The Effect of the Length on the Frequencies of Sound emitted by a Circular Cylinder with a Hemispherical Cap, a Rough Model of Japanese Bells

By

Keiji Yamashita and **Ichiro Aoki**

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Abstract

As rough models of Japanese bells two shells of different thickness are made of the same material as that of the Japanese bells. Both of them are circular cylinders with hemispherical caps and their thicknesses are uniform. The lengths of the cylindrical parts of the shells are changed and the effect of the length upon the frequencies of the vibration of the shells are examined experimentally. The results are then compared with those calculated by the formula which has been obtained by one[1] of the writers on the assumption that no line traced upon the middle surface of the shell undergoes extension when the shell is vibrating.

We[2] have found experimentally that the frequencies of the partial tones of the Japanese bell are nearly in the ratios

$$2^2 : 3^2 : 4^2 : 5^2 : \dots\dots\dots\dots\dots\dots .$$

The shape of the Japanese bell may roughly be represented by a circular cylinder with a hemispherical cap. But its thickness is not uniform everywhere, being very thin at the portion where the cylindrical part changes to the spherical form and very thick at the open end of the cylindrical part as well as at the pole of the spherical part. The non-uniformity of the thickness certainly has some effects on the vibration of the bell, but retaining the problem for a while let us examine, in the present paper, how the vibration of the shell is affected

1. K. Yamashita, These Memoirs, **15**, 315–322 (1932).
2. K. Yamashita and I. Aoki, These Memoirs, **15**, 323–326 (1932)

by the length of the cylindrical part when the thickness is uniform throughout the shell. Although the change of the length of the cylindrical part greatly influences the duration and intensity of the vibration, we shall here examine the effect of the length upon the frequencies of the partial tones emitted by the shell.

To investigate this problem experimentally two models of bells, named A and B, of the shape described above were made of the alloy composed of copper and tin in the ratio Cu 10 : Sn 1.5, of which the Japanese bells are usually made. The mean wall thickness of the bell A was 0.6 cm and the mean diameter of the cylindrical part and the spherical part was 27.4 cms. The bell B was much thicker than the bell A, its thickness and diameter being 1.4 cms and 26.8 cms respectively. The lengths of the cylindrical parts of the two bells were planned to be equal to the respective diameters, but the real lengths were about 0.95 times the diameters, i. e. 1.9 times the radii.

Figs. 1, 2, 3 in Plate I and Figs. 1, 2, 3 in Plate III show the sound-waves from the two bells recorded by a Low-Hilger audiometer. Analyzing the marked portions of the waves by the method of periodogram-analysis, we have obtained the following as the frequencies of the partial tones emitted by the two bells.

$$\text{Bell } A \text{ (thin)} \quad 158,\ 425,\ 670,\ 1250,\ 1800,\ 2700; \qquad (1)$$

$$\text{Bell } B \text{ (thick)} \quad 377,\ 1000,\ 1900. \qquad (2)$$

Comparing these frequencies with the ratios

$$2^2 : 3^2 : 4^2 : 5^2 : \ldots\ldots\ldots\ldots\ldots\ldots$$

which represent approximately the ratios of the frequencies of the partial tones emitted by Japanese bells, we see that in the bells of uniform thickness the frequency increases more rapidly with the order of partial tones than in the ordinary Japanese bells.

The frequencies[1] of the vibration of a circular cylinder with a hemispherical cap have been calculated on the assumption that no line traced upon the middle surface of the shell undergoes extension when the shell is vibrating, the thickness being assumed to be uniform throughout the shell. By the result the frequencies $p_n/2\pi$ of the partial tones are given by

$$p_n^2 = \frac{2\mu h^2}{3\rho a^4} \cdot \frac{P}{Q} \quad [n=2,\ 3,\ 4,\ldots\ldots\ldots],$$

1. K. Yamashita, *loc. cit.*

where
$$\left.\begin{aligned} P &\equiv \frac{n^2(n^2-1)^2}{1-\sigma}\left\{(3-2\sigma)\frac{l}{a}+n\frac{l^2}{a^2}+\frac{n^2}{3}\frac{l^3}{a^3}\right\} \\ &\quad +n(n^2-1)(2n^2-1), \\ Q &\equiv (n^2+2)\frac{l}{a}+n(n^2+1)\frac{l^2}{a^2}+\frac{1}{3}n^2(n^2+1)\frac{l^3}{a^3} \\ &\quad +\int_1^2\{(n-1)^2+2(n+1)x-x^2\}\frac{(2-x)^n}{x^n}dx, \end{aligned}\right\} \quad (3)$$

a being the radius of the cylindrical part and the spherical part, l the length of the cylindrical part, ρ the volume density of the material, σ its Poisson's ratio, and μ the modulus of rigidity.

In order to get the numerical values of the frequencies of the partial tones we have obtained the values of the integral

$$f(n)\equiv\int_1^2\{(n-1)^2+2(n+1)x-x^2\}\frac{(2-x)^n}{x^n}dx$$

for $n=2$, 3, 4, 5, and 6:

$$f(2)=20\log 2-12\frac{1}{3}=1.52961,$$

$$f(3)=57\frac{1}{3}-80\log 2=1.88156,$$

$$f(4)=200\log 2-136\frac{1}{3}=2.29609,$$

$$f(5)=280-400\log 2=2.74116,$$

$$f(6)=700\log 2-482=3.20297,$$

of which $f(2)$, $f(3)$ and $f(4)$ were calculated by Lord Kayleigh[1] in the vibration of a hemispherical cap.

