Coast Effect upon the Ocean Current and the Sea Level, II. Changing State

By

Takaharu Nomitsu

(Received June 24, 1934)

Abstract

In this second report on the coast effect, the writer will investigate the development of the surface slope (γ) and of the current (w) as it is influenced by land.

At the coast itself as soon as the primary cause begins to act the slope instantly springs up to a finite value $\gamma(0)$, and then approaches gradually to the steady value $\bar{\gamma}$ given in our first report on the problem. Whereas $\bar{\gamma}$ has different values according to the bottom condition, the value of $\gamma(0)$ is the same without distinction of the condition of bottom.

The slope thus produced at the coast propagates in the sea off the coast with a velocity of nearly $c=\sqrt{gH}$ and consequently the surface elevation at a distance y from the coast will be approximately given by

$$\zeta=(ct-y)\left[\bar{\gamma}-\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t}\right] \quad \text{for} \quad y<ct$$
$$=0 \quad \text{for} \quad y>ct.$$

The current influenced by the coast may be considered to consist of the primary current w_p and the secondary slope current w_γ. The former has already been described in our previous papers on the current in a boundless sea, and the latter can be calculated with the above obtained slope. The writer emphasizes here that both the slope and the secondary current have the form which is the same for every kind of primary cause, but different only in their final values.

Introduction

The changing state of the current and the sea surface influenced by land was first treated by Messrs Proudman and Doodson,[1] but confining themselves to one-directional motion they neglected Coriolis' force. They tried to make the current zero at the coast, but for that

1. "Time-relations in Meteorological Effects in the Sea", Proc. London Math. Soc., Ser. 2, **24**, 140 (1924)

intention their solution can be used only when the wind itself vanishes at the coast, while it can not be extended to cases where the wind is not zero at the coast, even to the simplest case of uniform wind all over the sea.

Horrocks,[1] by extending the method of the former investigators to a two-dimensional rotating sea, obtained so complicated a solution that it is inconvenient for numerical calculations. In reality, Horrocks gave a numerical example which was shown by Proudman to be considerably miscalculated.

Recently K. Hidaka[2] discussed the effect of wind on the surface slope in his paper "Non-stationary Ocean-currents", Chap. VI. He started, however, from the assumption that the total flow perpendicular to the coast is always nill even in the developing stage, so that his solution clearly can not be used except at the geometrical line of coast. Moreover, his result concerns with only the surface slope at the coast itself but not off the coast, so that the elevation of surface level and the current in the sea can not be known at all, and thus his result is of little practical use.

More recently Hidaka[3] published another paper, about which the same words may be said as for the paper of Proudman and Doodson, i. e., it deals with only one-dimensional motion without taking Coriolis' force into account, and his intention as a boundary problem has not been accomplished generally, even when the wind is uniform.

In all the above works, the bottom-condition is taken as that no slip velocity can exist at the sea bed. A different bottom-condition can be seen in Proudman's second paper[4] on the effect of traveling atmospheric pressure-wave. He assumed here the sea water as an ideal fluid, which means also that there is no bottom-friction at all.

The present writer intends to solve the changing state of the land effect upon the surface slope and the current of various kinds with various bottom-conditions, Coriolis' force being taken into consideration. He will try also to get the solution in as easy and convenient a form for numerical calculations as possible. The writer confesses that since

1. "Meteorological Perturbations of Tides etc.", Proc. Roy. Soc. London, **115**, 170 (1927)

2. Mem. Imp. Mar. Observatory, Kobe, **5**, 255 (1933)

3. "Motion of Lake Water generated by Winds", Geophys. Mag., Tokyo, **7**, 234 (1933)

4. "The Effects on the Sea of Changes in Atmospheric Pressure", M. N. R. A. S. Geophys. Suppl. **2**, 197 (1929)

he adopts the ordinary equation of motion used by Proudman and others, the obtained solution contains the weak point that the current does not vanish at the coast generally. The writer, however, kept this fallacy consciously because he knew that it is inevitable with the ordinary form of the equation of motion which neglects the vertical currents, and any effort must be in vain to make the current zero at the coast, whatever other method of solving may be used. The reason for it will be explained in § 9, and the correction due to the vertical motion in the immediate neighbourhood of the coast will be dealt with by Mr. Takegami[1] soon.

Moreover, since our physical common sense teaches us that the surface elevation ζ or slope γ will develope most rapidly at the coast and then propagate gradually off the coast, one of the chief objects of the present paper is to find not only the mode of generation of surface-slope at a definite place but also its propagation velocity which has not yet been discussed by any one.

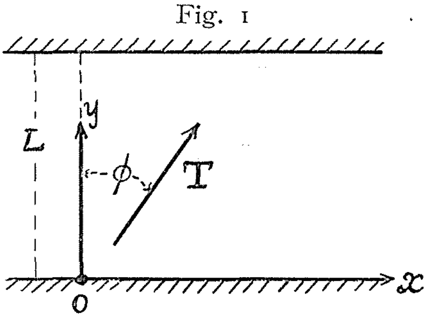
I. Wind Current

§ 1. Fundamental equations

First consider a channel limited by two long parallel straight coasts distant L from each other and suppose that a constant (but not necessarily uniform) wind begins to blow suddenly over the sea which is initially at rest.

Take x-axis along the coast on the undisturbed sea-surface and z-axis vertically downwards (Fig. 1), and denote the elevation of the free surface by ζ, the other notations being the same as those in our previous paper[2] on the steady state of the coast effect.

Fig. 1



If we neglect the vertical motion v_z compared with the horizontal velocity $w=v_x+iv_y$, then we have the equation of motion

$$\frac{\partial w}{\partial t}=\nu\frac{\partial^2 w}{\partial z^2}-2i\bar{\omega}w+ig\gamma, \qquad \gamma=-\frac{\partial\zeta}{\partial y}, \qquad (1)$$

1. These Memoirs, A, **17**, 305 (1934)
2. These Memoirs, A, **17**, 93 (1934)

with the equation of continuity

$$\left.\begin{aligned}\frac{\partial\zeta}{\partial t}&=-\frac{\partial S_y}{\partial y}=-\int_0^H\frac{\partial v_y}{\partial y}dz,\\ \text{or}\qquad \frac{\partial\gamma}{\partial t}&=\frac{\partial^2 S_y}{\partial y^2}=\int_0^H\frac{\partial^2 v_y}{\partial y^2}dz,\end{aligned}\right\}\tag{2}$$

and the initial and the boundary conditions

$$w=0,\ \gamma=0 \qquad \text{when} \quad t=0, \tag{3}$$

$$\left.\begin{aligned}\frac{\partial w}{\partial z}&=-i\frac{Te^{-i\phi}}{\mu} && \text{at}\quad z=0,\\ S_y&=0 && \text{at}\quad y=0\quad\text{and}\quad y=L.\end{aligned}\right\}\tag{4}$$

The bottom-condition may be different according to our assumption.

It should be remembered here that, even when the wind action T is constant and uniform all over the sea, the quantities ζ and γ will be functions of t and y, so that we may write $\zeta(y, t)$ and $\gamma(y,t)$.

§ 2. Case of no bottom-friction

If we adopt the bottom-condition

$$\partial w/\partial z=0 \qquad \text{at} \quad z=H, \tag{5}$$

then the solution of (1) which satisfies (3,) (4) and (5) is obviously the sum of already known "drift current"[1] w_T due to T and the "slope current"[2] w_γ due to γ in developing stage with the same bottom-condition, namely

$$\left.\begin{aligned}w&=w_T+w_\gamma\\ &=\frac{iTe^{-i\phi}}{\mu a}\,\frac{\cosh a(H-z)}{\sinh aH}\\ &\quad-\frac{Te^{-i\phi}}{\mu k^2 H}e^{-i\bar{\omega}t}\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1+i\beta_n^2/2k^2}{1+(\beta_n^2/2k^2)^2}\cos\beta_n z e^{-\nu\beta_n^2 t}\right\}\\ &\qquad\qquad+ig\int_0^t d\tau\,\gamma(y,\tau)e^{-2i\bar{\omega}(t-\tau)}\\ a&=(1+i)k,\quad k=\sqrt{\bar{\omega}/\nu},\ \beta_n=n\pi/H,\quad n=1,2,3,\ \dots\end{aligned}\right\}\tag{6}$$

which will be easily verified by direct substitution into the above equations. Thus, if γ is determined, the current w also can be easily calculated.

---

1. These Memoirs, A, **16**, 275 (1933), eqs. (6), (7) and (9).
2. Ditto, 333 (1933), eq. (9).

Let ${}_{T}S_y$ and ${}_{\Upsilon}S_y$ represent the parts of the total flow perpendicular to the coast due to the drift- and slope-current respectively. Then eq. (6) gives

$$\left.\begin{aligned} {}_{T}S_y &= -\frac{T\sin\phi}{2\mu k^2} + \frac{T}{2\mu k^2}\sin(2\bar{\omega}t + \phi) \\ &= \frac{T}{2\rho\omega}[\sin(2\bar{\omega}t + \phi) - \sin\phi] \\ {}_{\Upsilon}S_y &= gH\int_0^t d\tau\gamma(y, \tau)\cos 2\bar{\omega}(t-\tau). \end{aligned}\right\} \quad (7)$$

Slope at the coast:—At the coast itself, the total flow perpendicular to it must of course be zero, i. e.,

$$S_y = {}_{T}S_y + {}_{\Upsilon}S_y = 0 \qquad \text{at} \quad y = 0 \text{ and } L.$$

In the present case this gives

$$\frac{T}{2\rho\omega}[\sin(2\omega t + \phi) - \sin\phi] + gH\int_0^t d\tau\gamma(\tau)\cos 2\bar{\omega}(t-\tau) = 0. \quad (8)$$

Differentiating this with time t, we have

$$\gamma(t) = -\frac{T}{g\rho H}\cos(2\bar{\omega}t + \phi) + 2\bar{\omega}\int_0^t \gamma(\tau)\sin 2\bar{\omega}(t-\tau)d\tau, \quad (9)$$

which shows that

$$|\gamma|_{t\to 0} = -\frac{T}{g\rho H}\cos\phi \qquad \text{at the coast,} \quad (10)$$

and the surface slope at the coast takes this value as soon as the wind begins to blow.

