On Campbell's Theorem and its Applications to Relativistic Cosmology[1]

By

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Abstract

The object of this paper is to extend an interesting research done by Campbell on the solutions of Einstein's equations under certain initial conditions and to study some problems in relativistic cosmology by applying this extended result. Especially the instability and the cause of expansion of Einstein's universe are discussed in detail. For these purposes it seems to the writer that Campbell's method is more reasonable than the methods adopted by several authors. In agreement with Eddington's view, it is shown that the initial universe expands when and only when local condensations of matter which are closely connected by local diminutions of pressure occur more actively than local annihilations.

In his valuable treatise on differential geometry, J. E. Campbell[2] enabled us to solve the well-known Einstein equation

$$G_{\mu\nu}=0,$$

under certain initial conditions by means of successive approximation. In the present paper the writer tries first to extend his method to the equations of the form

$$G_{\mu\nu}-\lambda g_{\mu\nu}=-\varkappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right),$$

when $T_{\mu\nu}$ is related to $g_{\mu\nu}$ in some particular forms, and then to deal with its applications to relativistic cosmology.

For this object the most natural and plausible assumption will be that the actual universe was initially in an equilibrium state and began

1. This paper was read at the annual meeting of the Physico-Mathematical Society of Japan on April 2, 1934.

2. J. E. Campbell: A Course of Differential Geometry, §§ 150–154 (1926).

to expand or contract by some physical causes. Starting from this point of view it will easily be seen that Einstein's universe is the best for the initial state, whereas a universe having negative or zero curvature[1] and a cosmological solution without λ-term[2] are not suitable for the present purpose.

The last article of this work is devoted to the investigations on the instability of Einstein's universe first attacked by Eddington[3] and on the cause of the expansion of the universe which has recently been studied by several authors[4].

For the present research, the writer uses the extended Campbell's theorem obtained in § 1 under the assumptions that the energy density and pressure are uniformly distributed for the first problem, while their local changes and production of energy-flow really exist for the second one.

§ 1. Extended Campbell's theorem.

It was shown by Campbell[5] that if a quadratic differential form $a_{ik}dx^i dx^k (i, k, = 1, 2, 3)$[6] is given arbitrarily, where a_{ik} represents the functions of three independent variables x^1, x^2, x^3, one can construct, by means of successive approximation, a 4-dimensional Riemannian manifold whose fundamental form referred to x^1, x^2, x^3 and a new variable x^0 is given by

$$ds^2 = V^2(dx^0)^2 + g_{ik}dx^i dx^k, \tag{1.1}$$

which satisfies the Einstein equation for an empty world

$$G_{\mu\nu} = 0$$

under the initial condition

$$[ds^2] = [g_{ik}]dx^i dx^k = a_{ik}dx^i dx^k, \tag{1.2}$$

where V and g_{ik} are unknown functions of x^0, x^1, x^2, x^3 and [. . .] denotes quantities evaluated on the hypersurface $x^0 = 0$.

Let us consider a more extended case in which (1.1) satisfies the amplified field equations

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1. O. Heckmann: Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl, **15**, 126 (1931).
2. A. Einstein: Berlin. Berichte, 235 (1931).
   A. Einstein and W. de Sitter: Proc. Nat. Acad. Sci. U. S. A. **18**, 213 (1932).
3. A. S. Eddington: M. N., **90**, 668 (1930).
4. A. G. Lemaître: M. N., **91**, 490 (1931).
   W. H. McCrea and G. C. McVittie: M. N., **92**, 7 (1931).
   N. R. Sen: Proc. Roy. Soc. A., **140**, 269 (1933).
   T. Takéuchi: Proc. Phys.-math. Soc. Jap., **15**, 431 (1933).
5. J. E. Campbell: loc cit.
6. Latin suffixes take the values 1, 2, 3 and Greek 0, 1, 2, 3.

$$G_{\mu\nu}-\lambda g_{\mu\nu}=-\varkappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right) \tag{1.3}$$

under the same initial condition (1.2), the explicit dependency of the energy tensor $T_{\mu\nu}$ on $g_{\mu\nu}$ being given by

$$T_{00}=\rho V^2,\ \ T_{0i}=T_{i0}=VS_i,\ \ T_{ik}=-pg_{ik}. \tag{1.4}$$

The line of approach here adopted for this subject is in principle the the same as that of Campbell, but the calculations and results obtained become somewhat complicated due to four equations which should be satisfied by $T_{\mu\nu}$. Since two different kinds of quantities, i. e. quantities in the 4-dimensional Riemannian manifold with the fundamental form (1.1) and quantities in its immersed hypersurface $x^0=$const. whose fundamental form is given by

$$\varphi=g_{ik}dx^i dx^k,$$

arise in our analysis, we shall discriminate between them, if necessary, by a subscript " (4) " in order to prevent confusion. We shall further apply the operation of raising or lowering suffixes with respect to g_{ik} and its conjugate g^{ik} to 3-dimensional tensors defined on a hypersurface $x^0=$const.

