The Boundary Value Problem of the Wind Current in a Lake or a Sea

By

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Abstract

The effect of the coast on the wind current has been already investigated by W. Ekman, but his treatment is not an exact boundary value problem as he states[1].

Thereafter, several investigators give solutions of the similar problem under the boundary condition that the component current normal to the vertical sea wall is nil at the coast. But their solutions are appropriate only when the tractive force of the wind vanishes at the coast; otherwise they do not really satisfy the above boundary condition.

In the present paper, the writer investigates exactly the boundary value problem when the tractive force of the wind is nil at the coast, but is uniform all over the sea surface. Taking into account the term of the horizontal "Austausch", he succeeded in obtaining a complete solution for the steady state.

According to the results, the deviation of the Ekman's theory from our exact solution for the current and the surface slope is limited only to a distance from the coast equal to one or one half times the depth of the sea. The surface elevation is much the same in both.

The chief object of this paper

W. Ekman has already investigated the coast effect for the wind current in his paper of 1905 and 1923, but his solution is not an exact boundary value problem as he states. It is because he does not use the boundary condition that the component velocity normal to the vertical sea wall is nil at the coast ($x=0$, L), but only uses the equation of continuity that the total flow normal to the coast vanishes everywhere.

1. Ekman, V. W.: Über Horizontalzirkulationen bei winderzeugten Meeresströmungen. Ark. Mat. Astr. o. Fys. Bd. 17, Nr. 26. 1923.

J. Proudman and A. T. Doodson[1] studied the motion of water and the surface elevation generated by a special wind in a canal, and obtained a solution which satisfies the above boundary condition.

They used the equation of motion in usual form

$$\frac{\partial u}{\partial t}=-g\frac{\partial \zeta}{\partial x}+\mu\frac{\partial^2 u}{\partial z^2} \qquad \text{.........................(1)}$$

together with the boundary conditions

$$u=0 \qquad \text{at } x=0, \text{ and } L \qquad \text{........(2)}$$

$$\mu\frac{\partial u}{\partial z}=-T\sin kx \qquad \text{at } z=0 \quad \text{where } k=\frac{n\pi}{L}. \quad (3)$$

And they obtained a solution of the form, $u=f(z, t)\sin kx$, and stated that "The general case can be built up by the superposition of these particular cases with different value of k." But it is doubtful whether the purpose can be attained by superposing these particular cases, or not. Indeed, when the surface traction T is any function of x, by expanding $T(x)$ into a Fourier's sine series we can obtain a solution of form

$$u=\sum_{n=1}^{n=\infty} u(z, t)\sin kx. \qquad \text{..............................(4)}$$

The above series (4) vanishes certainly at $x=0$ and L, but it does not necessarily vanish when x converges to 0 or L. For the actual purpose, however, we must of course take the boundary value as the limiting value when x converges to 0 or L, but not the direct value at $x=0$, or L. Indeed in the present case. if T itself vanishes at $x=0$ or L, the series (4) will also vanish for $x\rightarrow 0$ or L, but otherwise or when T is uniform all over the sea, the series does not vanish. In other words, the condition (2) can not be satisfied. As Proudman does not treat the general case, we can not point out this fault. But in the recent papers of Dr. K. Hidaka[2] and Mr. K. Koenuma[3] on the problem of the drift current, the above weakness is present.

Thus the next step on the problem must be to find solution which generally satisfies the condition (2). Recall the theory of the conduction of heat, then its answer is very clear. Namely our purpose will be

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1. Proudman, J. and Doodson, A. T.: Time-relations in meteorological effects on the sea. *Proc. math. Soc.* London, Ser. 2, Bd. 24, Part 2. 1924.

2. K. Hidaka: Motion of Lake Water generated by Wind. Part I. Geophy. Mag. Cent. Meteo. Obs., Tokyo. Vol. VII, No. 3-4, 1933.

3. 肥沼貫一: 海と空 第十三卷 67 (1933); 226 (1933).

attained by raising again only one term of the horizontal "Austausch" $\mu\frac{\partial^2 u}{\partial x^2}$ in eq. (1), because, from the second derivative of u regarding x, two degrees of freedom appear, which supply the two boundary condition for x.

Actually in a lake or a sea whose depth is very small compared with the horizontal dimension, the term $\mu\frac{\partial^2 u}{\partial x^2}$ can be usually neglected in comparison with the vertical "Austausch" $\mu\frac{\partial^2 u}{\partial z^2}$, but in the immediate neighbourhood of the coast this term can not be neglected and becomes even larger than $\mu\frac{\partial u^2}{\partial z^2}$. Moreover, in the region where the wind traction changes very abruptly, this term also becomes large and it can not be neglected.