Eq. (8) or (9) is a Volterra integral equation, but to solve it we rather prefer to reduce it to a differential equation as below.

Differentiate (9) again with respect to t, then we get

$$gH\left[\frac{d\gamma}{dt} - 4\bar{\omega}^2\int_0^t d\tau\gamma\cos 2\bar{\omega}(t-\tau)\right] = \frac{T}{2\rho\bar{\omega}}4\bar{\omega}^2\sin(2\bar{\omega}t + \phi),$$

which, combined with (8), becomes

$$\frac{d\gamma}{dt} = \frac{T}{g\rho H}2\bar{\omega}\sin\phi.$$

Integrating this with respect to t and determining the integration constant by (10), we have

$$\gamma = \frac{T}{g\rho H}(2\bar{\omega}t\sin\phi - \cos\phi) \qquad \text{at the coast.} \quad (11)$$

Surface slope in the sea:—To obtain the surface slope off the coast, equation of continuity (2) must be used. From (7) we have

$$\frac{\partial\gamma}{\partial t}=\frac{d^2T}{dy^2}\frac{1}{2\rho\bar{\omega}}\left[\sin(2\bar{\omega}t+\phi)-\sin\phi\right]$$
$$+gH\int_0^t d\tau\frac{\partial^2\gamma(y,\tau)}{\partial y^2}\cos 2\bar{\omega}(t-\tau), \tag{12}$$

on supposition that the wind depends only upon the distance from the sea coast. We here notice that

$$\partial\gamma/\partial t=0 \qquad \text{when} \quad t=0. \tag{12a}$$

Now differentiate eq. (12) with respect to t, then

$$\frac{\partial^2\gamma}{\partial t^2}=\frac{1}{\rho}\frac{d^2T}{dy^2}\cos(2\bar{\omega}t+\phi)+gH\left[\frac{\partial^2\gamma(y,t)}{\partial y^2}\right.$$
$$\left.-2\bar{\omega}\int_0^t d\tau\frac{\partial^2\gamma(y,\tau)}{\partial y^2}\sin 2\bar{\omega}(t-\tau)\right], \tag{13}$$

which shows also that, when $t=0$,

$$\left(\frac{\partial^2\gamma}{\partial t^2}\right)_{t=0}=\frac{d^2T}{dy^2}\frac{\cos\phi}{\rho}+gH\left(\frac{\partial^2\gamma}{\partial y^2}\right)_{t=0}. \tag{13a}$$

Differentiating (13) once more, we obtain

$$\frac{\partial^3\gamma}{\partial t^3}=-\frac{2\bar{\omega}}{\rho}\frac{d^2T}{dy^2}\sin(2\bar{\omega}t+\phi)$$
$$+gH\left[\frac{\partial^3\gamma}{\partial y^2\partial t}-4\omega^2\int_0^t d\tau\frac{\partial^2\gamma(y,\tau)}{\partial y^2}\cos 2\bar{\omega}(t-\tau)\right]$$

which, by substitution of (12), becomes

$$\frac{\partial^3\gamma}{\partial t^3}=gH\frac{\partial^3\gamma}{\partial t\partial y^2}-4\bar{\omega}^2\frac{\partial\gamma}{\partial t}-\frac{2\bar{\omega}\sin\phi}{\rho}\frac{d^2T}{dy^2}. \tag{14}$$

This is evidently a wave equation and shows that the slope γ will propagate off from the coast.

Now since any wind stress T may be considered as a sum of (a) part which vanishes at the coast and (b) part that is uniform or a linear function of y, we shall deal separately with eq. (14) in two cases corresponding to such wind as (a) and (b).

Commence first with the case (a), then the slope γ also vanishes at the coasts, and we may expand T, d^2T/dy^2, γ and $\partial^2\gamma/\partial y^2$ in Fourier's sine series such as:

$$\left.\begin{aligned} T(y)&=\sum_m T_m \sin\frac{m\pi}{L}y, \\ \frac{d^2T(y)}{dy^2}&=\sum T''_m \sin\frac{m\pi}{L}y, \quad T''_m=-\left(\frac{m\pi}{L}\right)^2 T_m, \\ \gamma(y,t)&=\sum \gamma_m(t)\sin\frac{m\pi}{L}y, \\ \frac{d^2\gamma(y,t)}{\partial y^2}&=\sum \gamma''_m(t)\sin\frac{m\pi}{L}y, \quad \gamma''_m=-\left(\frac{m\pi}{L}\right)^2\gamma_m. \end{aligned}\right\} \quad (15)$$

For a given wind T, the coefficients T_m and T''_m are all known, and it remains only to determine the corresponding γ_m.

For that purpose, we may reduce eq. (12) into Volterra integral equation, as described later[1], by substituting (15) in (12) and integrating with time from 0 to t; but that method is tedious, and it is far more convenient to reduce (14) into ordinary differential equations and solve them. Namely, substitute (15) in (14) and equate the terms in both sides for each m, then

$$\frac{d^3\gamma_m}{dt^3}=-\left[gH\left(\frac{m\pi}{L}\right)^2+4\bar{\omega}^2\right]\frac{d\gamma_m}{dt}+\frac{2\bar{\omega}}{\rho}\left(\frac{m\pi}{L}\right)^2 T_m \sin\phi, \qquad (16)$$

whose general solution is obviously

$$\gamma_m=M\frac{T_m}{g\rho H}2\bar{\omega}t\sin\phi+A\sin\sigma_m t+B\cos\sigma_m t+C,$$

where

$$\left.\begin{aligned} M&=\left\{1+\frac{4\bar{\omega}^2}{gH\left(\frac{m\pi}{L}\right)^2}\right\}^{-1}, \\ \sigma_m&=\sqrt{gH\left(\frac{m\pi}{L}\right)^2+4\bar{\omega}^2}, \end{aligned}\right\} \qquad (17)$$

and A, B, C are arbitrary constants.

To determine these constants, we use three initial relations (3), (12_a), and (13_a):

$$0=\left|\gamma_m\right|_{t=0}, \qquad \therefore \quad C=-B.$$

1. p. 261, § 4.

$$0=\left|\frac{\partial\gamma_m}{\partial t}\right|_{t=0}=M\frac{T_m}{g\rho H}2\bar{\omega}\sin\phi+A\sigma_m,$$

$$\therefore\quad A=-\frac{T_m}{g\rho H}\frac{M}{\sigma_m}2\bar{\omega}\sin\phi.$$

$$0=\left|\frac{\partial^2\gamma_m}{\partial t^2}\right|_{t=0}=-B\sigma_m^2=-\left(\frac{m\pi}{L}\right)^2T_m\frac{\cos\phi}{\rho},$$

$$\therefore\quad B=\frac{T_m}{\rho\sigma_m^2}\left(\frac{m\pi}{L}\right)^2\cos\phi=\frac{T_m}{g\rho H}M\cos\phi.$$

Hence the solution of γ_m becomes

$$\gamma_m=\frac{T_m}{g\rho H}M\left[2\bar{\omega}t\sin\phi-\cos\phi+\cos\phi\cos\sigma_m t-\frac{2\bar{\omega}}{\sigma_m}\sin\phi\sin\sigma_m t\right]. \tag{18}$$

Thus, finally the required slope γ will be given by

$$\gamma=\sum_{m=1}^{\infty}M\frac{T_m}{g\rho H}\sin\frac{m\pi}{L}y\left[2\bar{\omega}t\sin\phi-\cos\phi+\cos\phi\cos\sigma_m t-\frac{2\bar{\omega}}{\sigma_m}\sin\phi\sin\sigma_m t\right], \tag{19}$$

and the surface elevation by

$$\zeta=\sum_{m=1}^{\infty}M\frac{T_m}{g\rho H}\frac{L}{m\pi}\cos\frac{m\pi}{L}y\left[2\bar{\omega}t\sin\phi-\cos\phi+\cos\phi\cos\sigma_m t-\frac{2\bar{\omega}}{\sigma_m}\sin\phi\sin\sigma_m t\right]. \tag{20}$$

From the above equations, we observe that:

1) When $\phi\neq 0$, i. e., the wind has a component parallel to the coast, the slope and the elevation will, by virtue of the first terms, increase with time so greatly that the vertical motion has to be taken into account and the equation of motion (1) must be altered.

2) If $\phi=0$ and the wind blows perpendicularly to the coast, the elementary slope γ_m makes an undamped oscillation with period $2\pi/\sigma_m$, i. e., $2\pi/\sqrt{gH(m\pi/L)^2+4\bar{\omega}^2}$, which is the seiche period of the channel.

§ 3. Uniform wind all over the sea with no bottom-friction

When the wind is uniform all over the sea or varies linearly with y only, the method of treating the problem must be modified.

In this case, since $d^2T/dy^2=0$, eq. (14) becomes

$$\frac{\partial^2\gamma}{\partial t^2}=gH\frac{\partial^2\gamma}{\partial y^2}-4\bar{\omega}^2\gamma, \tag{14'}$$

accompanied by coast condition (11) and initial relations (3) and (12_a). Moreover, the coefficients γ_m and γ''_m in the Fourier series for γ and $\frac{\partial^2\gamma}{\partial y^2}$ are not in the same relation as expressed in eq. (15). For, if we put

$$\left.\begin{aligned} T&=\sum_m T_m\sin\frac{m\pi}{L}y,\\ \gamma&=\sum_m \gamma_m\sin\frac{m\pi}{L}y, \qquad \gamma_m=\frac{2}{L}\int_0^L\gamma\sin\frac{m\pi}{L}y\,dy,\\ \frac{\partial^2\gamma}{\partial y^2}&=\sum_m\gamma''_m\sin\frac{m\pi}{L}y, \qquad \gamma''_m=\frac{2}{L}\int_0^L\frac{\partial^2\gamma}{\partial y^2}\sin\frac{m\pi}{L}y\,dy, \end{aligned}\right\} \tag{15'}$$

then, after Stokes, we have

$$\begin{aligned}\frac{L}{2}\gamma''_m&=\int_0^L\frac{\partial^2\gamma}{\partial y^2}\sin\frac{m\pi}{L}y\,dy\\ &=-\left[\frac{m\pi}{L}\gamma\cos\frac{m\pi}{L}y\right]_0^L-\left(\frac{m\pi}{L}\right)^2\int_0^L\gamma\sin\frac{m\pi}{L}y\,dy\end{aligned}$$

by integration by parts twice. Thus, generally there holds

$$\gamma''_m=-\frac{2m\pi}{L^2}\left[\gamma\cos\frac{m\pi}{L}y\right]_{y=0}^{y=L}-\left(\frac{m\pi}{L}\right)^2\gamma_m. \tag{21}$$

Substituting the coast values of γ in (11), the above relation becomes

$$\gamma''_m=\frac{2(T_0\pm T_L)}{g\rho H}(2\bar{\omega}t\sin\phi-\cos\phi)\frac{m\pi}{L^2}-\left(\frac{m\pi}{L}\right)^2\gamma_m, \tag{21'}$$

where T_0 and T_L denote the values of T at $x=0$ and $x=L$ respectively, and the double sign corresponds to odd or even values of m.