Now, if we introduce ω_{ik}, after Campbell, by the equations

$$\frac{\partial g_{ik}}{\partial x^0}=-2V\omega_{ik} \tag{1.5$_a$}$$

which, of course, may be written in the form

$$\frac{\partial g_{ik}}{\partial x^0}=-2Vg_{il}\omega^l_k, \tag{1.5$_b$}$$

then for the Christoffel symbols we have

$$\left.\begin{aligned}
&\underset{(4)}{\begin{Bmatrix} k \\ ij \end{Bmatrix}}=\begin{Bmatrix} k \\ ij \end{Bmatrix}, \qquad \underset{(4)}{\begin{Bmatrix} 0 \\ ij \end{Bmatrix}}=\frac{\omega_{ij}}{V}, \qquad \underset{(4)}{\begin{Bmatrix} j \\ i0 \end{Bmatrix}}=-V\omega^j_i, \\
&\underset{(4)}{\begin{Bmatrix} 0 \\ i0 \end{Bmatrix}}=\frac{1}{V}\frac{\partial V}{\partial x^i}, \quad \underset{(4)}{\begin{Bmatrix} i \\ 00 \end{Bmatrix}}=-Vg^{ih}\frac{\partial V}{\partial x^h}, \quad \underset{(4)}{\begin{Bmatrix} 0 \\ 00 \end{Bmatrix}}=\frac{1}{V}\frac{\partial V}{\partial x^0}.
\end{aligned}\right\} \tag{1.6}$$

From (1.6), we get

$$\left.\begin{aligned}
&\underset{(4)}{G_{ij}}=G_{ij}+\omega\omega_{ij}-2\omega^h_i\omega_{hj}+\frac{V_{;ij}}{V}-\frac{1}{V}\frac{\partial\omega_{ij}}{\partial x^0}, \\
&\underset{(4)}{G_{0i}}=\underset{(4)}{G_{i0}}=V(\omega^l_{i;l}-\omega_{;i}), \\
&\underset{(4)}{G_{00}}=g^{hk}V_{;hk}-Vg^{hk}\frac{\partial\omega_{hk}}{\partial x^0}-V^2\omega^k_h\omega^h_k,
\end{aligned}\right\} \tag{1.7}$$

where ; denotes the covariant differentiation with respect to g_{ik}.

Applying the first equations of (1.7) to (1.3) and then raising one suffix, we get

$$\frac{\partial \omega_i^k}{\partial x^0} = V(G_i^k + \omega\omega_i^k) + V_i^k - V\delta_i^k\left\{\lambda + \frac{\varkappa}{2}(\rho - p)\right\}, \quad (1.8)$$

where $V_i^k = g^{hk} V_{;ih}$, while making use of the second equations of (1.7), (1.3) gives

$$\omega_{i;l}^l - \omega_{;i} = -\varkappa S_i, \quad (1.9)$$

for $\mu = i$, $\nu = 0$.

If we combine two equations obtained by contracting (1.8) and by applying the third equation of (1.7) to (1.3), we get

$$G = \omega_h^k \omega_k^h - \omega^2 + 2(\lambda + \varkappa\rho). \quad (1.10)$$

Thus, in consequence of $(1.5)_a$ or $(1.5)_b$, the field equations (1.3) with the energy tensor given by (1.4) are resolved into three sets of equations (1.8), (1.9) and (1.10). On the other hand, since the energy tensor must satisfy the law of conservation, its divergence vanishes, i. e.

$$T_{\mu,\nu}^{\nu} = 0,$$

where , denotes 4-dimensional covariant differentiation with respect to $g_{\mu\nu}$.