Thus, in the following section we shall give a solution which satisfies the condition (3) at the same time taking $\mu\frac{\partial^2 u}{\partial x^2}$ into account.

Solution

Now, for simplicity, let us consider only the state overlooking for the time being the effect of the earth's rotation. This simplest case is sufficient for our main object aimed at in the present paper.

Conceive a sea of uniform depth and of semi-infinite extent, whose one end is bounded by a straight vertical wall. If a constant wind blows uniformly all over the sea, the surface elevation or depression caused by the wind will become infinite at the coast; but it can not be admitted physically. Hence we shall consider a case where the wind blows over only a finite region from $x=0$ to $x=l$ and in the direction normal to the coast.

Choose the coordinate axes such that the z-axis is directed vertically upwards and the x-axis lies on the sea surface and perpendicularly to the coast, and x is measured from the coast seawards.

Let the following notations be used:

T=the tractive force of the wind per unit area of the sea surface, and it is uniform for $0<x<l$, and vanishes for $x>l$,
h=the depth of sea,
ρ=the density of sea water which is here assumed to be unity,
μ=the coefficient of the eddy viscosity of water,
u=the velocity of current in the direction of x,

f= the coefficient of friction at the bottom, assuming the resistance proportional to the slip velocity.

Then the current and the surface elevation produced by the wind will be represented by the solution of the eq.

$$g\frac{\partial \zeta}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad \text{for } x>0 \text{ and } -h<z<0, \qquad (5)$$

with the eq. of continuity

$$\int_{-h}^{0} u dz=0, \qquad \dots\dots(6)$$

and under the conditions

$$\left.\begin{aligned} &\mu\frac{\partial u}{\partial z}=-T && \text{at } z=0 \text{ for } 0<x<l, \\ &\mu\frac{\partial u}{\partial z}=0 && \text{at } z=0 \text{ for } x>l, \end{aligned}\right\} \dots(7)$$

$$\frac{\partial u}{\partial z}=Ku \qquad \text{at } z=-h, \qquad \dots\dots(8)$$

$$\text{where} \qquad K=\frac{f}{\mu},$$

$$u=0 \qquad \text{at } x=0, \qquad \dots\dots(9)$$

$$\left.\begin{aligned} &u,\ \mu\frac{\partial u}{\partial x}, \text{ and } \zeta \text{ are continuous at } x=l, \\ &\text{and are zero } x=\infty. \end{aligned}\right\} \dots\dots(10)$$

To solve the eq. (5) we shall use Stokes method. Let u, $\frac{\partial \zeta}{\partial x}$ and T be expressed by Fourier's integral, then

$$\frac{\mu}{gh^2}u=\frac{2}{\pi}\int_0^\infty \sin ax\, v(a)\, da \qquad \dots\dots(11)$$

$$\text{where} \qquad v(a)\equiv\frac{\mu}{gh^2}\int_0^\infty u(\lambda)\sin a\lambda\, d\lambda,$$

$$\frac{\partial \zeta}{\partial x}=\frac{2}{\pi}\int_0^\infty \gamma(a)\sin ax\, dx \qquad \dots\dots(12)$$

$$\text{where} \qquad \gamma(a)\equiv\int_0^\infty \frac{\partial \zeta}{\partial \lambda}\sin a\lambda\, d\lambda,$$

$$\frac{T}{gh}=\frac{2}{\pi}\int_0^\infty T'(a)\sin ax\, dx \qquad \dots\dots(13)$$

$$\text{where} \qquad T'(a)\equiv\frac{T}{gh}\,\frac{1-\cos al}{a}.$$

Express $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial x^2}$ also with Fourier's integral

$$\frac{\partial^2 u}{\partial z^2}=\frac{2}{\pi}\int_0^\infty \sin ax\, da\int_0^\infty \frac{\partial^2 u}{\partial z^2}\sin a\lambda\, d\lambda=\frac{gh^2}{\mu}\,\frac{2}{\pi}\int_0^\infty \frac{\partial^2 v}{\partial z^2}\sin ax\, dx,$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{2}{\pi}\int_0^\infty \sin ax\, da\int_0^\infty \frac{\partial^2 u}{\partial \lambda^2}\sin a\lambda\, d\lambda.$$

Now

$$\int_0^\infty \frac{\partial^2 u}{\partial \lambda^2}\sin a\lambda\, d\lambda=\int_0^l \frac{\partial^2 u}{\partial \lambda^2}\sin a\lambda\, d\lambda+\int_l^\infty \frac{\partial^2 u}{\partial \lambda^2}\sin a\lambda\, d\lambda$$

$$=\left[\frac{\partial u}{\partial \lambda}\sin a\lambda\right]_0^l+\left[\frac{\partial u}{\partial \lambda}\sin a\lambda\right]_l^\infty-a\int_0^l \frac{\partial u}{\partial \lambda}\cos a\lambda\, d\lambda-a\int_l^\infty \frac{\partial u}{\partial \lambda}\cos a\lambda\, d\lambda.$$