Substitute (15′) and (21′) in eq. (14′), and equate the coefficients of the m-th term in both sides, then

$$\frac{d^2\gamma_m}{dt^2}+\left[gH\left(\frac{m\pi}{L}\right)^2+4\omega^2\right]\gamma_m=\frac{2(T_0\pm T_l)}{\rho}(2\bar{\omega}t\sin\phi-\cos\phi)\frac{m\pi}{L^2}.$$

The solution of this, subjected to the initial conditions (3) and (12_a), is obviously

$$\gamma_m=\frac{(T_0\pm T_L)}{g\rho H}M\frac{2}{m\pi}(2\bar{\omega}t\sin\phi-\cos\phi+\cos\phi\cos\sigma_m t$$

$$-\frac{2\bar{\omega}}{\sigma_m}\sin\phi\sin\sigma_m t), \qquad (18')$$

and accordingly

$$\gamma=\sum_m\frac{T_0\pm T_L}{g\rho H}M\frac{2}{m\pi}\sin\frac{m\pi}{L}y(2\bar{\omega}t\sin\phi-\cos\phi+\cos\phi\cos\sigma_m t$$

$$-\frac{2\bar{\omega}}{\sigma_m}\sin\phi\sin\sigma_m t), \quad (19')$$

where M and σ_m are the same as in (17).

We notice here that the previous formula (19) includes (19′), because, by the well-known formula for Fourier expansion of a constant and a linear quantity, we shall have for a uniform wind ($T_0=T_L=T$)

$$T_m=T\frac{4}{m\pi}, \quad m=\text{odd only},$$

and for a linear wind $\quad T=T_0+\frac{T_L-T_0}{L}y,$

$$T_m=2(T_0+T_L)/m\pi \qquad \text{for odd } m,$$
$$T_m=2(T_0-T_L)/m\pi \qquad \text{for even } m.$$

Propagation of surface slope:— In order to get a clear idea of propagation of the surface slope from the coast, let us first consider, for the sake of simplicity, an *unrotating sea of semi-infinite extent* over which a uniform wind is blowing, so that $\bar{\omega}=0$, $L=\infty$ and $T_0=T_L=T$.

Then, putting

$$m=2n+1 \text{ (odd integer)}, \qquad \frac{m\pi}{L}=\frac{(2n+1)\pi}{L}=s,$$

$$2\pi/L=ds,$$

we can write (19′) as:

$$\gamma=-\frac{T\cos\phi}{g\rho H}+\frac{T\cos\phi}{g\rho H}\frac{2}{\pi}\int_0^\infty\frac{\sin sy\cos sct}{s}ds,$$

where $\quad c=\sqrt{gH}$.

But since

$$\int_0^\infty\frac{\sin sy\cos sct}{s}ds=\frac{1}{2}\int_0^\infty\frac{\sin s(y+ct)}{s}ds+\frac{1}{2}\int_0^\infty\frac{\sin s(y-ct)}{s}ds$$

$$=0 \qquad \text{for } y<ct$$
$$=\pi/2 \qquad \text{for } y>ct,$$

the above equation becomes

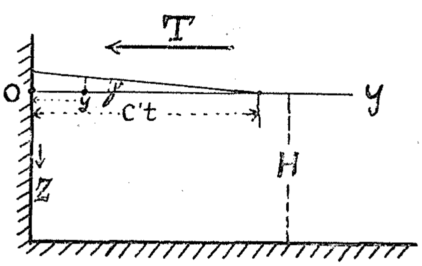
$$\left.\begin{aligned} \gamma &= \frac{-T}{g\rho H}\cos\phi && \text{for} \quad y<ct \\ &= 0 && \text{for} \quad y>ct. \end{aligned}\right\} \qquad (22)$$

The surface elevation ζ is obtained from (22) or directly from (20)

$$\left.\begin{aligned} \zeta &= -\frac{T\cos\phi}{g\rho H}(ct-y) && \text{for} \quad y<ct \\ &= 0 && \text{for} \quad y>ct. \end{aligned}\right\} \qquad (23)$$

Thus, we see that the surface-slope γ takes its steady value $-\frac{T}{g\rho H}\cos\phi$ instantly at the coast and propagates off with a constant velocity $c=\sqrt{gH}$, and that the surface elevation at a point increases linearly with time after the slope-wave has reached there (Fig. 2).

Fig. 2



Next, for a *rotating sea* also, a similar propagation of the slope will occur, but differing in its final value. For, in an actual sea, $4\bar{\omega}^2$ is always very small compared with $gH\left(\frac{m\pi}{L}\right)^2$, as exemplified by Proudman[1]. Hence we may put approximately $M \doteqdot 1$, and $\sigma_m \doteqdot \frac{m\pi}{L}\sqrt{gH}$, and we shall have as the first approximation

$$\left.\begin{aligned} \gamma &\doteqdot (2\bar{\omega}t\sin\phi - \cos\phi)\frac{T}{g\rho H} + \frac{T\cos\phi}{g\rho H}\,\frac{2}{\pi}\int_0^\infty \frac{\sin sy\cos sct}{s}\,ds \\ &\doteqdot (2\bar{\omega}t\sin\phi - \cos\phi)\frac{T}{g\rho H} && \text{for} \quad y<ct, \\ &\doteqdot 2\bar{\omega}t\sin\phi\,\frac{T}{g\rho H} && \text{for} \quad y>ct. \end{aligned}\right\} \qquad (24)$$

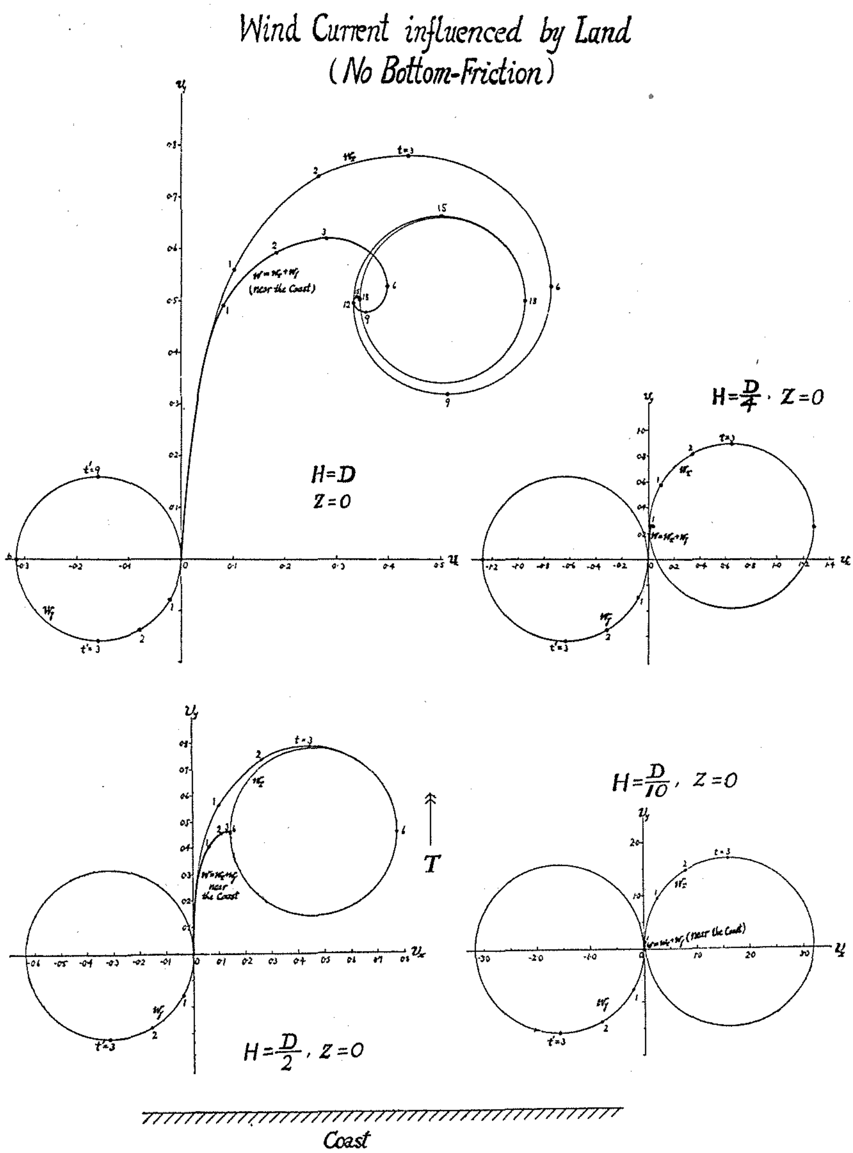
Thus if a component wind parallel to the coast exists, the final value of the slope will become so large that the equation of motion (1) must be altered.

Current influenced by the coast:— Having determined the slope

1. M. N. R. A. S. Geophys. Supplement **2**, 204 (1929)

γ, we can calculate the secondary slope current, and adding it to the primary pure drift current already known in our previous paper[1] we obtain the composite current affected by the coast.

