity in a polytropic star.

Summary

1) The steady states of distorted outer layers of the polytrope $n=3$ in the presence of the rotation and the tidal effect are studied as an extended Roche model defined in §3 and the results are compared with the investigations of Chandrasekhar.

2) The divergence among the two solutions comes into appearance rapidly for the value of $v=\dfrac{\omega^2}{2\pi G\rho_c}$ greater than 0.001 (Fig. 1. and Fig. 2); for example, the oblateness of the boundary surface is about double of his value for the critical rotation ($v=0.004$), although even in this case his solution gives a good approximation for inner regions (say $\xi<4.5$).

3) For a binary, the characteristics of the equator of a component at the critical point are calculated for different values of the mass ratio and the results are collected in Table III.

4) The effect of the deformation on the light curve is estimated by taking into account both the variation in the surface brightness and the darkening to the limb. The result is compared with Shapley's data concerning the eclipsing variables, and found to be in satisfactory accord (§12).

5) The reflection effect is also considered (§10). The deformation is proportional to $\left(\dfrac{\xi_1}{a}\right)^3$ while the reflection is proportional to $\left(\dfrac{\xi_1}{a}\right)^2$ but the coefficient being greater, the former effect becomes significant for close binaries.

6) Apart from this the dependency on the mass ratio is remarkable, the reflection effect being predominate for the component whose mass M is a little smaller than the other. In Fig. 4 the combined phase effect is illustrated for different values of M'/M, $\frac{\xi_1}{a}$ being fixed. It is shown that the deformation effect becomes appreciable when $1.7M$ becomes greater than M', and increases its relative importance for close binaries, although ultimately its absolute amount begins to decrease proportionally to M'/M (§ 11).