Let us now compare the frequencies calculated by (3) with the frequencies (1) and (2) which were obtained experimentally. If we assume $\sigma=1/3$ as the reasonable value and if we put $l/a=2$ in (3), we get

$$p_n^2=\frac{2\mu h^2}{3\rho a^4}\frac{P}{Q},$$

where
$$P\equiv n^2(n^2-1)^2\{7+6n+4n^2\}+n(n^2-1)(2n^2-1),$$
$$Q\equiv 2(n^2+2)+4n(n^2+1)+\frac{8}{3}n^2(n^2+1)+f(n)$$

and therefore

1. Lord Rayleigh, *Proc. London Math. Soc.* **13** (1881); Scientific Papers, vol. 1, p. 557.

$$p_2^2 = \frac{2\mu h^2}{3\rho a^4} \times 12.18,$$

$$p_3^2 = \frac{2\mu h^2}{3\rho a^4} \times 92.59,$$

$$p_4^2 = \frac{2\mu h^2}{3\rho a^4} \times 332.0,$$

$$p_5^2 = \frac{2\mu h^2}{3\rho a^4} \times 856.6,$$

$$p_6^2 = \frac{2\mu h^2}{3\rho a^4} \times 1828.$$

If the density and the modulus of rigidity be assumed to be

$$\rho = 8.5, \qquad \mu = 3.5 \times 10^{11},$$

and for the bell *A*,

$$2h = 0.6, \qquad 2a = 27,$$

and for the bell *B*,

$$2h = 1.4, \qquad 2a = 26.8,$$

then the frequencies of the partial tones ($\nu_n = p_n/2\pi$) are as follows:

Bell *A*	Bell *B*
$\nu_2 = 147.1$	$\nu_2 = 358.8$
$\nu_3 = 405.6$	$\nu_3 = 989.1$
$\nu_4 = 768.0$	$\nu_4 = 1873$
$\nu_5 = 1234$	$\nu_5 = 3009$
$\nu_6 = 1802$	$\nu_6 = 4396$

Dividing the experimental frequencies (1) and (2) by the corresponding values calculated above, we get

$158 \div 147.1 = 1.07,$	$377 \div 358.8 = 1.05,$
$425 \div 405.6 = 1.05,$	$1000 \div 989.1 = 1.01,$
$670 \div 768.0 = 0.87,$	$1900 \div 1873 = 1.01.$
$1250 \div 1234 = 1.01,$	
$1800 \div 1802 = 1.00;$	

With one exception $670 \div 768.0 = 0.87$, these quotients are all nearly equal to unity and so we see that the experimental frequencies coincide with the calculated values.

The length of the cylindrical parts of the two bells *A* and *B* were diminished in order to see the effect of the length on the frequencies of the partial tones, and the sounds emitted from the bells were recorded in three cases where $l/a = 1, 0.5, 0$ by the same method as in the case where $l/a=1.9$. The waves are arranged in Plates I, II, III, IV.. The diagrams preceding these plates show the intensities and the frequencies obtained by analyzing the waves.

The frequencies of the partial tones corresponding to the above experimental values are calculated by the formula (3) and are shown in Tables I and II. The experimental data are also tabulated for the convenience of comparison.

Table I.

Bell *A*. Thickness $2h=0.6$ cm. Diameter $2a=27.4$ cms.

$l/a=2$ (1.9)		$l/a=1$		$l/a=0.5$		$l/a=0$	
Cal.	Exp.	Cal.	Exp.	Cal.	Exp.	Cal.	Exp.
147.1	158	159.9	184	178.5	211	220.9	258
405.6	425	430.8	490	477.1	570	620.9	690
768.0	670	802.1	850	877.6	950	1200	1225
1234	1205	1275	1400	1374	1600	1952	1900
1802	1800	1848	......	1968	2200	2876	2600

Table II.

Bell *B*. Thickness $2h=1.4$ cms. Diameter $2a=26.8$ cms.

$l/a=2$ (1.9)		$l/a=1$		$l/a=0.5$		$l/a=0$	
Cal.	Exp.	Cal.	Exp.	Cal.	Exp.	Cal.	Exp.
358.8	377	389.9	413	435.3	451	538.7	512
989.1	1000	1051	1100	1164	1180	1514	1325
1873	1900	1956	2100	2140	2100	2926	2400
3009	......	3109	3000	3351	3000	4761	......
4396	......	4508	......	4801	......	7014	......

The equation $l/a=2$ (1.9), the ratio of the length l of the cylindrical part and the radius a, in Tables I and II means that $l/a=2$ for calculated values and $l/a=1.9$ for experimental values, and of course $l/a=0$ represents a hemisphere. Figs. 1 and 2 are the graphs of these frequencies. Each curve shows the change of frequency of a partial tone when the length of the cylindrical part is diminished. The broken lines are the straight line-segment connecting the exprimental values of the frequencies; and the continuous curves are the graphs obtained by the formula (3), the ratio l/a being taken as the abscissa and the frequency $p_n/2\pi$ as the ordinate.