Fig. 3



1. These Memoirs, A, **16**, 275 (1933)

When the wind blows perpendicularly to the coast, for instance, a constant slope $\frac{T}{g\rho H}$ will be produced after $t=y/c$ and hence the hodograph of the secondary slope current will be a circle of radius

$$\frac{1}{2}\frac{g\gamma}{\bar{\omega}}=\frac{1}{2}\frac{T}{\bar{\omega}\rho H}=\frac{T}{\mu k}\cdot\frac{1}{2\pi}\left(\frac{D}{H}\right). \qquad (25)$$

In Fig. 3 the circles in the left-hand quadrants indicate the hodographs of the secondary slope current, the numbers representing the time after the slope-wave has reached, $t'=t-y/c$.

The vector sum of this and the pure drift current in the first quadrant gives the resultant current required. The composite surface currents in the neighbourhood of the coast thus obtained are shown on the figures with thick lines. The current at any distance y can be obtained in a similar way graphically. We here notice that the quasi-steady values attained in a few hours in Fig. 3 are different a little from the final steady currents in our previous paper. This shows that the limit of $2\bar{\omega}t\sin\phi$ for $\phi\rightarrow 0$ and $t\rightarrow\infty$ is not zero but takes a finite value.

§ 4. Deduction of (19) by integral equation

As stated before, eq. (19) can be obtained from (12) by the method of integral equation also.

Substitution of (15) into (12) gives

$$\frac{d\gamma_m}{dt}=-\left(\frac{m\pi}{L}\right)^2\frac{T_m}{2\rho\bar{\omega}}[\sin(2\bar{\omega}t+\phi)-\sin\phi]$$
$$-gH\left(\frac{m\pi}{L}\right)^2\int_0^t\gamma_m(\tau)\cos 2\bar{\omega}(t-\tau)d\tau. \qquad (26)$$

Integrating with time from 0 to t, and putting $|\gamma_m|_{t=0}=0$, we have

$$\gamma_m=-\left(\frac{m\pi}{L}\right)\frac{T_m}{4\rho\bar{\omega}^2}[\cos\phi-\cos(2\bar{\omega}t+\phi)-2\bar{\omega}t\sin\phi]$$
$$-gH\left(\frac{m\pi}{L}\right)^2\int_0^t dt\int_0^t\gamma_m(\tau)\cos 2\bar{\omega}(t-\tau)d\tau.$$

But
$$\int_0^t dt\int_0^t\gamma_m(\tau)\cos 2\bar{\omega}(t-\tau)d\tau=\int_0^t d\xi\int_0^\xi\gamma_m(\tau)\cos 2\bar{\omega}(\xi-\tau)d\tau$$
$$=\int_0^t d\tau\,\gamma_m(\tau)\int_\tau^t\cos 2\bar{\omega}(\xi-\tau)d\xi.$$

$$\therefore\quad \gamma_m = -\left(\frac{m\pi}{L}\right)^2 \frac{T_m}{4\rho\bar{\omega}^2}[\cos\phi - \cos(2\bar{\omega}t+\phi) - 2\bar{\omega}t\sin\phi]$$
$$-\left(\frac{m\pi}{L}\right)^2 gH\int_0^t d\tau\, \gamma_m(\tau)\int_\tau^t \cos 2\bar{\omega}(\xi-\tau)d\xi,$$

which is a Volterra equation of the second kind of the form

$$\gamma_m = \psi_m(t) + \int_0^t K_m(t,\tau)\,\gamma_m d\tau, \tag{27}$$

where

$$\left.\begin{aligned} \psi_m(t) &= -a\frac{T_m}{g\rho H}[\cos\phi - \cos(2\bar{\omega}t+\phi) - 2\bar{\omega}t\sin\phi], \\ K_m(t,\tau) &= -gH\left(\frac{m\pi}{L}\right)^2\int_\tau^t \cos 2\bar{\omega}(\xi-\tau)d\xi \\ &= -gH\left(\frac{m\pi}{L}\right)^2 \frac{\sin 2\bar{\omega}(t-\tau)}{2\bar{\omega}} = -a\,2\bar{\omega}\sin 2\bar{\omega}(t-\tau), \\ a &\equiv \frac{gH}{4\bar{\omega}^2}\left(\frac{m\pi}{L}\right)^2. \end{aligned}\right\} \tag{28}$$

Since the kernel of the present integral equation is a sine function, the kernel of the solution will be

$$S_m(t,\tau) = \frac{2\bar{\omega}a}{\sqrt{1+a}}\sin\sqrt{1+a}\,2\bar{\omega}(t-\tau),$$

and the solution is

$$\gamma_m(t) = \psi_m(t) - \int_0^t S_m(t,\tau)\psi_m(\tau)d\tau.$$

By actual integration and with notice that $\sqrt{1+a} = 2\bar{\omega}\sigma_m$, we have

$$-\int_0^t S_m(t,\tau)\psi_m(\tau)d\tau = \frac{T_m}{g\rho H}\left[\frac{a^2}{1+a}\cos\phi - a\cos(2\bar{\omega}t+\phi)\right.$$
$$\left. -\frac{a^2}{1+a}2\bar{\omega}t\sin\phi + \frac{a}{1+a}\cos\phi\cos\sigma_m t - \frac{a}{\sqrt{(1+a)^3}}\sin\phi\sin\sigma_m t\right]$$

and therefore

$$\gamma_m = -\frac{T_m}{g\rho H}\frac{a}{1+a}\left[\cos\phi - 2\bar{\omega}t\sin\phi - \cos\phi\cos\sigma_m t\right.$$
$$\left. + \frac{1}{\sqrt{1+a}}\sin\phi\sin\sigma_m t\right], \tag{29}$$

which can be reduced to eq. (19) exactly.

§ 5. Case of no bottom-current

If we assume

$$w=0 \qquad \text{at} \quad z=H, \tag{5$_b$}$$

then, using my formulae for drift and slope-current with the same condition, we shall have

$$\left.\begin{aligned} w &= \frac{2iTe^{-i\phi}}{\rho H}\sum_{n=0}^{\infty}\cos\beta_n z\int_0^t dt\, e^{-(\nu\beta_n^2+i2\bar{\omega})t} \\ &\quad + \frac{i2g}{H}\sum_{n=0}^{\infty}\frac{(-1)^n}{\beta_n}\cos\beta_n z\int_0^t d\tau\,\gamma(y,\tau)e^{-(\nu\beta_n^2+i2\bar{\omega})(t-\tau)}, \\ \beta_n &= (n+\frac{1}{2})\frac{\pi}{H}, \qquad n=0,\ 1,\ 2,\ 3,\dots\dots, \end{aligned}\right\} \tag{6$_b$}$$

and

$$\left.\begin{aligned} {}_TS_y &= \frac{2T}{\rho H}\sum_{n=0}^{\infty}\frac{(-1)^n}{\beta_n}\int_0^t e^{-\nu\beta_n^2 t}\cos(2\bar{\omega}t+\phi)dt = T.{}_TS_y'(\text{put}), \\ {}_\gamma S_y &= \frac{2g}{H}\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\int_0^t \gamma(\tau)e^{-\nu\beta_n^2(t-\tau)}\cos 2\bar{\omega}(t-\tau)d\tau, \end{aligned}\right\} \tag{7$_b$}$$

instead of (6) and (7).

Hence eq. of continuity (2) will become

$$\frac{\partial\gamma}{\partial t} = \frac{d^2T}{dy^2}\frac{2}{\rho H}\sum_{n=0}^{\infty}\frac{(-1)^n}{\beta_n}\int_0^t e^{-\nu\beta_n^2 t}\cos(2\bar{\omega}t+\phi)dt$$
$$+\frac{2g}{H}\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\int_0^t \frac{\partial^2\gamma(y,\tau)}{\partial y^2}e^{-\nu\beta_n^2(t-\tau)}\cos 2\bar{\omega}(t-\tau)d\tau \tag{12$_b$}$$

instead of (12).

By the reason stated in p. 254, consider first the *case where T vanishes at the coast* (Case of Horrocks and Proudman). Then T and γ can be expanded in sine series as in (15), so that (12$_b$) will give

$$\frac{\partial\gamma_m}{\partial t} = -T_m\left(\frac{m\pi}{L}\right)^2\frac{2}{\rho H}\sum_{n=0}^{\infty}\frac{(-1)^n}{\beta_n}\int_0^t e^{-\nu\beta_n^2 t}\cos(2\bar{\omega}t+\phi)dt$$
$$-\frac{2g}{H}\left(\frac{m\pi}{L}\right)^2\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\int_0^t \gamma_m(\tau)e^{-\nu\beta_n^2(t-\tau)}\cos 2\bar{\omega}(t-\tau)d\tau. \tag{26$_b$}$$

This may be solved by the method of integral equation similar to that in § 4. Namely, integrating (26_b) with time from o to t and putting $|\gamma_m|_{t=0}=0$, we get an integral equation

$$\gamma_m=\psi_m(t)+\int_0^t K_m(t,\tau)\,\gamma_m\,d\tau, \qquad (27_b)$$

where

$$\left.\begin{aligned}\psi_m(t)&=-T_m\frac{2}{\rho H}\left(\frac{m\pi}{L}\right)^2\sum_{n=0}^{\infty}\frac{(-1)^n}{\beta_n}\int_0^t dt\int_0^t e^{-\nu\beta_n^2 t}\cos(2\bar{\omega}t+\phi)dt\\&=-T_m\left(\frac{m\pi}{L}\right)^2{}_T S'_y,\\K_m(t,\tau)&=-\frac{2g}{H}\left(\frac{m\pi}{L}\right)^2\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\int_\tau^t e^{-\nu\beta_n^2(\xi-\tau)}\cos 2\bar{\omega}(\xi-\tau)d\xi.\end{aligned}\right\}\qquad(28_b)$$

The solution of this integral equation is

$$\left.\begin{aligned}&\gamma_m(t)=\psi_m(t)-\int_0^t S_m(t,\tau)\psi_m(\tau)\,d\tau,\\&\text{where}\\&S_m(t,\tau)=\sum_{s=1}^{\infty}K_m^{(s)}(t,\tau),\\&K_m^{(1)}(t,\tau)=K_m(t,\tau),\\&K_m^{(s)}(t,\tau)=\int_\tau^t K_m(t,\tau)\,K_m^{(s-1)}(t,\tau)\,d\tau.\end{aligned}\right\}\qquad(29_b)$$

This solution is perfect formally, but it is nothing but the result of successive approximation in which $\psi_m(t)$ is taken as the first approximate value, and it is very inconvenient for practical calculations. The method analogous to the latter part of § 2 will give a far more convenient solution.