From the conditions that $\left[\mu\frac{\partial u}{\partial \lambda}\right]_{\lambda=l}$ is continuous and that $\left[\mu\frac{\partial u}{\partial \lambda}\right]_{\lambda=\infty}=0$, the above expression becomes

$$=-a\int_0^l \frac{\partial u}{\partial \lambda}\cos a\lambda\, d\lambda-a\int_l^\infty \frac{\partial u}{\partial \lambda}\cos a\lambda\, d\lambda$$

$$=-a\Big[u\cos a\lambda\Big]_0^l-a\Big[u\cos a\lambda\Big]_l^\infty-a^2\int_0^\infty u\sin a\lambda\, d\lambda.$$

Here again remembering that $[u(\lambda)]_{\lambda=0}=[u(\lambda)]_{\lambda=\infty}=0$ and $u(\lambda)$ is continuous at $\lambda=l$, we get

$$\frac{\partial^2 u}{\partial x^2}=-\frac{2}{\pi}\int_0^\infty \sin ax\, da\int_0^\infty a^2 u(a)\sin a\lambda d\lambda=-\frac{gh^2}{\mu}\,\frac{2}{\pi}\int_0^\infty a^2 v(a)\sin ax\, da.$$

Substitute the above values of $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial z^2}$, and $\frac{\partial \zeta}{\partial x}$ into (5), and equate the factors of $\sin ax$, then we get

$$\gamma=\frac{d^2 v}{d\xi^2}-\sigma^2 v, \qquad \text{.................................}(5')$$

where $\xi=\frac{z}{h}$ and $\sigma=ah$.

Corresponding to (5′), the eq. of continuity (6) and the boundary conditions (7) and (8) become as follows:

$$\int_{-1}^0 v\, d\xi=0, \qquad \text{.......................................}(6')$$

$$\left.\begin{aligned} &\frac{dv}{d\xi}=-\frac{T}{g}\,\frac{1-\cos\sigma\frac{l}{h}}{\sigma} && 0<x<l, \\ & && \text{at}\quad \xi=0 \\ &\frac{dv}{d\xi}=0 && x>l, \end{aligned}\right\} \quad (7')$$

and

$$\frac{dv}{d\xi}=\overline{K}v \qquad \text{at} \quad \xi=-1 \qquad \text{......}(8')$$

where $\overline{K}=hK$.

Since γ is independent of ξ, the general solution of $(7')$ is obviously

$$v=\frac{1}{\sigma^2}\{A_1\sinh\sigma(1+\xi)+A_2\cosh\sigma(1+\xi)-\gamma\}.$$

Determining the constants A_1, A_2 and γ by $(6')$, $(7')$ and $(8')$, we obtain

$$\begin{aligned} v=-\frac{T}{g}\,&\frac{1-\cos\sigma\frac{l}{h}}{\sigma^2\{\sigma^2\sinh\sigma+\overline{K}(\sigma\cosh\sigma-\sinh\sigma)\}} \\ &\times\left[\overline{K}(\sigma-\sinh\sigma)\sinh\sigma(1+\xi)+\left\{\sigma^2+\overline{K}(\cosh\sigma-1)\right\}\cosh\sigma(1+\xi)\right. \\ &\left.-\sigma\sinh\sigma-\overline{K}(\cosh\sigma-1)\right]. \qquad \text{........................}(14) \end{aligned}$$

and

$$\gamma=-\frac{T}{g}\,\frac{\left\{\sigma\sinh\sigma+\overline{K}(\cosh\sigma-1)\right\}\left\{1-\cos\sigma\frac{l}{h}\right\}}{\sigma^2\sinh\sigma+\overline{K}(\sigma\cosh\sigma-\sinh\sigma)}. \qquad (15)$$

Therefore the current and the surface slope produced by the wind traction are given by the following integral forms.

$$\frac{\mu}{gh^2}u=\frac{2}{\pi h}\int_0^\infty v(\sigma,\xi)\sin\sigma\frac{x}{h}d\sigma, \qquad \text{...........}(14')$$

and

$$\frac{\partial\zeta}{\partial x}=\frac{2}{\pi h}\int_0^\infty \gamma\sin\sigma\frac{x}{h}d\sigma. \qquad \text{.....................}(15')$$