Fig. 1

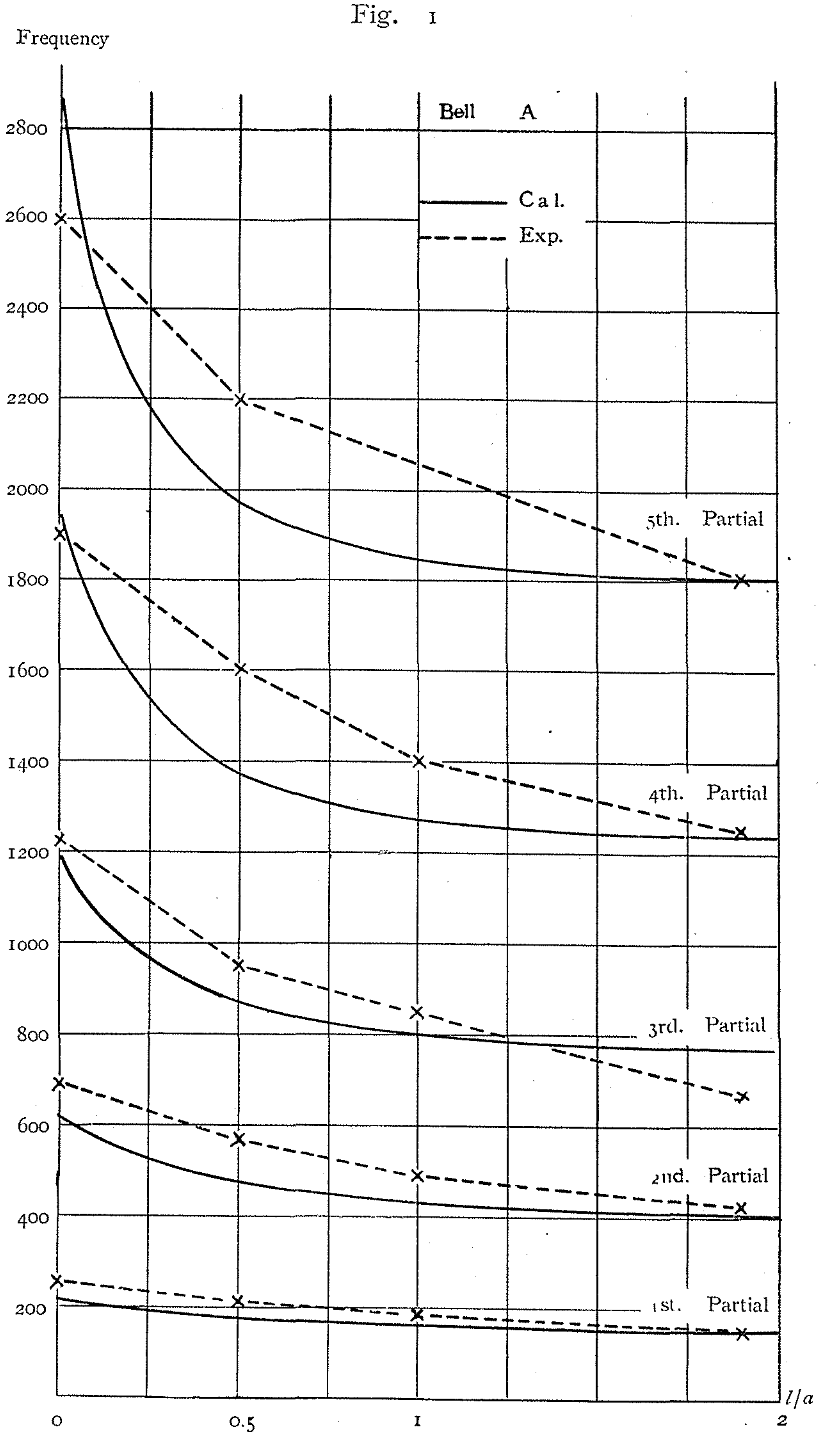
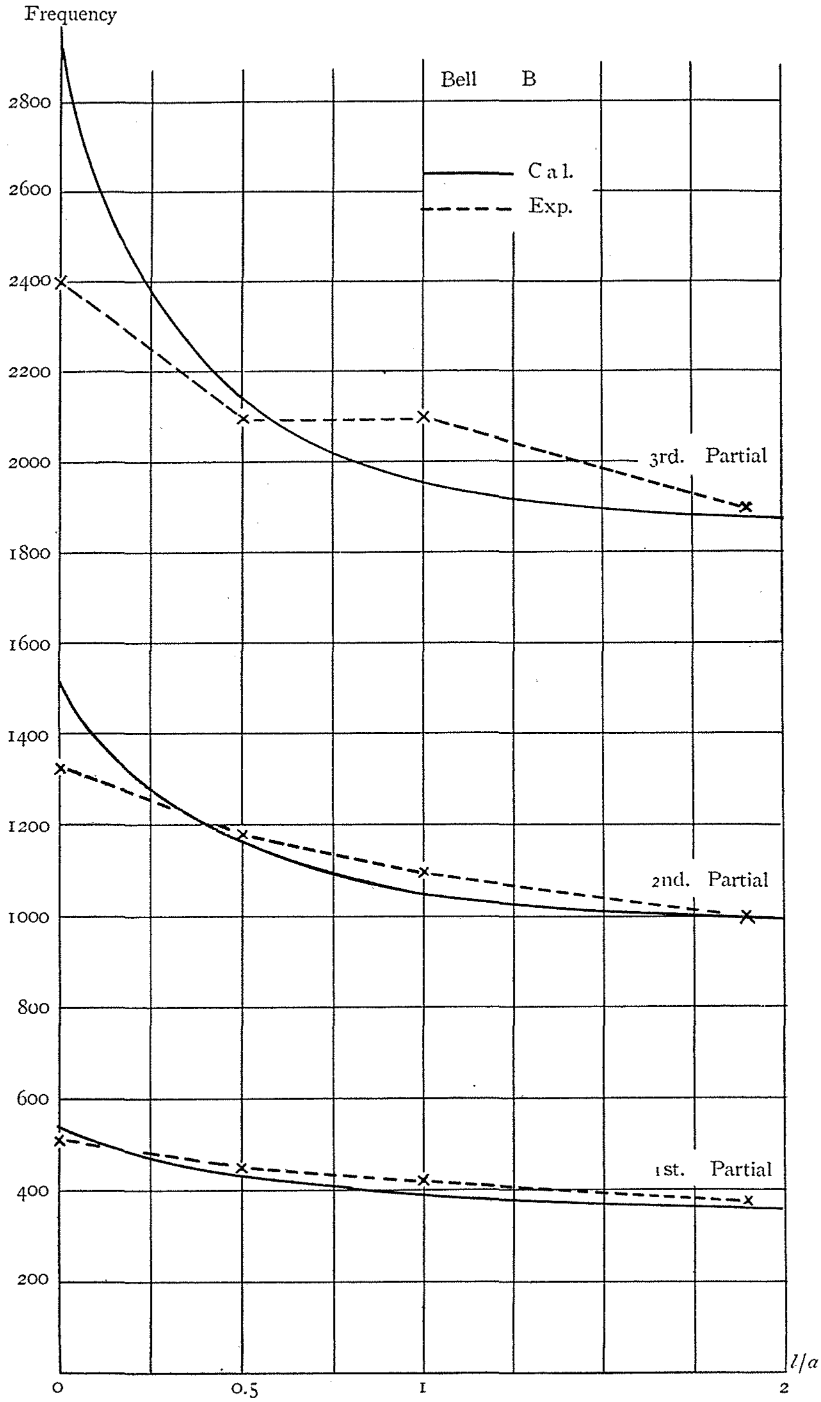