By means of (1.6), these conservation-equations for the case given by (1.4) can be written in the form

$$\frac{\partial \rho}{\partial x^0} = V\omega(p + \rho) - 2S^k \frac{\partial V}{\partial x^k} - VS^k_{;k}, \quad (1.11)$$

$$\frac{\partial S_i}{\partial x^0} = V\frac{\partial p}{\partial x^i} + (p + \rho)\frac{\partial V}{\partial x^i} + V\omega S_i. \quad (1.12)$$

If, for simplicity, four sets of equations $(1.5)_b$, (1.8), (1.11), (1.12) and two sets of equations (1.9), (1.10) are called the sets (A) and (B) respectively, it will be seen that the x^0-derivatives of g_{ik}, ω_i^k, ρ, S_i do not appear in (B) but merely in (A), while those of p, V never occur in both sets. From this we can infer that, if, taking arbitrarily any functions p and V of x^1, x^2, x^3 and x^0, we regard the set (A) as generating equations of g_{ik}, ω_i^k, ρ, S_i, we can determine them under certain initial conditions on the hypersurface $x^0 = 0$.

Among these initial conditions, it is evident that

$$[g_{ik}] = a_{ik}, \ [\rho] = \rho_0, \ [S_i] = \sigma_i \quad (1.13)_a$$

are considered to be those imposed on g_{ik}, ρ and S_i, where ρ_0 and σ_i are any given functions of x^1, x^2, x^3. On the other hand, since ω_i^k are the mixed components of the symmetric tensor ω_{ik} and moreover satisfy (1.9) and (1.10), their initial values Ω_i^k given by

$$[\omega_i^k]=\Omega_i^k \tag{1.13$_b$}$$

must satisfy the differential equations

$$\Omega_{i\,|\,l}^l-\Omega_{\,|\,i}=-\varkappa\sigma_i, \tag{1.13$_c$}$$

where $\Omega=\Omega_l^l$ and $|$ denotes the covariant differentiation with respect to a_{ik}, and the algebraic equations

$$\left.\begin{aligned} &a_{ih}\Omega_k^h=a_{kh}\Omega_i^h,\\ &\underset{(a)}{G}=\Omega_k^h\Omega_h^k-\Omega^2+2(\lambda+\varkappa\rho_0), \end{aligned}\right\} \tag{1.13$_d$}$$

where $\underset{(a)}{G}$ is the scalar curvature formed by a_{ik}.

Differentiating two quantities A and B_i defined by

$$A=G-\omega_k^h\omega_h^k+\omega^2-2(\lambda+\varkappa\rho),$$

and
$$B_i=\omega_{i\,;\,l}^l-\omega_{;\,i}+\varkappa S_i,$$

with respect to x^0 and then eliminating $\dfrac{\partial g_{ik}}{\partial x^0}$, $\dfrac{\partial \omega_i^k}{\partial x^0}$, $\dfrac{\partial\rho}{\partial x^0}$, $\dfrac{\partial S_i}{\partial x^0}$ by means of the set (A), we obtain

$$\frac{\partial A}{\partial x^0}=4g^{hk}V_{;\,k}B_h+2V\omega A+2Vg^{hk}B_{h\,;\,k},$$

$$\frac{\partial B_i}{\partial x^0}=-V_{;\,i}A-\frac{V}{2}A_{;\,i}+V\omega B_i,$$

in consequence of the Ricci identity

$$V_{i\,;\,j}^j-V_{j\,;\,i}^j=-V_{;\,k}G_i^k.$$

Hence, from the form of these equations, it will easily be seen that if A and B_i evaluated on the hypersurface $x^0=0$ vanish, the equations $A=0$ and $B_i=0$ will remain true when x^0 takes any value whatever. Since (1.13)$_c$ and (1.13)$_d$ represent the vanishing of A and B_i on $x^0=0$ and the equations $A=0$ and $B_i=0$ are nothing but the set (B), it follows that, when g_{ik}, ω_i^k, ρ, S_i are generated by the equations (A) from their initial values (1.13), the set (B) is satisfied by these functions not only for the value $x^0=0$ but for any value of x^0. Thus we get the following theorem which will be called the extended Campbell's theorem:

Let any functions p, V of x^1, x^2, x^3 and a new variable x^0 be taken arbitrarily and moreover any functions a_{ik}, ρ_0, σ_i of x^1, x^2, x^3 be given arbitrarily, where a_{ik} is symmetric and its determinant does not vanish identically. If Ω_i^k is found to satisfy

$$\Omega_{i\,|\,l}^l - \Omega_{\,|\,i} = -\varkappa\sigma_i,$$

$$a_{ih}\Omega_k^h = a_{kh}\Omega_i^h,$$

$$\underset{(a)}{G} = \Omega_k^h\Omega_h^k - \Omega^2 + 2(\lambda + \varkappa\rho),$$

and if g_{ik}, ω_i^k, ρ and S_i are successively generated by means of the equations

$$\frac{\partial g_{ik}}{\partial x^0} = -2Vg_{il}\omega_k^l,$$

$$\frac{\partial \omega_i^k}{\partial x^0} = V(G_i^k + \omega\omega_i^k) + V_i^k - V\delta_i^k\left\{\lambda + \frac{\varkappa}{2}(\rho - p)\right\},$$

$$\frac{\partial \rho}{\partial x^0} = V\omega(p+\rho) - 2S^k\frac{\partial V}{\partial x^k} - VS^k{}_{;\,k},$$

$$\frac{\partial S_i}{\partial x^0} = V\frac{\partial p}{\partial x^i} + (p+\rho)\frac{\partial V}{\partial x^i} + V\omega S_i,$$

in such a manner that a_{ik}, Ω_i^k, ρ_0 and σ_i are their initial values when $x^0 = 0$ respectively, then the functions thus obtained, i. e.

$$g_{ik} = a_{ik} - (2[V]a_{kl}\Omega_i^l)x^0 + \ldots\ldots,$$

$$\rho = \rho_0 + \left\{[V]\Omega([p] + \rho_0) - 2\sigma_i\frac{\partial[V]}{\partial x^i} - [V]\sigma^i{}_{|\,i}\right\}x^0 + \ldots\ldots,$$

$$S_i = \sigma_i + \left\{[V]\frac{\partial[p]}{\partial x^i} + ([p] + \rho_0)\frac{\partial[V]}{\partial x^i} + [V]\Omega\sigma_i\right\}x^0 + \ldots\ldots,$$

are the solutions of the field equations

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

for which the initial fundamental form and the energy tensor are given by

$$[ds^2] = a_{ik}dx^i dx^k,$$

and $\quad T_{00} = \rho V^2,\ \ T_{0i} = T_{i0} = VS_i,\ \ T_{ik} = -pg_{ik}$

respectively.

§ 2. Some properties of a non-statical universe.

In this article, following the ordinary view, we will confine ourselves to the case $S_i = 0$. Then, from (1.12), we have

$$\frac{\partial}{\partial x^i}\{(p+\rho)V\} = V\frac{\partial \rho}{\partial x^i}.$$

This gives

$$V=\frac{\partial \Phi(\rho,x^0)}{\partial \rho}, \qquad p=\frac{\Phi(\rho,x^0)}{\frac{\partial \Phi}{\partial \rho}}-\rho, \tag{2.1}$$

where $\Phi(\rho, x^0)$ is an arbitrary function of ρ and x^0, unless $p+\rho=0$ and ρ is independent of x^1, x^2, x^3. In consequence of (2.1), when ρ depends only on x^0 so that p and V are also functions of x^0 alone, we can, without any loss of generality, take $V=1$. When we put $V=1$ conversely, we find from (2.1), however, that p is an arbitrary function of x^0 only, while ρ may be a function of x^0, x^1, x^2, x^3.

Consequently we can state Campbell's theorem for the case $V=1$ and $S_i=0$, as we have done in § 1. In this case the generating equations are

$$\left.\begin{aligned} &\frac{\partial g_{ik}}{\partial x^0}=-2g_{il}\omega^l_k, \\ &\frac{\partial \omega^k_i}{\partial x^0}=G^k_i+\omega\omega^k_i-\delta^k_i\left\{\lambda+\frac{\varkappa}{2}(\rho-p)\right\}, \\ &\frac{\partial \rho}{\partial x^0}=\omega(p+\rho), \end{aligned}\right\} \tag{2.2$_a$}$$

where p is any given function of x^0 alone, and the initial conditions are given by

$$[g_{ik}]=a_{ik}(x^1, x^2, x^3),\ [\rho]=\rho_0(x^1, x^2, x^3),\ [\omega^k_i]=\Omega^k_i(x^1, x^2, x^3), \tag{2.2$_b$}$$

where Ω^k_i represents functions satisfying

$$\left.\begin{aligned} &\Omega^l_{i\,|\,l}-\Omega_{\,|\,i}=0, \\ &a_{ih}\Omega^h_k=a_{kh}\Omega^h_i, \\ &\underset{(a)}{G}=\Omega^h_k\Omega^k_h-\Omega^2+2(\lambda+\varkappa\rho_0). \end{aligned}\right\} \tag{2.2$_c$}$$

It is obvious that solutions obtained by this theorem include all those of relativistic cosmology except that of de Sitter, whose solution is subjected to the conditions that $p+\rho=0$ and ρ is independent of x^1, x^2, x^3.