For convenience, to evaluate these infinite integrals we transform them into

$$\frac{\mu}{gh^2}u = -\frac{2}{\pi}\frac{T}{gh}\int_0^\infty v'(\sigma,\xi)\sin\sigma\frac{x}{h}d\sigma + \frac{1}{\pi}\frac{T}{gh}\int_0^\infty v'(\sigma,\xi)\sin\sigma\frac{x+l}{h}d\sigma$$

$$\mp\frac{1}{\pi}\frac{T}{gh}\int_0^\infty v'(\sigma,\xi)\sin\left(\pm\sigma\frac{l-x}{h}\right)d\sigma, \qquad \text{.........}(14'')$$

$$\frac{\partial\zeta}{\partial x} = -\frac{2}{\pi}\frac{T}{gh}\int_0^\infty \gamma'(\sigma)\sin\sigma\frac{x}{h}d\sigma + \frac{1}{\pi}\frac{T}{gh}\int_0^\infty \gamma'\sin\sigma\frac{x+l}{h}d\sigma$$

$$\mp\frac{1}{\pi}\frac{T}{gh}\int_0^\infty \gamma'(\sigma)\sin\left(\pm\sigma\frac{l-x}{h}\right)d\sigma, \qquad \text{...........}(15'')$$

$$\text{where} \quad v' = -\frac{g}{T}v, \quad \text{and} \quad \gamma' = -\frac{g}{T}\gamma,$$

and the upper and the lower sign must be taken according to $0<x<l$ and $x>l$.

Evaluation of the above integral can be easily carried out by Cauchy's integral expression. The results are as follows:

$$\frac{\mu}{gh^2}u = \frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s x}{h}} + e^{-\frac{\beta_s l}{h}}\sinh\frac{\beta_s x}{h}\right\}$$

$$\times\left\{\beta_s\sin\beta_s(1+\xi) + \beta_s\cot\beta_s\cos\beta_s(1+\xi)\right\}$$

$$-\frac{T}{gh}\left\{\frac{3(\overline{K}+2)}{4(\overline{K}+3)}\xi^2 + \xi + \frac{\overline{K}+4}{4(\overline{K}+3)}\right\} \quad \text{for} \quad 0<x<l,$$

$$\frac{\mu}{gh^2}u = \frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s x}{h}} - e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}\right\}$$

$$\times\left\{\beta_s\sin\beta_s(1+\xi) + \beta_s\cot\beta_s\cos\beta_s(1+\xi) - 1\right\} \quad \text{for} \quad x>l,$$

$$\text{...........}(17)$$

$$\text{where} \qquad B_s = 2\frac{\beta_s\tan\beta_s + \overline{K}(\sec\beta_s - 1)}{\beta_s^3\{\beta_s + (\overline{K}+2)\tan\beta_s\}},$$

and β_s are roots of eq. $\tan\beta = \frac{\overline{K}\beta}{\overline{K}+\beta^2}$ which involves an infinite number of real root.

$$\frac{\partial\zeta}{\partial x} = -\frac{T}{gh}\sum_1^\infty C_s\left\{e^{-\frac{\beta_s x}{h}} + e^{-\frac{\beta_s l}{h}}\sinh\frac{\beta_s x}{h}\right\} - \frac{T}{gh}\frac{3(2+\overline{K})}{2(3+\overline{K})}$$

$$\text{for} \quad 0<x<l,$$

$$\frac{\partial\zeta}{\partial x}=-\frac{T}{gh}\sum_{1}^{\infty}C_s\left\{e^{-\frac{\beta_s x}{h}}-e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}\right\} \qquad \text{for} \quad x>l,$$

$$\dots\dots\dots\dots(18)$$

where $$C_s=2\frac{\beta_s\tan\beta_s+\overline{K}(\sec\beta_s-1)}{\beta_s\{\beta_s+(\overline{K}+2)\tan\beta_s\}}.$$

Integrate (18) with x and determine the integration constant by condition (10), then we have surface elevation ζ generated by the wind.

$$\zeta=-\frac{T}{g}\sum_{1}^{\infty}D_s\left\{e^{-\frac{\beta_s l}{h}}\cosh\frac{\beta_s x}{h}-e^{-\frac{\beta_s x}{h}}\right\}-\frac{T}{g}\,\frac{3(2+\overline{K})}{2(3+\overline{K})}\,\frac{x-l}{h},$$
$$\text{for} \quad 0<x<l,$$

$$\zeta=-\frac{T}{g}\sum_{1}^{\infty}D_s\left\{e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}-e^{-\frac{\beta_s x}{h}}\right\} \qquad \text{for} \quad x>l,$$

$$\dots\dots\dots\dots(19)$$

where $$D_s=2\frac{\beta_s\tan\beta_s+\overline{K}(\sec\beta_s-1)}{\beta_s^2\{\beta_s+(\overline{K}+2)\tan\beta_s\}}.$$