Frequency
2800
2600
2400
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
Bell A
Cal.
Exp.
5th. Partial
4th. Partial
3rd. Partial
2nd. Partial
1st. Partial
0
0.5
1
2
l/a

Fig. 2


Frequency
2800
2600
2400
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
Bell B
Cal.
Exp.
3rd. Partial
2nd. Partial
1st. Partial
l/a
0
0.5
1
2

From Tables, I, II or from Figs. 1, 2 it follows that so long as the length of the cylindrical part is comparable with the diameter, the calculated frequencies and the experimental values nearly coincide with each other, but that they become a little different as the length of the cylindrical part diminishes.

The writers wish to express their gratitude to Professor K. Tamaki for his suggestion and guidance.

Diagram I

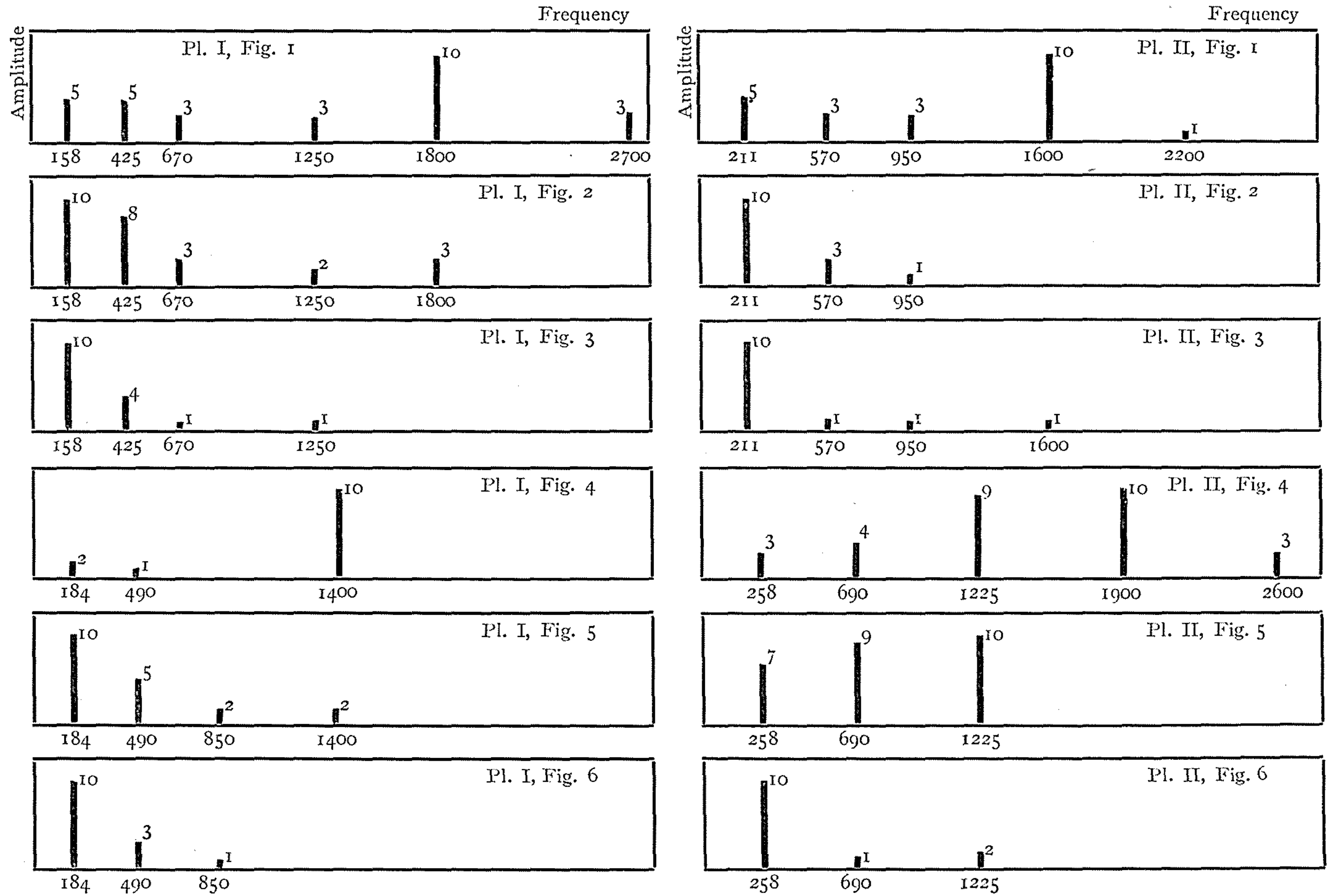

Frequency
Amplitude
Pl. I, Fig. 1
5 5 3 3 10 3
158 425 670 1250 1800 2700
Pl. I, Fig. 2
10 8 3 2 3
158 425 670 1250 1800