Let χ, θ, ϕ be chosen as coordinates denoted hitherto by x^1, x^2, x^3 and t be written for x^0. We regard

$$\left.\begin{array}{l}\begin{pmatrix} -R_0^2 & 0 & 0 \\ 0 & -R_0^2\dfrac{\sin^2(\sqrt{k}\chi)}{k} & 0 \\ 0 & 0 & -R_0^2\dfrac{\sin^2(\sqrt{k}\chi)}{k}\sin^2\theta \end{pmatrix}, \\ \qquad\qquad\qquad \rho_0(=\text{const.}), \\ \qquad\qquad\qquad u_0\delta_i^k, \end{array}\right\} \quad (2.3)$$

as a_{ik}, ρ_0, Ω_i^k of $(2.2)_b$ respectively, where R_0, ρ_0, u_0 are three constants and k represents $+1$, 0 or -1 according as the curvature of the hypersurface $x^0=0$ is positive, zero or negative[1]. From the third equation of $(2.2)_c$, these constants are then connected by

$$u_0^2=\frac{1}{3}(\lambda+\varkappa\rho_0)-\frac{k}{R_0^2}. \quad (2.4)$$

Starting from (2.2) with these initial values and using a given function $p(t)$, we obtain

$$g_{ik}=a_{ik}\Big\{1-2e\Big[\frac{1}{3}(\lambda+\varkappa\rho_0)-\frac{k}{R_0^2}\Big]^{\frac{1}{2}}t+\Big[\frac{\varkappa}{6}(\rho_0-3p_0)+\frac{2}{3}\lambda-\frac{k}{R_0^2}\Big]t^2$$
$$-\frac{2}{3\,!}\Big[\frac{\varkappa}{2}\dot{p}_0-\frac{e}{6}(\varkappa\{\rho_0+9p_0\}+8\lambda)\Big(\frac{1}{3}\{\lambda+\varkappa\rho_0\}$$
$$-\frac{k}{R_0^2}\Big)^{\frac{1}{2}}\Big]t^3+\dots\Big\}, \quad (2.5)$$

where $\dot{p}_0$ denotes $\left[\dfrac{dp}{dt}\right]$ and e is $+1$ or -1 according as $u_0>0$ or $u_0<0$, after tedious calculation. Of course, this can also be obtained by assuming

$$ds^2=dt^2-R(t)^2\Big\{d\chi^2+\frac{\sin^2(\sqrt{k}\chi)}{k}(d\theta^2+\sin^2\theta d\phi^2)\Big\}. \quad (2.6)$$

Comparing (2.5) with (2.6) we have

$$R(t)^2=R_0^2\Big\{1-2e\Big[\frac{1}{3}(\lambda+\varkappa\rho_0)-\frac{k}{R_0^2}\Big]^{\frac{1}{2}}t+\Big[\frac{\varkappa}{6}(\rho_0-3p_0)$$
$$+\frac{2}{3}\lambda-\frac{k}{R_0^2}\Big]t^2-\frac{2}{3\,!}\Big[\frac{\varkappa}{2}\dot{p}_0-\frac{e}{6}(\varkappa\{\rho_0+9p_0\}+8\lambda)$$
$$\times\Big(\frac{1}{3}(\lambda+\varkappa\rho_0)-\frac{k}{R_0^2}\Big)^{\frac{1}{2}}\Big]t^3+\dots\Big\}. \quad (2.7)$$

In order to interpret the correlation between distances and radial

1. This device of representing the sign of the curvature is due to W. de Sitter: Proc. Akad. Wetensch. Amsterd., **35**, 596 (1932).

velocities of extragalactic nebulae, it is sufficient to take into consideration up to the second term in the expansion of (2.7), making $t=0$ to correspond to the present state of the universe. But as we study the instability of the initial state of our universe, we will take $t=0$ at the instant when the equilibrium begins to be destroyed.