Lastly from the horizontal velocity u given by eq. (17) and the eq. of continuity $\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$, we can obtain the vertical velocity of the current, w. Namely

$$\frac{\mu}{gh^2}w=\frac{T}{gh}\sum_{1}^{\infty}B_s\left\{e^{-\frac{\beta_s l}{h}}\cosh\frac{\beta_s x}{h}-e^{-\frac{\beta_s x}{h}}\right\}$$
$$\times\left\{\beta_s\cot\beta_s\sin\beta_s(1+\xi)-\beta_s\cos\beta_s(1+\xi)-\beta_s\xi\right\} \qquad \text{for} \quad 0<x<l,$$

$$\frac{\mu}{gh^2}w=\frac{T}{gh}\sum_{1}^{\infty}B_s\left\{e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}-e^{-\frac{\beta_s x}{h}}\right\}$$
$$\times\left\{\beta_s\cot\beta_s\sin\beta_s(1+\xi)-\beta_s\cos\beta_s(1+\xi)-\beta_s\xi\right\} \qquad \text{for} \quad x>l.$$

$$\dots\dots\dots\dots(20)$$

In the above obtained eqs. (17), (18) and (19), the 2nd parts are the solutions when the term of the horizontal "Austausch" $\mu\frac{\partial^2 u}{\partial x^2}$ is neglected, namely they are identical with the solutions given by Ekman. And the 1st parts may be considered as the correction term for Ekman's solution due to the consideration of $\mu\frac{\partial^2 u}{\partial x^2}$.

Now we shall regard the two extreme cases when $\overline{K}=\infty$ (no bottom-current) and $\overline{K}=0$ (no bottom-friction). Then in these cases (17), (18), (19) and (20) become respectively:

(a) *Case of no bottom-current*, $\overline{K}=\infty$.

$$\frac{\mu}{gh^2}u=\frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s x}{h}}+e^{-\frac{\beta_s l}{h}}\sinh\frac{\beta_s x}{h}\right\}$$
$$\times\left\{\beta_s\sin\beta_s(1+\xi)+\cos\beta_s(1+\xi)-1\right\}-\frac{T}{gh}\left\{\frac{3}{4}\xi^2+\xi+\frac{1}{4}\right\}$$
$$\text{for}\quad 0<x<l,$$

$$\frac{\mu}{gh^2}u=\frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s x}{h}}-e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}\right\}$$
$$\times\left\{\beta_s\sin\beta_s(1+\xi)+\cos\beta_s(1+\xi)-1\right\}\qquad\text{for}\quad x>l,$$
$$\dots\dots\dots\dots(17')$$

$$\text{where}\qquad B_s=2\frac{\sec\beta_s-1}{\beta_s^4},$$

and β_s are roots of eq. $\tan\beta=\beta$ which involves an infinite number of real roots.

$$\frac{\partial\zeta}{\partial x}=-\frac{T}{gh}\sum_1^\infty C_s\left\{e^{-\frac{\beta_s x}{h}}+e^{-\frac{\beta_s l}{h}}\sinh\frac{\beta_s x}{h}\right\}-\frac{3}{2}\cdot\frac{T}{gh}$$
$$\text{for}\quad 0<x<l,$$

$$\frac{\partial\zeta}{\partial x}=-\frac{T}{gh}\sum_1^\infty C_s\left\{e^{-\frac{\beta_s x}{h}}-e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}\right\}\qquad\text{for}\quad x>l,$$
$$\dots\dots\dots\dots(18')$$

$$\text{where}\qquad C_s=2\frac{\sec\beta_s-1}{\beta_s^2}.$$

$$\zeta=-\frac{T}{g}\sum_1^\infty D_s\left\{e^{-\frac{\beta_s l}{h}}\cosh\frac{\beta_s x}{h}-e^{-\frac{\beta_s x}{h}}\right\}-\frac{T}{g}\,\frac{3}{2}\,\frac{x-l}{h}$$
$$\text{for}\quad 0<x<l,$$

$$\zeta=-\frac{T}{g}\sum_1^\infty D_s\left\{e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}-e^{-\frac{\beta_s x}{h}}\right\}\qquad\text{for}\quad x>l,$$
$$\dots\dots\dots\dots(19')$$

where $D_s = 2\frac{\sec\beta_s - 1}{\beta_s^3}$.