Practical formula convenient for numerical calculations:— Start again from (12_b) and notice in it that the term with $n=0$ is far more predominant compared with any other terms. For, even when $t=0$ and $\gamma\doteqdot 0$, the successive terms are in ratio

$$1:\frac{2}{3\times5}:\frac{2}{7\times9}:\frac{2}{11\times13}:\ldots\ldots,\qquad 1:\frac{1}{3^2}:\frac{1}{5^2}:\frac{1}{7^2}:\ldots\ldots,$$

and when t gets a moderate value, the factor $e^{-\nu\beta_n^2 t}$ is vanishingly small for any finite n in comparison with for $n=0$. The writer's experience has proved it actually in the occasion of his numerical calculations for the rising stage of drift current. Moreover this character is conspicuous especially in a sea of continental shelf which is comparatively shallow and for which the problem of the coast effect will be most important.

Thus, with sufficient approximation for practical purpose we may take only the terms with $n=0$, and eq. (12_b) can be replaced by

$$\left.\begin{aligned}\frac{\partial\gamma}{\partial t} &\doteq \frac{d^2T}{dy^2}E\left\{1-\frac{\cos(2\bar{\omega}t+\phi+\varphi_0)}{\cos(\phi+\varphi_0)}e^{-\nu\beta_0^2t}\right\}\\ &\quad+\frac{2g}{H\beta_0^2}\int_0^t\frac{\partial^2\gamma(y,\tau)}{\partial y^2}e^{-\nu\beta_0^2(t-\tau)}\cos 2\bar{\omega}(t-\tau)d\tau,\\ \text{where}\qquad&\\ \beta_0=\frac{\pi}{2H},\quad E&=\frac{2\cos(\phi+\varphi_0)}{\rho H\beta_0\sqrt{(\nu\beta_0^2)^2+4\bar{\omega}^2}},\quad \varphi_0=\text{arc tan}\frac{2\bar{\omega}}{\nu\beta_0^2}.\end{aligned}\right\}\quad(12_b')$$

(i) *Rotating sea* (*case of Horrocks*). To solve $(12_b')$, put

$$\gamma=e^{-\nu\beta_0^2t}X(y,t), \tag{30}$$

then we get

$$\frac{\partial X}{\partial t}-\nu\beta_0^2X=\frac{d^2T}{dy^2}E\left\{e^{\nu\beta_0^2t}-\frac{\cos(2\bar{\omega}t+\phi+\varphi_0)}{\cos(\phi+\varphi_0)}\right\}$$
$$+\frac{8gH}{\pi^2}\int_0^t\frac{\partial^2X(y,\tau)}{\partial y^2}\cos 2\bar{\omega}(t-\tau)d\tau. \qquad (12_b'')$$

Differentiating $(12_b'')$ twice with respect to t and combining the result with the original equation, we obtain

$$\frac{\partial^3X}{\partial t^3}-\nu\beta_0^2\frac{\partial^2X}{\partial t^2}+4\bar{\omega}^2\frac{\partial X}{\partial t}-4\bar{\omega}^2\nu\beta_0^2X-\frac{8gH}{\pi^2}\frac{\partial^3X}{\partial t\partial y^2}$$
$$=\frac{d^2T}{dy^2}E\{(\nu\beta_0^2)^2+4\bar{\omega}^2\}e^{\nu\beta_0^2t}, \qquad (14_b)$$

accompanying with the initial relations

$$\left.\begin{aligned}&|X|_{t=0}=0,\qquad |\partial X/\partial t|_{t=0}=0,\\ &\left|\frac{\partial^2X}{\partial t^2}\right|_{t=0}=\frac{d^2T}{dy^2}E\sqrt{(\nu\beta_0^2)^2+4\bar{\omega}^2}\frac{\cos\phi}{\cos(\phi+\varphi_0)}.\end{aligned}\right\}\quad(13_b)$$

Now if we expand T and γ in Fourier sine series

$$\left.\begin{aligned}T&=\sum T_m\sin\frac{m\pi}{L}y, &\gamma&=\sum\gamma_m\sin\frac{m\pi}{L}y,\\ X&=\sum X_m\sin\frac{m\pi}{L}y, &X_m&=\gamma_m e^{\nu\beta_0^2t},\end{aligned}\right\}$$

then (14_b) requires

$$\frac{d^3X_m}{dt^3}-\nu\beta_0^2\frac{d^2X_m}{dt^2}+\left(4\omega^2+\frac{8gHm^2}{L^2}\right)\frac{dX_m}{dt}-4\bar{\omega}^2\nu\beta_0^2X_m$$
$$=-T_mE\left(\frac{m\pi}{L}\right)^2\{(\nu\beta_0^2)^2+4\bar{\omega}^2\}e^{\nu\beta_0^2t} \qquad (16_b)$$

whose solution is obviously

$$X_m=-T_m\frac{E\pi^2}{8gH}\{(\nu\beta_0^2)^2+4\bar{\omega}^2\}e^{\nu\beta_0^2t}+e^{\frac{1}{3}\nu\beta_0^2t}[B_1e^{\sigma'_mt}+B_2e^{\sigma''_mt}+B_3e^{\sigma'''_mt}].$$
$$(18_b)$$

Indices σ'_m σ''_m σ'''_m are the roots of Cardin's cubic equation

$$\left.\begin{aligned}&\sigma_m^3+3p\sigma_m+2q=0,\\ \text{where}\quad &3p=\frac{8gHm^2}{L^2}+4\bar{\omega}^2-\tfrac{1}{3}(\nu\beta_0^2)^2,\\ &2q=\frac{1}{3}\nu\beta_0^2\left\{\frac{8m^2}{L^2}gH-8\bar{\omega}^2-\frac{2}{9}(\nu\beta_0^2)^2\right\}.\end{aligned}\right\} \qquad (17_b)$$

For actual sea, both p and q are usually positive so that the roots will be one real and two complex; i. e., if we put

$$\sqrt[3]{-q+\sqrt{q^2+p^3}}\equiv r, \qquad \sqrt[3]{-q-\sqrt{q^2+p^3}}\equiv s,$$

the roots will be given by

$$\sigma'_m=r+s, \qquad \sigma''_m=-\frac{r+s}{2}+\frac{r-s}{2}\sqrt{-3},$$

$$\sigma'''_m=-\frac{r+s}{2}-\frac{r-s}{2}\sqrt{-3}.$$

Further if q^2 is very small compared with p^3 as in Horrocks' example, the above expressions reduce to

$$\sigma'_m=0, \qquad \sigma''_m=i\sqrt{3p}, \qquad \sigma'''_m=-i\sqrt{3p}. \qquad (17'_b)$$

Three integration constants B_1, B_2, B_3 can be determined by three initial relations (13_b).

Thus, X_m being determined completely, the elementary slope γ_m will be given by

$$\gamma_m=\bar{\gamma}_m+e^{-\frac{2}{3}\nu\beta_0^2t}\{B_1e^{\sigma'_mt}+B_2e^{\sigma''_mt}+B_3e^{\sigma'''_mt}\}, \qquad (18'_b)$$

where $\bar{\gamma}_m$ means the final steady value of γ_m, which is known already in our prevous paper, I. Steady State.

This approximate solution is generally quite enough for practical use. If we want, however, to have more accurate solution, we proceed to construct successive approximations.

We write $\gamma_m^{(1)}$ for the above first approximate value and substitute it in the integral sign of eq. (27), then the second approximation will be

$$\gamma_m^{(2)}=\phi_m(t)+\int_0^t K_m(t,\tau)\gamma_m^{(1)}(\tau)\,d\tau.$$

Similarly the sth approximation is

$$\gamma_m^{(s)}=\phi_m(t)+\int_0^t K_m(t,\tau)\gamma_m^{(s-1)}(\tau)d\tau.$$

This expression tends to (29_b) for $s\to\infty$, but converges far more rapidly than (29_b) and the second approximation is almost perfect in any case.

Let us here recalculate the period of Horrocks' sea of $L=40$ km., $H=100$ m, $\lambda=55°$ and $\nu=80$. As the longest period of the sea (for $m=1$), Horrocks[1] found 0.19 day=4.6 hours, while Proudman[2] stated that such a figure must be the result of error, and gave himself 0.71 hour.

According to our present formulae,

$$p=2.021\times10^{-6},\quad q=1.897\times10^{-12},$$

and q^2 is negligible compared with p^3. Hence we have

$$\text{period}\doteqdot 2\pi/\sqrt{3p}=2552\text{ sec.}=0.709\text{ hour},$$

which coincides with Proudman's result.

(ii) *Non-rotating sea (case of Proudman).* In this case ($12_b'$) becomes

$$\frac{\partial\gamma}{\partial t}\doteqdot\frac{d^2T}{dy^2}\frac{2\cos\phi}{\mu H\beta_0^3}(1-e^{-\nu\beta_0^2 t})+\frac{2g}{H\beta_0^2}\int_0^t\frac{\partial^2\gamma(y,\tau)}{\partial y^2}e^{-\nu\beta_0^2(t-\tau)}d\tau.$$

Differentiating this with time, we obtain

$$\frac{\partial^2\gamma}{\partial t^2}+\nu\beta_0^2\frac{\partial\gamma}{\partial t}-\frac{2g}{H\beta_0^2}\frac{\partial^2\gamma}{\partial y^2}=\frac{2\cos\phi}{\rho H\beta_0}\frac{d^2T}{dy^2},\qquad(14_c)$$

which indicates a damped wave motion.