When ρ_S, p_S and R_S represent the energy density, the pressure, and the radius of the statical universe respectively, it can easily be shown, from the second equations of $(2.2)_a$ and the third equation of $(2.2)_c$, that these constants must satisfy

$$\left.\begin{aligned} &\frac{1}{3}(\lambda+\kappa\rho_S)-\frac{k}{R_S^2}=0,\\ \text{and}\qquad &\kappa(\rho_S+3p_S)-2\lambda=0. \end{aligned}\right\} \qquad (2.8)_a$$

Eliminating λ between $(2.8)_a$, we get

$$\kappa\rho_S=\frac{2k}{R_S^2}-\kappa p_S.$$

This indicates that, if $k=0$ or -1, ρ_S must not be positive. Hence we should take $k=+1$ and moreover, from the second equation of $(2.8)_a$, we see that λ must be positive[1]. Consequently the equilibrium conditions $(2.8)_a$ may be written in the form

$$\left.\begin{aligned} &\frac{1}{3}(\lambda+\kappa\rho_S)-\frac{1}{R_S^2}=0,\\ \text{and}\qquad &\kappa(\rho_S+3p_S)-2\lambda=0,^2 \end{aligned}\right\} \qquad (2.8)_b$$

which show that the initial state is identical with that of Einstein.

Since $t=0$ now corresponds to the instant when the equilibrium begins to be destroyed, we have

$$\rho_0=\rho_S,\ p(0)=p_S,\ R_0=R_S.$$

Substituting these values in (2.7), we obtain by means of $(2.8)_b$

$$R(t)^2=\begin{cases} R_S^2, & \text{for } t\leqq 0,\\ R_S^2\left\{1-\dfrac{\kappa}{3\,!}p_0t^3+\dots\right\}, & \text{for } t\geqq 0, \end{cases} \qquad (2.9)$$

provided we assume that the universe begins to expand or contract from an initial state of equilibrium.

It will be seen from (2.9) that the Einstein's universe is instable

1. Cf. McCrea and McVittie: op. cit.
2. Cf. S. Kunii: These Memoirs, **15**, 97 (1932).

and expands or contracts according as $\dot{p}_0 < 0$ or $\dot{p}_0 > 0$ when t runs through from negative to positive value. Hence the time rate of change of p, which is closely related to condensations of matter as shown in § 3, plays an essential rôle in the expansion. And we will not able to measure accurately the instability and expansion of the initial state, if we neglect the cosmological pressure p[1].

§ 3. Cause of expansion.

We have just seen that, if the cosmological pressure decreases, the Einstein's universe expands so far as the change takes place uniformly. In this article we will inquire how the initial state behaves when local changes of pressure as well as energy density and production of energy-flow begin to take place. Regarding $t=0$ as this instant, the line element and the energy tensor of our world when $t \leqq 0$ can be put in the ordinary form

$$ds^2 = dt^2 - R_S^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\} \qquad (3.1)_a$$

and

$$T_{\mu\nu} = \begin{pmatrix} \rho_S & 0 & 0 & 0 \\ 0 & p_S R_S^2 & 0 & 0 \\ 0 & 0 & p_S R_S^2 \sin^2\chi & 0 \\ 0 & 0 & 0 & p_S R_S^2 \sin^2\chi \sin^2\theta \end{pmatrix} \qquad (3.1)_b$$

respectively, where ρ_S, p_S and R_S are connected by the equations

$$\frac{1}{R_S^2} = \frac{1}{3}(\lambda + \varkappa\rho_S), \quad 2\lambda = \varkappa(\rho_S + 3p_S). \qquad (3.2)$$

For simplicity, suppose that the disturbance of the equilibrium of the initial state to have radial symmetry about a point P on the hyper-surface $t=0$. Then, without any loss of generality, we may consider P as the pole (i. e. $\chi=0$) of this hypersurface. These simplifications will not impair the applicability of our results to real phenomena.

Transforming χ into r by the equation $r = R_S \sin\chi$ and denoting new components of the energy tensor again by $T_{\mu\nu}$, we obtain from $(3.1)_a$ and $(3.1)_b$

$$ds^2 = dt^2 - \left\{\frac{dr^2}{1 - \frac{r^2}{R_S^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\} \qquad (3.3)_a$$

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1. Cf. Eddington: op. cit.

and

$$T_{\mu\nu}=\begin{pmatrix} \rho_S & 0 & 0 & 0 \\ 0 & p_S\dfrac{1}{1-\dfrac{r^2}{R_S^2}} & 0 & 0 \\ 0 & 0 & p_S r^2 & 0 \\ 0 & 0 & 0 & p_S r^2\sin^2\theta \end{pmatrix}, \qquad (3.3)_b$$

provided $t\leqq 0$ and $x^1=r,\ x^2=\theta,\ x^3=\phi$.