$$\frac{\mu}{gh^2}w = \frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s l}{h}}\cosh\frac{\beta_s x}{h} - e^{-\frac{\beta_s x}{h}}\right\}$$

$$\times\left\{\sin\beta_s(1+\xi) - \beta_s\cos\beta_s(1+\xi) - \beta_s\xi\right\} \qquad \text{for} \quad 0<x<l,$$

$$\frac{\mu}{gh^2}w = \frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h} - e^{-\frac{\beta_s x}{h}}\right\}$$

$$\times\left\{\sin\beta_s(1+\xi) - \beta_s\cos\beta_s(1+\xi) - \beta_s\xi\right\} \qquad \text{for} \quad x>l,$$

............................(20′)

(b) *Case of no bottom-friction*, $\overline{K}=0$.

$$\frac{\mu}{gh^2}u = \frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s x}{h}} + e^{-\frac{\beta_s l}{h}}\sinh\frac{\beta_s x}{h}\right\}\cos\beta_s\xi$$

$$-\frac{T}{gh}\left\{\frac{1}{2}\xi^2 + \xi + \frac{1}{3}\right\} \qquad \text{for} \quad 0<x<l,$$

$$\frac{\mu}{gh^2}u = \frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s x}{h}} - e^{-\frac{\beta_s x}{h}}\cosh\frac{\beta_s l}{h}\right\} \qquad \text{for} \quad x>l,$$

............................(17″)

where $B_s = \frac{2}{\beta_s^2}$,

and β_s are roots of eq. $\sin\beta=0$ which involves an infinite number of real roots.

$$\left.\begin{aligned} &\frac{\partial\zeta}{\partial x} = -\frac{T}{gh} && \text{for} \quad 0<x<l, \\ &\frac{\partial\zeta}{\partial x} = 0 && \text{for} \quad x>l, \end{aligned}\right\} \quad \text{......(18″)}$$

$$\left.\begin{aligned} &\zeta = -\frac{T}{g}\frac{x-l}{h} && \text{for} \quad 0<x<l, \\ &\zeta = 0 && \text{for} \quad x>l, \end{aligned}\right\} \quad \text{......(19″)}$$

$$\frac{\mu}{gh^2}w = \frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s l}{h}}\cosh\frac{\beta_s x}{h} - e^{-\frac{\beta_s x}{h}}\right\}\sin\beta_s\xi$$

$$\text{for} \quad 0<x<l,$$

$$\frac{\mu}{gh^2}w=\frac{T}{gh}\sum_1^\infty B_s\left\{e^{-\frac{\beta_s l}{h}}\cosh\frac{\beta_s l}{h}-e^{-\frac{\beta_s x}{h}}\right\}\sin\beta_s\xi \quad \text{for} \quad x>l, \qquad \text{(20'')}$$

By these formulae for two extreme cases ($\overline{K}=\infty$ and $\overline{K}=0$), we can calculate numerically the horizontal and the vertical current u and w, the surface elevation ζ and the surface slope $\frac{\partial\zeta}{\partial x}$ without much labour. Tables 1 and 2 (which give only the values of the velocity near the sea wall, but the values near $x=l$ are omitted) and Figs 1, 2 and 3 show the results thus obtained, units of the current and the surface slope being taken as $\frac{Th}{\mu}$ and $\frac{T}{g}$ respectively.

In these figures, the scale of depth is taken to be the same as that of the horizontal length, and the vectors show the direction and magnitude of the current velocity.

Table 1

$\overline{K}=\infty$ (Case of no bottom-current)