---

1. loc. cit.
2. M. N. R. A. S. Geophys. Suppl. **2**, 205 (1929)

Now if we expand T and γ in Fourier sine series, then the elementary slope of mth order must satisfy

$$\frac{d^2\gamma_m}{dt^2}+\nu\beta_0^2\frac{d\gamma_m}{dt}+\frac{2g}{H\beta_0^2}\left(\frac{m\pi}{L}\right)^2\gamma_m=-\frac{2}{\rho H\beta_0}\left(\frac{m\pi}{L}\right)^2T_m, \quad (16_c)$$

the solution of which is

$$\gamma_m=-\frac{\pi}{2}\frac{T_m}{g\rho H}+e^{-\frac{1}{2}\nu\beta_0^2t}\{C_1e^{\sigma_mt}+C_2e^{-\sigma_mt}\}, \quad (18_c)$$

where

$$\sigma_m=\sqrt{-\frac{2g}{H\beta_0^2}\left(\frac{m\pi}{L}\right)^2+\left(\frac{\nu\beta_0^2}{2}\right)^2}=\frac{\nu\beta_0^2}{2}\sqrt{1-\frac{0.533}{a}}. \quad (17_c)$$

This σ_m is real or imaginary according as

$$a\left(\begin{matrix}\text{in Proudman's}\\ \text{notation}\end{matrix}\right)\equiv\frac{\nu^2}{g(m\pi/L)^2H^5}\gtrless\frac{2^9}{\pi^6}(=0.533),$$

which is almost coincident with Proudman's value 0.537.

Two integration constants C_1 and C_2 can be determined by the initial conditions $|\gamma|_{t=0}=0$ and $\left|\frac{\partial\gamma}{\partial t}\right|_{t=0}=0$, and eq. (18_c) finally takes form

$$\gamma_m(t)=\bar{\gamma}_m\left[1-e^{-\frac{1}{2}\nu\beta_0^2t}\{(1+1/\sqrt{1-0.533/a})\ \sinh\sigma_mt+e^{-\sigma_mt}\}\right], \quad (18_c')$$

where $\bar{\gamma}_m$ denotes the steady value of γ_m. The present approximation gives $\bar{\gamma}_m=\frac{\pi}{2}\frac{T_m}{g\rho H}$, but more accurately we should use $\bar{\gamma}_m=\frac{3}{2}\frac{T_m}{g\rho H}$ as seen in our previous paper on the steady states of coast effect.

When $a<0.533$ and $\sigma_m=i\sigma_m'$, the above equation shall be written

$$\left.\begin{aligned}\gamma_m(t)&=\bar{\gamma}_m\left[1-e^{-\frac{1}{2}\nu\beta_0^2t}\left\{\cos\sigma_m't+\frac{\sin\sigma_m't}{\sqrt{\frac{0.533}{a}-1}}\right\}\right]\\ &=\bar{\gamma}_m\left[1-\sqrt{\frac{0.533}{0.533-a}}e^{-\frac{1}{2}\nu\beta_0^2t}\cos(\sigma_m't-\varepsilon)\right],\\ &\text{where}\\ \sigma_m'&=\frac{\nu\beta_0^2}{2}\sqrt{\frac{0.533}{a}-1},\quad \tan\varepsilon=\sqrt{\frac{a}{0.533-a}}.\end{aligned}\right\} \quad (18_c'')$$

Thus our solution is very simple, while it keeps a quite sufficient accuracy in practical use. For instance, let us recalculate the examples in Proudman's paper.[1]

For a given α, the calculation of speed $2\xi\eta$ of Proudman is very tedious, but its approximate value will be given by our simple expression $\frac{\pi^2}{2^3}\sqrt{1-\frac{0.533}{\alpha}}$. Table 1 is a comparison of the two methods.

Table 1

α	0.473	0.338	0.213	0.128	0.0615	0.0244	0.00936	0.00268
Proudman's $2\xi\eta$	0.452	0.947	1.52	2.20	3.44	5.67	9.34	17.8
Ours $\frac{\pi^2}{2^3}\sqrt{1-\frac{0.533}{\alpha}}$	0.438	0.938	1.51	2.19	3.42	5.64	9.24	17.0

Moreover, for a sea of $\alpha=0.0615$, our formula will give (taking $\frac{T_m}{g\rho H}=1$)

$$\gamma_{m=1}=1.5-1.595e^{-1.235\frac{\nu}{H^2}t}\cos\left(3.42\frac{\nu}{H^2}t-19.^\circ 9\right)$$

instead of Proudman's result

$$\gamma_{m=1}=1.5-1.613\cos(3.435\nu t/H^2-22^\circ.8)e^{-1.272\frac{\nu}{H^2}t}$$

$$-0.014e^{-22.14\frac{\nu}{H^2}t}+\ldots\ldots$$

These two expressions are represented in Fig. 4. Table 2 and Fig. 4 contain two other examples, one for an aperiodic case ($\alpha=1.969$) and the other for the limiting case of $\alpha=0.533$.

All the above comparisons show that our first approximate formula is entirely reliable in practice, while its numerical calculation is exceedingly easier than Proudman's.

To make our solution perfect from the theoretical standpoint also, the successive approximation method may be used just as stated for rotating sea.

1. loc. cit. Proc. London Math. Soc. Ser. 2, **24**, 147-148 (1924)

Fig. 4

Development of the elementary slope $\gamma_{m=1}$. Time unit is $0.864H^2/\nu$.

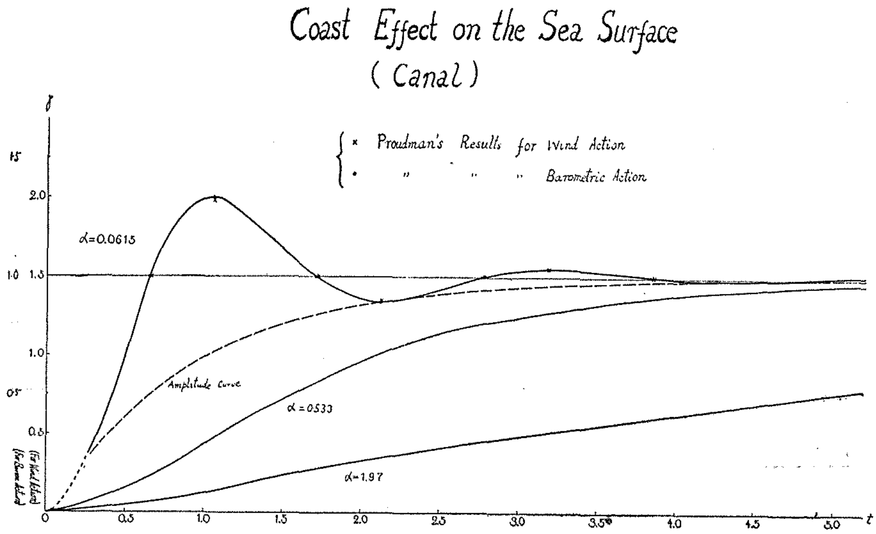


Table 2.

Development of γ_1, time unit being taken as $0.864H^2/\nu$ after Proudman

$\alpha=0.0615$ ∴ period$=2.126$		$\alpha=0.533$		$\alpha=1.967$	
t	γ_1	t	γ_1	t	γ_1
0	0	0	0	0	0
0.648	1.500	0.5	0.149	0.5	0.041
1.060	2.013	1.0	0.434	1.0	0.119
1.712	1.500	1.5	0.712	1.5	0.239
2.126	1.333	2.0	0.961	2.0	0.324
2.775	1.500	3.0	1.244	3.0	0.487
3.189	1.553	4.0	1.390	4.0	0.638
3.838	1.500	5.0	1.454	5.0	0.761
4.252	1.483	10.0	1.500	10.0	1.165

§ 6. Uniform wind over a sea with no bottom-current

Even when $\frac{d^2T}{dy^2}=0$, the foregoing solution (18) or (18′) holds good; but the proof must be altered, because, in Fourier expansion of γ and $\frac{\partial^2\gamma}{\partial y^2}$, the coefficient γ''_m is not equal to $-\left(\frac{m\pi}{L}\right)^2\gamma_m$.

Surface slope at the coast. At the coast itself the total flow S_y must be zero, i. e.,

$$_TS_y+{}_\gamma S_y={}_TS_y+\frac{2g}{H}\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\int_0^t\gamma(\tau)e^{-\nu\beta_n^2(t-\tau)}\cos2\bar{\omega}(t-\tau)d\tau=0, \quad (8_b)$$

where $_TS_y$ is a known function of t by eq. (7_b).

As long as t is very small, we can write

$$\left.\begin{aligned} {}_TS_y &\doteqdot \frac{2T}{\rho H}\sum_{n=0}^{\infty}\frac{(-1)^n}{\beta_n}\cos\phi\cdot t=\frac{T\cos\phi}{\rho}t \\ \text{and}\qquad {}_\gamma S_y &\doteqdot \frac{2g}{H}\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\gamma(t)t=gH\gamma(t)t, \end{aligned}\right\}$$

since $$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{\pi}{4} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

Thus, if $\gamma(0)$ denotes the value of γ at $t\to0$ or immediately after the wind begins to blow, then condition (8_b) gives

$$\gamma(0)=\frac{T\cos\phi}{g\rho H}, \quad (10_b)$$

which is first found by Hidaka and coincides with that for no bottom-friction.

The solution of (8_b) at any other time can be formally obtained by the usual method of Volterra's integral equation of the first kind, but the result will be too complicated and useless for practical computation.

Hence let us here make the first approximate solution, taking only terms of $n=0$ in (8_b).

Put $\gamma(t)=e^{-\nu\beta_0^2t}X(t)$,

then we have from (8_b) approximately

$$TE\left\{e^{\nu\beta_0^2t}-\frac{\cos(2\bar{\omega}t+\phi+\varphi_0)}{\cos(\phi+\varphi_0)}\right\} + \frac{2g}{H\beta_0^2}\int_0^t\frac{\partial^2X(\tau)}{\partial y^2}\cos2\bar{\omega}(t-\tau)d\tau=0,$$

where E and φ_0 are constants given in $(12_b')$.