Pl. I, Fig. 3
10 4 1 1
158 425 670 1250
Pl. I, Fig. 4
2 1 10
184 490 1400
Pl. I, Fig. 5
10 5 2 2
184 490 850 1400
Pl. I, Fig. 6
10 3 1
184 490 850
Frequency
Amplitude
Pl. II, Fig. 1
5 3 3 10 1
211 570 950 1600 2200
Pl. II, Fig. 2
10 3 1
211 570 950
Pl. II, Fig. 3
10 1 1 1
211 570 950 1600
Pl. II, Fig. 4
3 4 9 10 3
258 690 1225 1900 2600
Pl. II, Fig. 5
7 9 10
258 690 1225
Pl. II, Fig. 6
10 1 2
258 690 1225

Diagram II

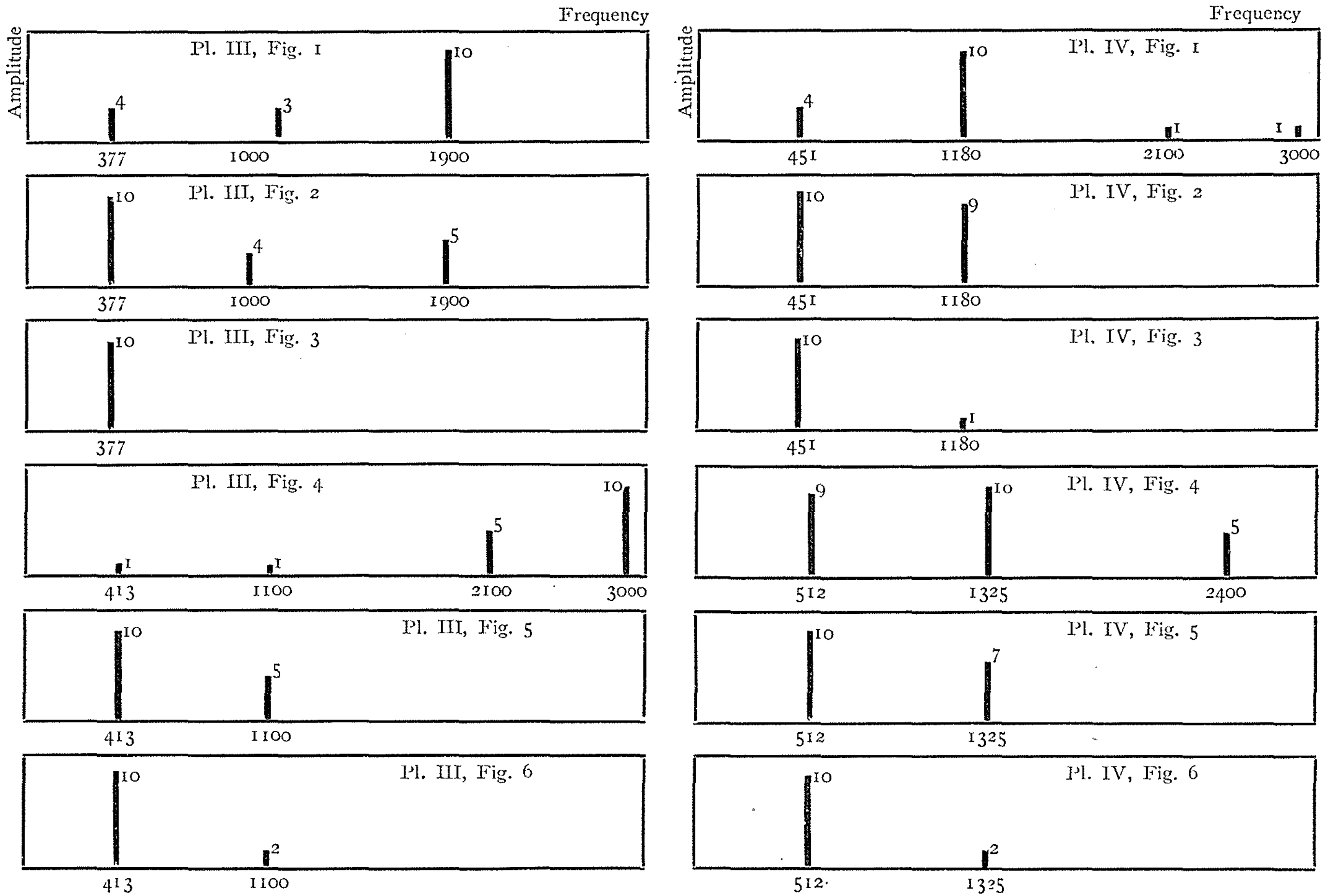

Frequency
Amplitude
Pl. III, Fig. 1
4 3 10
377 1000 1900
Pl. III, Fig. 2
10 4 5
377 1000 1900
Pl. III, Fig. 3
10
377
Pl. III, Fig. 4
1 1 5 10
413 1100 2100 3000
Pl. III, Fig. 5
10 5
413 1100
Pl. III, Fig. 6
10 2
413 1100
Frequency
Amplitude
Pl. IV, Fig. 1
4 10 1 1
451 1180 2100 3000
Pl. IV, Fig. 2
10 9
451 1180
Pl. IV, Fig. 3
10 1
451 1180
Pl. IV, Fig. 4
9 10 5
512 1325 2400
Pl. IV, Fig. 5
10 7
512 1325
Pl. IV, Fig. 6
10 2
512 1325