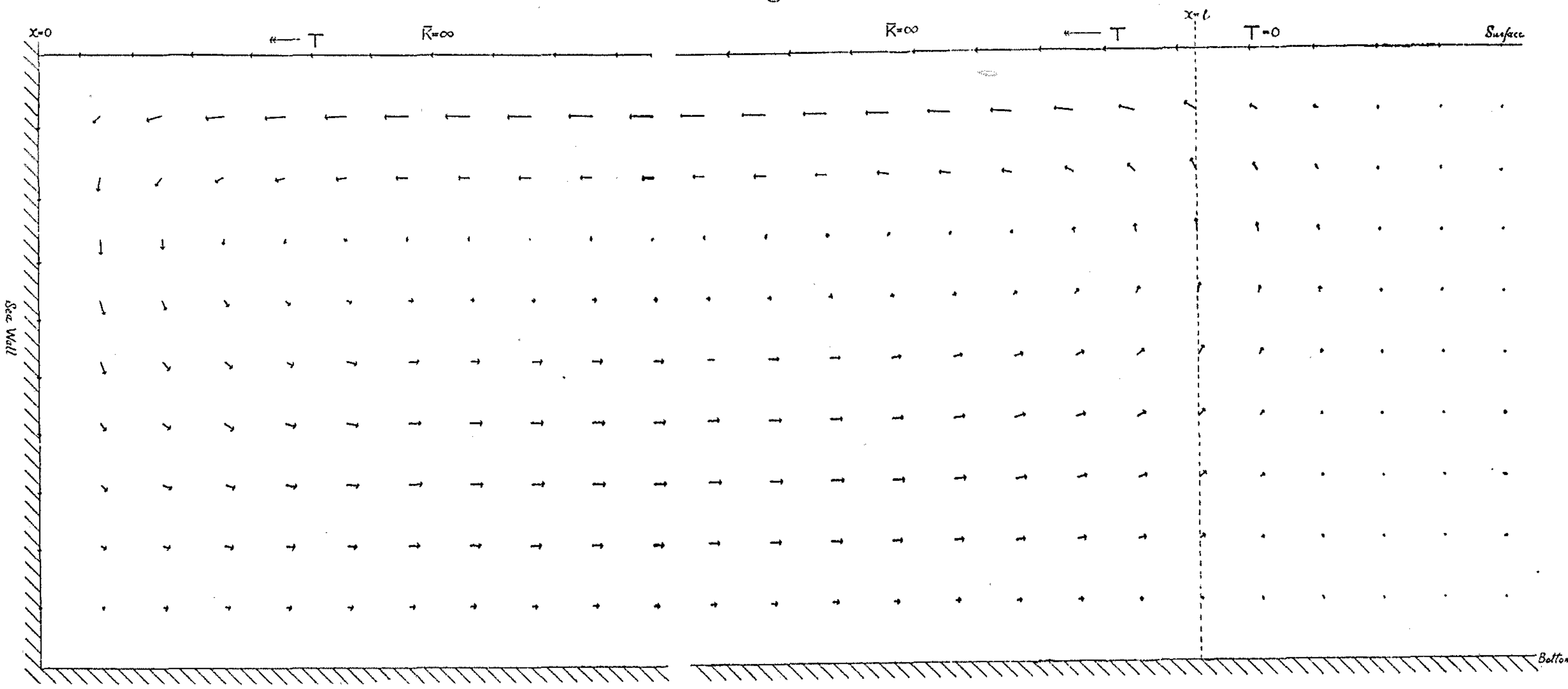
z/h \ x/h	0		0.1		0.2		0.3		0.4		0.5	
	u	w	u	w	u	w	u	w	u	w	u	w
0	0	0	.142	0	.186	0	.211	0	.228	0	.236	0
0.1	0	−.088	.058	−.048	.100	−.029	.122	−.014	.137	−.009	.145	−.006
0.2	0	−.153	.018	−.088	.039	−.052	.054	−.028	.064	−.017	.070	−.011
0.3	0	−.165	−.002	−.102	.001	−.063	.006	−.037	.008	−.023	.011	−.015
0.4	0	−.140	−.023	−.093	−.022	−.061	−.027	−.040	−.030	−.025	−.030	−.016
0.5	0	−.109	−.028	−.075	−.042	−.052	−.052	−.038	−.050	−.024	−.062	−.015
0.6	0	−.082	−.031	−.056	−.046	−.038	−.059	−.039	−.067	−.019	−.072	−.012
0.7	0	−.061	−.035	−.039	−.050	−.024	−.062	−.021	−.068	−.013	−.074	−.008
0.8	0	−.035	−.028	−.019	−.045	−.013	−.053	−.010	−.057	−.006	−.062	−.004
0.9	0	−.011	−.015	−.006	−.026	−.003	−.031	−.003	−.032	−.002	−.035	−.001
0.1	0	0	0	0	0	0	0	0	0	0	0	0

Table 2

$\overline{K}=0$ (Case of no bottom-friction)