This equation, combined with its second derivative with respect to t, gives

$$\frac{2g}{H\beta_0^2}\frac{dX}{dt}+TE\{(\nu\beta_0^2)^2+4\bar{\omega}^2\}e^{\nu\beta_0^2 t}=0,$$

whose solution is

$$X(t)=-\frac{HTE}{2g\nu}\{(\nu\beta_0^2)^2+4\bar{\omega}^2\}e^{\nu\beta_0^2 t}+C.$$

This corresponds to

$$\gamma(t)=-\frac{TEH}{2g\nu}\{(\nu\beta_0^2)^2+4\bar{\omega}^2\}+Ce^{-\nu\beta_0^2 t}$$

$$\fallingdotseq\bar{\gamma}+\{\gamma(0)-\bar{\gamma}\}e^{-\nu\beta_0^2 t} \qquad (11_b)$$

where $\bar{\gamma}$ and $\gamma(0)$ denote the final and initial values of γ.

Surface slope in the sea:— In the sea off the coast, the equation of continuity in the case of uniform wind will take the form

$$\frac{\partial\gamma}{\partial t}\fallingdotseq\frac{2g}{H\beta_0^2}\int_0^t\frac{\partial^2\gamma(y,\tau)}{\partial y^2}e^{-\nu\beta_0^2(t-\tau)}\cos 2\bar{\omega}(t-\tau)d\tau,$$

instead of $(12_b')$.

Subsequently, instead of (14_b) or (14_c) we shall have

$$\left.\begin{aligned}\frac{\partial^3X}{\partial t^3}-\nu\beta_0^2\frac{\partial^2X}{\partial t^2}+4\bar{\omega}\frac{\partial X}{\partial t}-4\bar{\omega}^2\nu\beta_0^2X\fallingdotseq\frac{8gH}{\pi^2}\frac{\partial^3X}{\partial t\,\partial y^2},\\ X(y,t)=\gamma\,e^{\nu\beta_0^2 t},\end{aligned}\right\} \qquad (14_b')$$

or if the sea is a narrow canal, more simply

$$\frac{\partial^2\gamma}{\partial t^2}+\nu\beta_0^2\frac{\partial\gamma}{\partial t}\fallingdotseq\frac{8gH}{\pi^2}\frac{\partial^2\gamma}{\partial y^2}. \qquad (14_c')$$

Expanding T, γ and $\frac{\partial^2\gamma}{\partial y^2}$ in Fourier sine series as in $(15')$, and using the relation (21) and the coast value of γ $(11_b'')$, we have

for even m, $\gamma_m''=-(m\pi/L)^2\gamma_m$,

for odd m, $\gamma_m''=-(m\pi/L)^2\gamma_m-\frac{4}{L}\left(\frac{m\pi}{L}\right)\left[\bar{\gamma}+\{\gamma(0)-\bar{\gamma}\}e^{-\nu\beta_0^2 t}\right].$

Substitute this γ''_m into $(14'_b)$ and $(14'_c)$, then there will result equations identical to (14_b) and (14_c).

Thus we see that (18_b) and $(18''_b)$ are true even when the wind is uniform all over the sea. Similary we can prove that the same solution is also applicable for a linearly varying wind.

Propagation of slope :— For an unrotating sea of semi-infinite extent ($L=\infty$), from eq. $(18''_c)$ we have

$$\gamma=\sum\gamma_m \sin\frac{m\pi}{L}y$$

$$=\bar{\gamma}\left[1-e^{-\frac{1}{2}\nu\beta_0^2 t}\int_0^\infty \frac{\cos\{t\sqrt{c'^2s^2-(\nu\beta_0^2)^2}\}+\sin\{t\sqrt{c'^2s^2-(\nu\beta_0^2)^2}\}}{\sqrt{c'^2s^2-(\nu\beta_0^2)^2}}\sin sy\, ds\right],$$

where we put $m\pi/L=s$, and $c'=\frac{\sqrt{8}}{\pi}\sqrt{gH}$.

Evaluating the above integral we get

$$\left.\begin{aligned}\gamma&=\bar{\gamma}\left\{1-e^{-\frac{1}{2}\nu\beta_0^2 t}\frac{1}{2c'}I_0\left(\nu\beta_0^2\sqrt{t^2-\frac{y^2}{c'^2}}\right)\right\} && y>c't\\ &=\bar{\gamma} && y<c't,\end{aligned}\right\}$$

where I_0 is a cylindrical function usually indicated with such notation.

This solution is in a similar form as (24), and it is natural because eq. $(14'_c)$, by substitution $\gamma=Xe^{-\frac{1}{2}\nu\beta_0^2 t}$, becomes

$$\frac{\partial^2 X}{\partial t^2}-\frac{1}{2}\nu\beta_0^2 X=c'^2\frac{\partial^2 X}{\partial y^2},$$

which is in the same form as $(14')$.

The above solution shows that the slope propagates with a velocity c' as the first approximation, and accurately it will be very near to $\sqrt{gH}$. Thus we may consider practically that the slope (11_b) propagates off the coast with a velocity $\sqrt{gH}$. Then, the elevation of surface ζ may be calculated approximately by

$$\left.\begin{aligned}\zeta&=\left[\bar{\gamma}-\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t}\right](ct-y) && \text{for} \quad y<ct\\ &=0 && \text{for} \quad y>ct\end{aligned}\right\} \qquad (23_b)$$

§ 7. Case of finite bottom-friction

Similarly as before, we can solve the coast effect when the bottom-resistance is proportional to the slip-velocity, but we will explain it together with the barometric and density current in the next chapter.

II. Barometric and Convection Current

§ 8. Surface slope

The coast effect upon the barometric or the convection current may be dealt with in a way entirely similar as that used for the wind current.

Whatever the causal force may be, either the wind traction T, the barometric gradient γ_0, or the density gradient α, denote it generally by F. Then the primary current w_F due to the motive force can be represented by

$$w_F = Fe^{-i\phi}\sum_{n=0}^{\infty} A_n \cos\beta_n z \{1 - e^{-(\nu\beta_n^2 + i2\bar{\omega})t}\}$$

where $\beta_n = \left(\frac{1}{2} + n\right)\frac{\pi}{H}$ for no bottom-current,

$= n\pi/H$ for no bottom-friction,

$\beta_n \tan\beta_n H = \frac{f'\rho}{\mu}$ for finite bottom-friction,

and A_n is a complex constant given in our previous papers.[1]

The total flow due to w_F perpendicular to the coast is

$${}_FS_y = F\sum_{n=0}^{\infty}\sqrt{A_{nx}^2 + A_{ny}^2}\,\frac{\sin\beta_n H}{\beta_n}\left\{\cos(\phi+\varphi_n) - e^{-\nu\beta_n^2 t}\cos(2\bar{\omega}t + \phi + \varphi_n)\right\},$$

where $A_{nx} + iA_{ny} = A_n$,

$\tan\varphi_n = A_{nx}/A_{ny}$.

The secondary slope-current produced by the influence of the coast will be

$$w_\Upsilon = \sum_{n=0}^{\infty} B_n \cos\beta_n z \int_0^t \gamma(\tau) e^{-(\nu\beta_n^2 + i2\bar{\omega})(t-\tau)} \cos\{2\bar{\omega}(t-\tau) + \phi + \varphi_n'\}\,d\tau,$$

where B_n is a new complex constant such that

$B_n = B_{nx} + iB_{ny}$, and $\tan\varphi_n' = B_{nx}/B_{ny}$.

1. These Memoirs, A, **16**, 161, 203, 261, 275, 309, 333, 383, 397 (1933)

We must here notice that the secondary current and its total flow do not differ in form whatever the primary cause may be, and that the constant A_n only differs according to the kind of primary current.

Thus all the foregoing discussions for the wind current may be followed exactly for barometric and convection current also.

Surface slope at the coast:— By the condition $S_y=0$ at the coast, we shall have

$$\gamma(0)=|\gamma|_{t\to 0}=-T\cos\phi/g\rho H \quad \text{for wind current,}$$
$$=-\gamma_0\cos\phi \quad \text{for barometric current,}$$
$$=-\frac{\nu H}{2g}\alpha\cos\phi \quad \text{for convection current,}$$

whatever bottom-condition may be taken.

After a time t,

$$\gamma=\gamma(0)(1+2\bar{\omega}t\tan\phi) \quad \text{when the bottom is free,}$$

while $$\gamma=\bar{\gamma}-\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t} \quad \text{when the bottom is not free,}$$

$\bar{\gamma}$ being the final steady value of γ given in our first report on the present problem of the coast effect.

Surface slope in the sea:— From the equation of continuity (2), we get (18), (18′), (18_b), (18_c) and all other results in the foregoing chapter, the only necessary alteration being that instead of the letter T we must put $g\rho H\gamma_0$ for the barometric current and $\frac{\mu H^2}{2}\alpha$ for the convection current.

§ 9. Composite current

Having determined the slope γ, we can calculate the secondary current w_Υ and subsequently the composite current w_F+w_Υ.

For instance, Table 3 gives the surface current with no bottom-current corresponding to a varying slope $\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t}$, $\frac{g\{\bar{\gamma}-\gamma(0)\}}{\bar{\omega}}$ being taken unity. Tables 4 and 5 give the composite wind current with no bottom-current for $\phi=0$ and $z=0$, the former representing the current near the coast and the latter that at a very great distance from the coast. Figs. 5 and 6 are the hodographs obtained by the tables.

Table 3.

Surface slope current due to a slope $\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t}$, $g\frac{\bar{\gamma}-\gamma(0)}{\bar{\omega}}$ being taken unity.