Plate I (Bell A)

Fig. 1 |←—1/50 sec. —→| l/a=1.9

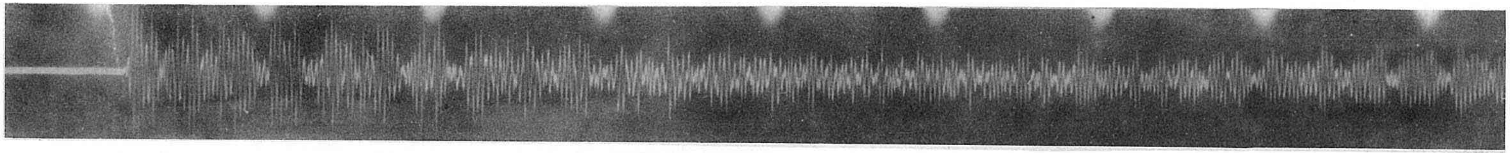


Fig. 2 2.5 secs. after it is struck. l/a=1.9

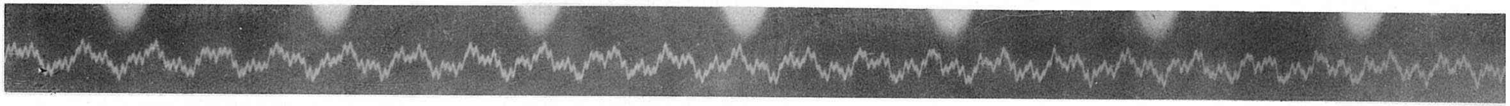


Fig. 3 5 secs. l/a=1.9

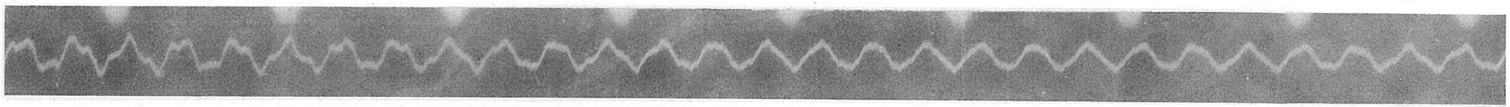


Fig. 4 l/a=1

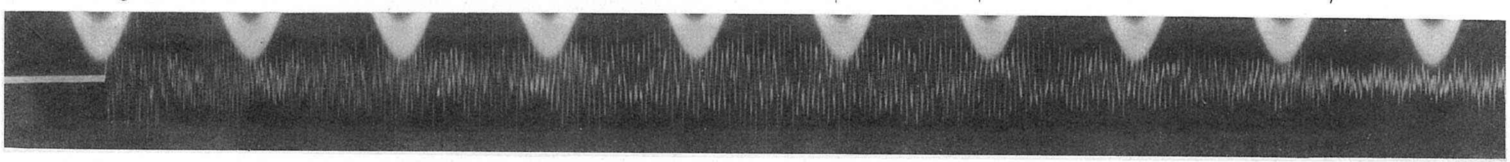


Fig. 5 1.5 secs. l/a=1

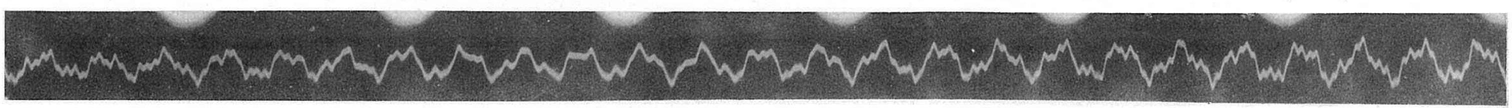


Fig. 6 2.5 secs. l/a=1

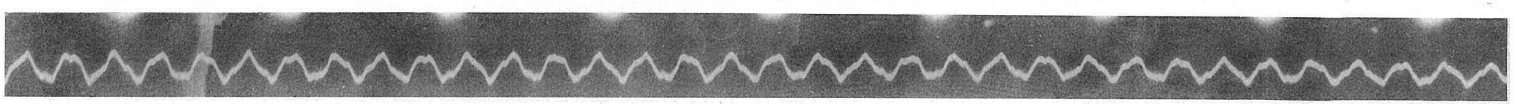


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Plate II (Bell A)