The condition of radial symmetry now requires that S_2 and S_3 vanish identically and moreover p and V must be independent of θ and ϕ. When $t=0$, p and V should be equal to p_S and $+1$ respectively, in order that they may be continuously connected with their values given by (3.3). However, by assuming the existence of universal time coordinate, we may put $V=1$. Thus the generating equations in § 1 can be written in the form

$$\left.\begin{aligned} &\frac{\partial g_{ik}}{\partial t}=-2g_{il}\omega^l_k, \\ &\frac{\partial \omega^k_i}{\partial t}=G^k_i+\omega\omega^k_i-\delta^k_i\left\{\lambda+\frac{\varkappa}{2}(\rho-p)\right\}, \\ &\frac{\partial \rho}{\partial t}=\omega(p+\rho)-S^l_{:l}, \\ &\frac{\partial S_1}{\partial t}=\frac{\partial p}{\partial t}+\omega S_1. \end{aligned}\right\} \qquad (3.4)$$

On the other hand, in comparison with § 1, it is obvious from (3.3) that the initial values of g_{ik}, ρ and S_1 are given by

$$a_{ik}=\begin{pmatrix} -\dfrac{1}{1-\dfrac{r^2}{R_S^2}} & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \quad \rho_0=\rho_S,\ \sigma_1=0 \qquad (3.5)$$

respectively. From the first equations of (3.5) we have for the nonvanishing components of the contracted curvature tensor formed by a_{ik}

$$\underset{(a)}{G^1_1}=\underset{(a)}{G^2_2}=\underset{(a)}{G^3_3}=\frac{2}{R_S^2}. \qquad (3.6)$$

Applying the third equation of (3.5) to $(1.13)_c$ and (3.6) to the second equation of $(1.13)_d$, the condition of radial symmetry gives

$$\Omega_i^k = 0, \tag{3.7}$$

by means of (3.2).

Putting $t=0$ in (3.4) and then using (3.2), (3.5) and (3.6), we get

$$\left[\frac{\partial g_{ik}}{\partial t}\right]=0,\quad \left[\frac{\partial \omega_i^k}{\partial t}\right]=0,\quad \left[\frac{\partial \rho}{\partial t}\right]=0,\quad \left[\frac{\partial S_1}{\partial t}\right]=0. \tag{3.8$_a$}$$

Differentiating (3.4) with respect to t and then putting $t=0$, the second derivatives at $t=0$ of these quantities are seen to be

$$\left.\begin{aligned} &\left[\frac{\partial^2 g_{ik}}{\partial t^2}\right]=0,\quad \left[\frac{\partial^2 \omega_i^k}{\partial t^2}\right]=\frac{\varkappa}{2}\delta_i^k\left[\frac{\partial p}{\partial t}\right],\\ &\left[\frac{\partial^2 \rho}{\partial t^2}\right]=0,\quad \left[\frac{\partial^2 S_1}{\partial t^2}\right]=\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right], \end{aligned}\right\} \tag{3.8$_b$}$$

by making use of (3.5), (3.7) and (3.8)$_a$.

Proceeding further in the same way, we find

$$\left.\begin{aligned} &\left[\frac{\partial^3 g_{ik}}{\partial t^3}\right]=-\varkappa a_{ik}\left[\frac{\partial p}{\partial t}\right],\\ &\left[\frac{\partial^3 \rho}{\partial t^3}\right]=\frac{3}{R_S^2}\left[\frac{\partial p}{\partial t}\right]+\left(1-\frac{r^2}{R_S^2}\right)\frac{\partial^2}{\partial r^2}\left[\frac{\partial p}{\partial t}\right]\\ &\qquad\qquad +\frac{2}{r}\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right]-\frac{3r}{R_S^2}\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right],\\ &\left[\frac{\partial^3 S_1}{\partial t^3}\right]=\frac{\partial}{\partial r}\left[\frac{\partial^2 p}{\partial t^2}\right]. \end{aligned}\right\} \tag{3.8$_c$}$$

Thus g_{ik}, ρ and S_1 are expressible as the power series in t

$$\left.\begin{aligned} &g_{ik}=a_{ik}-\frac{t^3}{3\,!}\varkappa a_{ik}\left[\frac{\partial p}{\partial t}\right]+\dots,\\ &\rho=\rho_S+\frac{t^3}{3\,!}\left\{\frac{3}{R_S^2}\left[\frac{\partial p}{\partial t}\right]+\left(1-\frac{r^2}{R_S^2}\right)\frac{\partial^2}{\partial r^2}\left[\frac{\partial p}{\partial t}\right]\right.\\ &\qquad\left.+\frac{2}{r}\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right]-\frac{3r}{R_S^2}\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right]\right\}+\dots,\\ &\text{and}\\ &S_1=\frac{t^2}{2\,!}\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right]+\frac{t^3}{3\,!}\frac{\partial}{\partial r}\left[\frac{\partial^2 p}{\partial t^2}\right]+\dots. \end{aligned}\right\} \tag{3.9}$$