	$H=D$		$H=D/2$		$H=D/4$	
t (p. h)	v_x	v_y	v_x	v_y	v_x	v_y
0	0	0	0	0	0	0
1	0.0667	0.250	0.065	0.245	0.030	0.112
2	0.240	0.431	0.189	0.327	0.039	0.068
3	0.486	0.486	0.291	0.291	0.028	0.027
6	0.860	0	0.265	0	0.002	0
9	0.353	−0.353	0.060	−0.061	0.000	0.000
12	0	0	0	0		
15	0.239	0.238	0.012	0.012		
18	0.392	0	0.011	0		
21	0.161	−0.161	0.003	−0.003		
24	0	0	0	0		
30	0.179	0	0.000	0		
36	0	0				
42	0.082	0				

	$H=D/10$	
t	v_x	v_y
0	0	0
$\frac{1}{4}$	0.0011	0.0150
$\frac{1}{2}$	0.0008	0.0062
1	0.0001	0.0004

Time is measured in pendulum hours.

Table 4.

Composite wind-current near a coast with no bottom-current ($\phi=0$, $z=0$).

	$H=D$		$H=D/2$		$H=D/4$	
t	v_x	v_y	v_x	v_y	v_x	v_y
0	0	0	0	0	0	0
1	0.078	0.483	0.057	0.404	0.030	0.258
2	0.187	0.594	0.118	0.462	0.036	0.237
3	0.283	0.625	0.170	0.467	0.037	0.212
6	0.440	0.529	0.210	0.381	0.022	0.195
9	0.418	0.426	0.167	0.352	0.021	0.195
12	0.355	0.480	0.142	0.370		
15	0.407	0.540	0.148	0.376		
18	0.470	0.510	0.150	0.372		
21	0.456	0.470	0.147	0.372		
24	0.433	0.490	0.147	0.373		
30	0.489	0.504				
36	0.468	0.494				
42	0.502	0.501				
∞	0.498	0.498	0.147	0.373	0.021	0.195

Table 5.

Composite wind-current at a great distance from the coast

	$H=D/2$		$H=D/4$	
t'	v_x	v_y	v_x	v_y
0	0.545	0.545	0.260	0.657
1	0.493	0.348	0.178	0.299
2	0.371	0.224	0.086	0.199
3	0.239	0.187	0.041	0.184
6	0.062	0.328	0.020	0.193
9	0.133	0.410	0.021	0.195
12	0.164	0.381		
15	0.151	0.374		
18	0.143	0.370		
21	0.146	0.374		
24	0.148	0.372		

Fig. 5

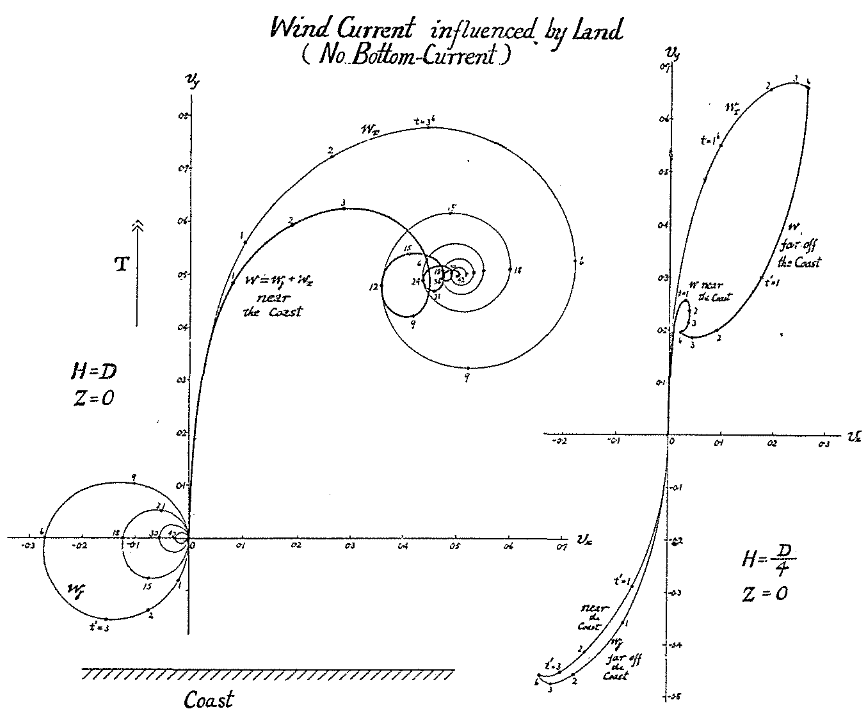
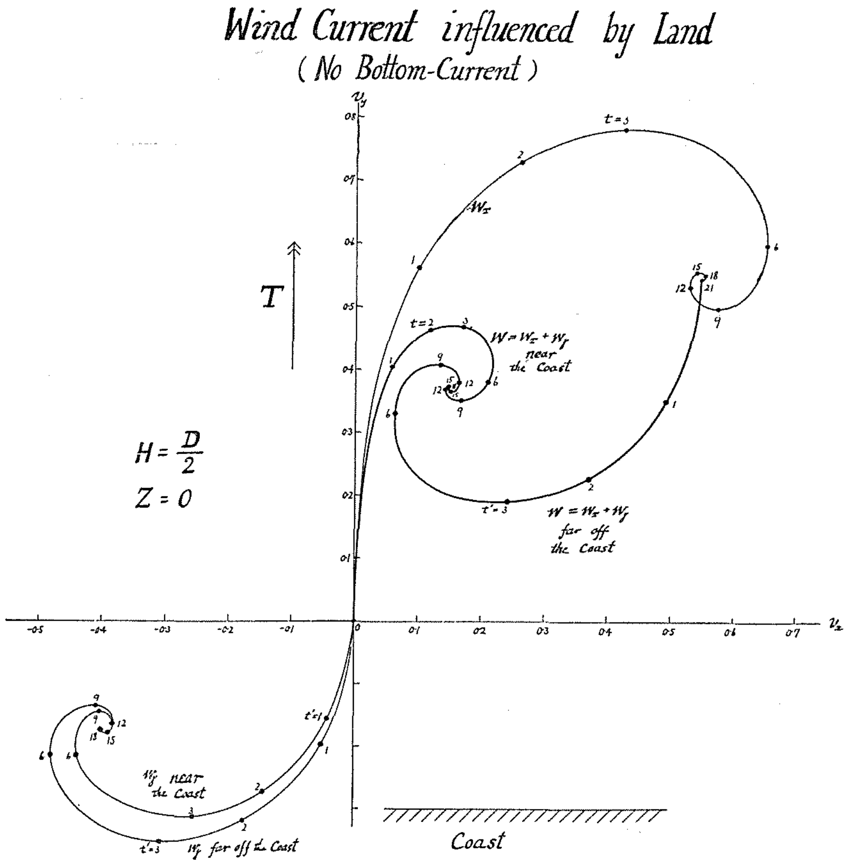


Fig. 6



At the coast, our current formula satisfies $S_y = 0$, but not $v_y = 0$ except when the primary cause itself vanishes there. This weakness, however, is inevitable as long as we neglect the vertical motion even in the immediate vicinity of the coast and use the equation of motion like (1). The reason is as follows.

As the equation of continuity, eq. (2) is necessary but not sufficient. Strictly speaking

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

is necessary and sufficient for incompressible liquid.

In the case where the primary cause is uniform all over the sea, we must have $\partial v_x / \partial x = 0$, so that

$$\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0, \quad \text{and} \quad \frac{\partial^2 v_y}{\partial y^2}=-\frac{\partial^2 v_z}{\partial y \partial z}.$$

Now, if $v_z=0$, then v_y must be independent of y. Thus, as long as we neglect vertical motion, we must get a horizontal velocity equal everywhere in the same level and not vanishing at the coast. The assumptions that $v_z=0$ and that $v_y=0$ at the coast are inconsistent except when the causal force itself vanishes there.

In reality, in the vicinity of the coast, v_z will not be negligible but will be rather more predominant than v_y, so that $\frac{\partial^2 v_y}{\partial y^2}=-\frac{\partial^2 v_z}{\partial y \partial z}$ is not neccessarily small compared with $\frac{\partial^2 v_y}{\partial z^2}$. Thus, of the original expression for viscosity-force $\nu\nabla^2 w$, we can not neglect the part $\nu\frac{\partial^2 w}{\partial y^2}$ compared with $\nu\frac{\partial^2 w}{\partial z^2}$, and that part must be restored in the equation of motion. With such equation of motion, it will be possible to make $v_y=0$ at the coast.

Summary

From the foregoing discussions we see that:

1) As soon as the primary causal force begins to act, there appears at the coast a finite slope $\gamma(0)$ which is equal to $-T\cos\phi/g\rho H$ for the wind current, $-\gamma_0\cos\phi$ for the barometric and $-\nu H\alpha\cos\phi/2g$ for the convection current, but with no distinction of bottom-conditions.

If $\gamma(0)$ differs from the steady slope $\bar{\gamma}$, the slope at the coast will change gradually from $\gamma(0)$ to $\bar{\gamma}$ in such a way as

$$\gamma=\bar{\gamma}-\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t},$$

where $\bar{\gamma}$ should be taken from our first paper on the coast effect.

2) The slope thus produced at the coast propagates in the sea off the coast with a velocity of approximately $c=\sqrt{gH}$ for any bottom-condition.

Consequently the surface elevation ζ at a given place can be calculated approximately by

$$\begin{aligned}\zeta&=(ct-y)\left[\bar{\gamma}-\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t}\right] && \text{for} \quad y<ct,\\ &=0 && \text{for} \quad y>ct.\end{aligned}$$

3) When the primary cause is nil at the coast, the surface slope and elevation will oscillate with a stationary wave type as in Horrocks' and Proudman's papers.

4) The current influenced by the coast may be considered to consist of the primary current and the secondary slope current. The former is already known in our previous papers on the current in a boundless sea, and the latter is nothing but the current due to the above obtained slope.

Here we notice that the secondary current has the same form, whatever the primary cause may be, the only difference being in its general coefficient.

5) As to the secondary current, the part due to the steady slope is already known in the first report[1], and the part to be calculated newly is only that due to the slope $\{\bar{\gamma}-\gamma(0)\}e^{-\nu\beta_0^2 t}$. The current corresponding to a simple slope $e^{-\nu\beta_0^2 t}$ is given in Table 3.

The writer wishes to express his thanks to the Hattori Hôkôkwai for the grant given for the present research.

1. These Memoirs, A, **17**, 93 (1934)