Fig. 1 — 1/a=0.5

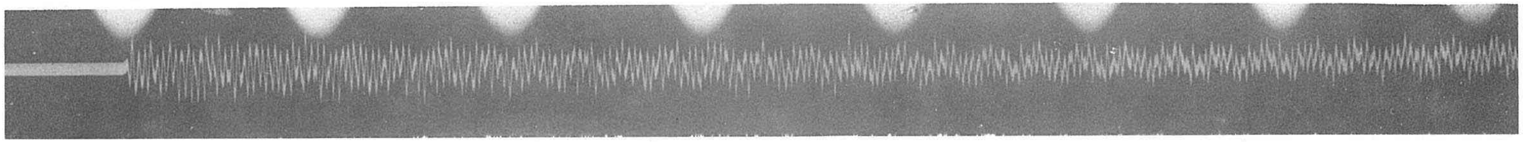

Fig. 2 1.5 secs. — 1/a=0.5

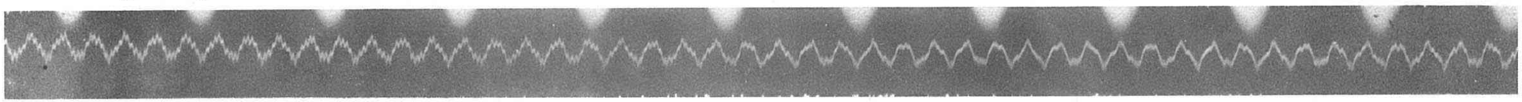

Fig. 3 2.5 secs. — 1/a=0.5

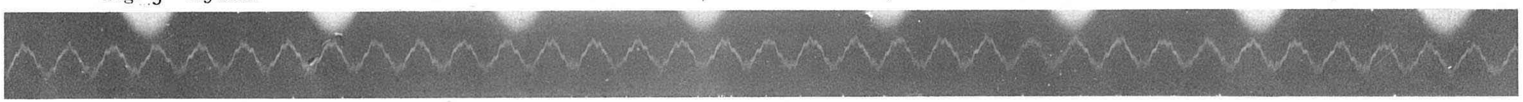

Fig 4 — 1/a=0

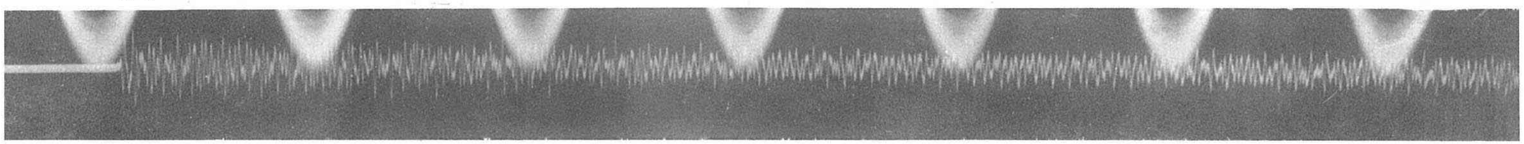

Fig. 5 1 sec. — 1/a=0

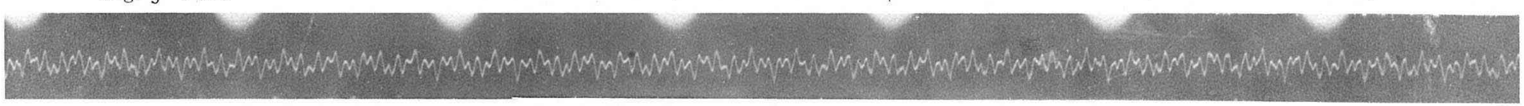

Fig. 6 2 secs. — 1/a=0

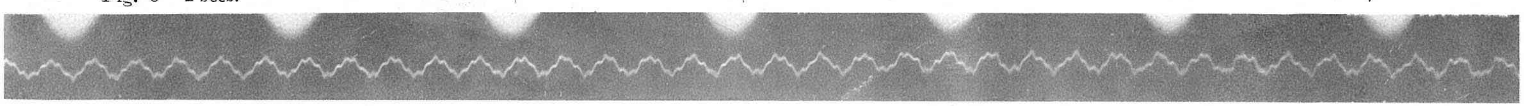

Plate III (Bell B)