Consequently the line element of our universe when $t\geqq 0$ is given by

$$ds^2=dt^2-\left(1-\frac{t^3}{3\,!}\varkappa\left[\frac{\partial p}{\partial t}\right]\right)\left\{\frac{dr^2}{1-\dfrac{r^2}{R_S^2}}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right\} \tag{3.10$_a$}$$

within the accuracy of the third power of t. Transforming r back to χ, this is converted

$$ds^2 = dt^2 - R_S^2\left(1 - \frac{t^3}{3!}\varkappa\left[\frac{\partial p}{\partial t}\right]\right)\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}. \quad (3.10)_b$$

This formula for our line element implies that $R_S\left(1 - \frac{t^3}{3!}\varkappa\left[\frac{\partial p}{\partial t}\right]\right)^{\frac{1}{2}}$ i. e. $R_S\left(1 - \frac{t^3}{12}\varkappa\left[\frac{\partial p}{\partial t}\right]\right)$ may be regarded, within the accuracy of the third power of t, as the radius at the point χ of our universe which will be called the local radius at χ. Consequently the local radius at any point χ expands or contracts from its original value R_S, according as $\left[\frac{\partial p}{\partial t}\right]$ in χ is negative or positive. From the second and third equations of (3.9), it will easily be seen that the (macroscopic) density ρ of the total energy varies with the time in the same order as the local radius, but the variation depends on $\left[\frac{\partial^2 S_1}{\partial t^2}\right]$ and $\frac{\partial}{\partial r}\left[\frac{\partial^2 S_1}{\partial t^2}\right]$ other than $\left[\frac{\partial p}{\partial t}\right]$. Hence the change of local radius has nothing to do with it directly. In this sense we find the expansion of local radius is not caused by the fluctuation of the total energy-density ρ but it is caused by the fluctuation of the pressure.

Denoting the (microscopic) density of matter by ρ_M, we have

$$\rho_M = \rho - 3p.$$

Hence, from (3.9) its explicit form is given by

$$\begin{aligned}\rho_M = (\rho_M)_S - \frac{3t}{1!}\left[\frac{\partial p}{\partial t}\right] - \frac{3t^2}{2!}\left[\frac{\partial^2 p}{\partial t^2}\right] + \frac{t^3}{3!}\Bigg\{\frac{3}{R_S^2}\left[\frac{\partial p}{\partial t}\right] \\ + \left(1 - \frac{r^2}{R_S^2}\right)\frac{\partial^2}{\partial r^2}\left[\frac{\partial p}{\partial t}\right] + \frac{2}{r}\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right] \\ - \frac{3r}{R_S^2}\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial t}\right] - 3\left[\frac{\partial^3 p}{\partial t^3}\right]\Bigg\} + \dots, \qquad (3.11)_a\end{aligned}$$

where $(\rho_M)_S$ represents the (microscopic) density of matter in the initial universe.

From $(3.11)_a$ we obtain

$$\left[\frac{\partial \rho_M}{\partial t}\right] = -3\left[\frac{\partial p}{\partial t}\right], \qquad (3.11)_b$$

that is to say, when $t=0$, the increasing rate of density of matter is equal to three times the decreasing rate of pressure. Hence the local radius at χ can be written in the form

$$R_s\left(1+\frac{t^3}{36}\varkappa\left[\frac{\partial\rho_M}{\partial t}\right]\right),$$

and consequently it begins to expand wherever condensation of matter is formed. From this it will be seen that the mean value of local radii becomes enlarged if condensation of matter takes place, as a mean, in Einstein's universe. Accordingly the expansion-process of our universe can be stated as follows:

If at first variations of pressure and of density of matter, which are closely related to each other by $(3.11)_b$, occur in the initial equilibrium state, the flow of energy is then produced, but the (macroscopic) density of total energy and the initial radius remain unchanged. In the next stage the radius begins to expand or contract locally according as condensation or annihilation of matter takes place, while at the same time the density of total energy also begins to vary in a somewhat complicated way. And if the mean condensation of matter is positive, our universe will make an expansion as a whole.

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