Fig. 1 $1/a=1.9$

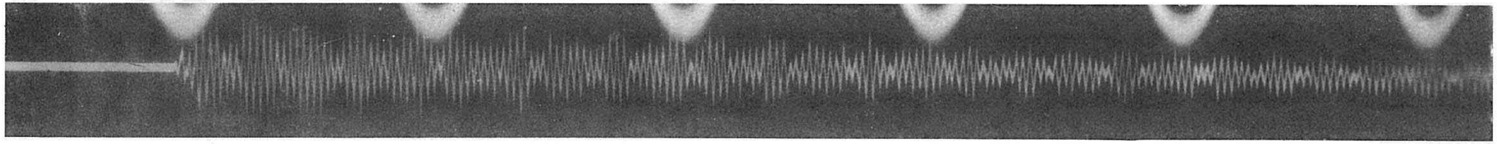

Fig. 2 1 sec. $1/a=1.9$

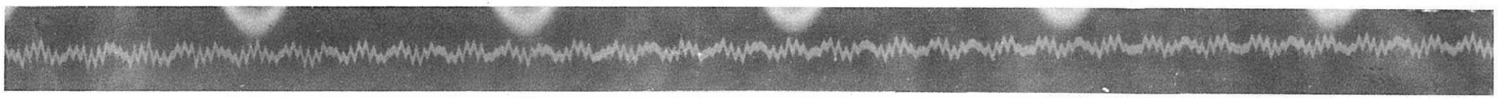

Fig. 3 2.5 secs $1/a=1.9$

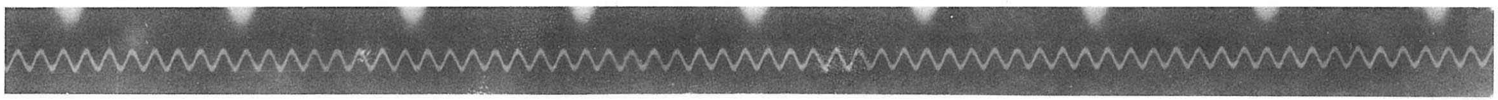

Fig. 4 $1/a=1$

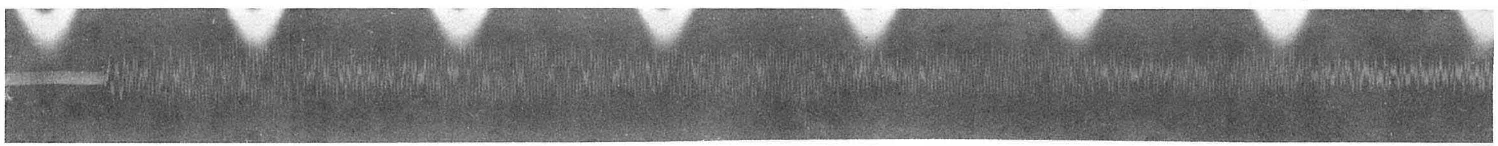

Fig. 5 1 sec. $1/a=1$

Fig. 6 1.5 secs. $1/a=1$

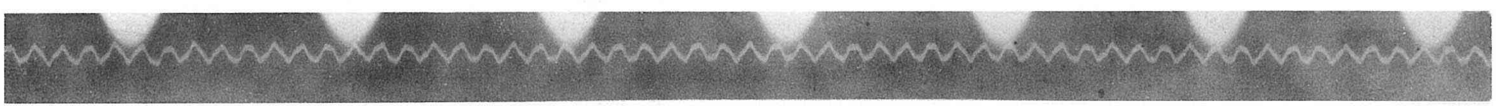

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Plate IV (Bell B)

Fig 1 — 1/a=0.5

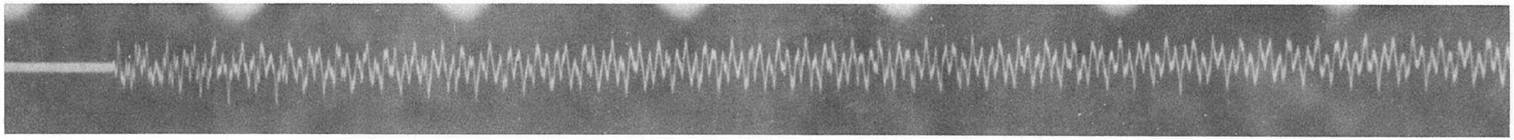


Fig. 2 1 sec. — 1/a=0.5

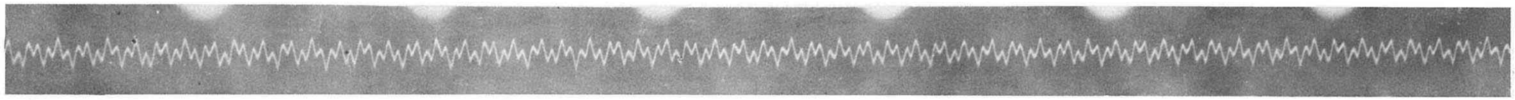


Fig. 3 1.5 secs. — 1/a=0.5

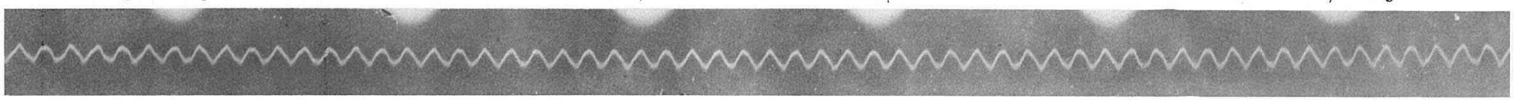


Fig. 4 — 1/a=0

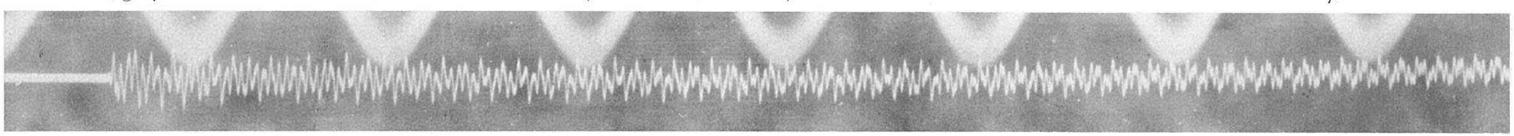


Fig. 5 1 sec. — 1/a=0

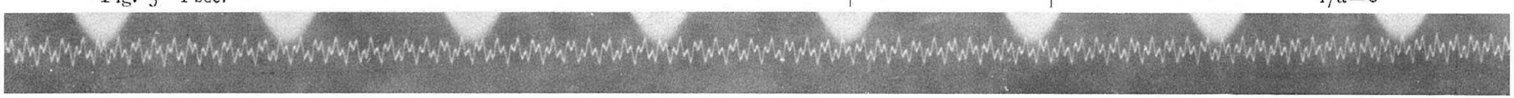
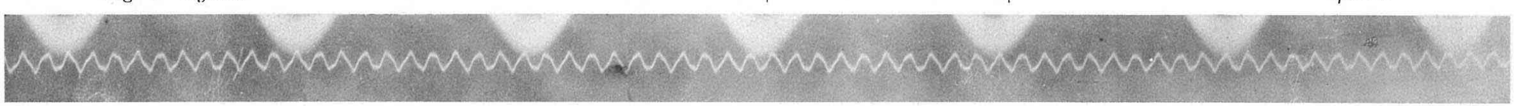


Fig. 6 1.5 secs — 1/a=0



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