z/h \ x/h	0		0.1		0.2		0.3		0.4		0.5		0.6		0.7		0.8		0.9		1.0	
	u	w	u	w	u	w	u	w	u	w	u	w	u	w	u	w	u	w	u	w	u	w
0	0	0	.145	0	.210	0	.246	0	.272	0	.291	0	.302	0	.311	0	.317	0	.321	0	.324	0
0.1	0	−.112	.070	−.061	.124	−.042	.157	−.029	.180	−.018	.198	−.013	.209	−.010	.217	−.007	.223	−.005	.227	−.004	.230	−.003
0.2	0	−.196	.030	−.113	.061	−.078	.086	−.055	.105	−.039	.119	−.025	.128	−.018	.135	−.013	.135	−.010	.143	−.007	.146	−.005
0.3	0	−.212	.021	−.145	.019	−.101	.043	−.072	.041	−.046	.053	−.034	.060	−.025	.065	−.018	.068	−.013	.071	−.010	.073	−.007
0.4	0	−.216	−.012	−.162	−.011	−.114	−.006	−.081	−.003	−.055	.000	−.040	.003	−.029	.006	−.021	.007	−.015	.009	−.011	.010	−.008
0.5	0	−.176	−.019	−.148	−.027	−.108	−.033	−.079	−.038	−.057	−.042	−.042	−.042	−.031	−.042	−.022	−.042	−.016	−.042	−.012	−.042	−.009
0.6	0	−.161	−.028	−.124	−.042	−.093	−.057	−.070	−.066	−.055	−.074	−.040	−.077	−.029	−.081	−.021	−.082	−.015	−.083	−.011	−.084	−.008
0.7	0	−.114	−.029	−.093	−.058	−.073	−.072	−.056	−.081	−.046	−.097	−.034	−.104	−.025	−.109	−.018	−.112	−.013	−.115	−.010	−.117	−.007
0.8	0	−.088	−.038	−.061	−.069	−.050	−.086	−.039	−.103	−.039	−.113	−.025	−.122	−.018	−.129	−.013	−.134	−.010	−.137	−.007	−.140	−.005
0.9	0	−.036	−.036	−.023	−.072	−.021	−.092	−.018	−.109	−.018	−.112	−.013	−.133	−.010	−.141	−.007	−.147	−.005	−.151	−.004	−.154	−.003
1.0	0	0	−.041	0	−.074	0	−.096	0	−.114	0	−.145	0	−.136	0	−.145	0	−.151	0	−.155	0	−.158	0

Fig. 1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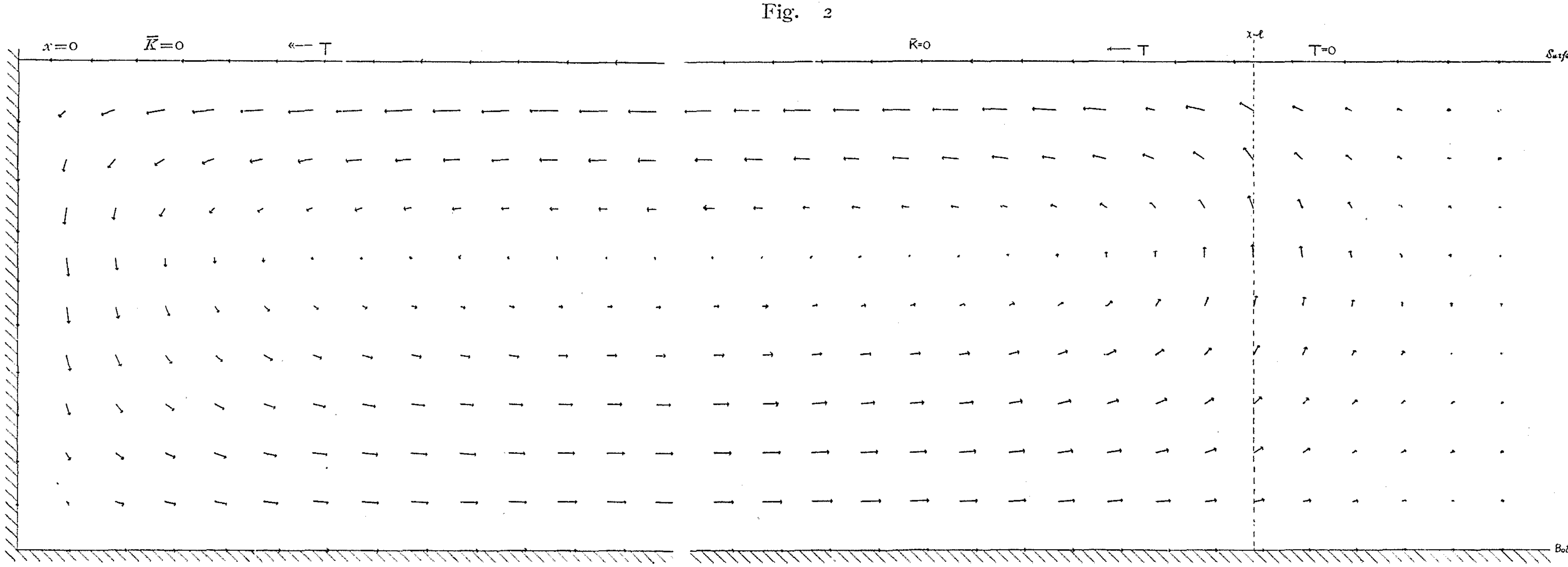
← T
K̄=∞
K̄=∞
← T
x=l
T=0
Surface
Sea Wall
Bottom

Fig. 2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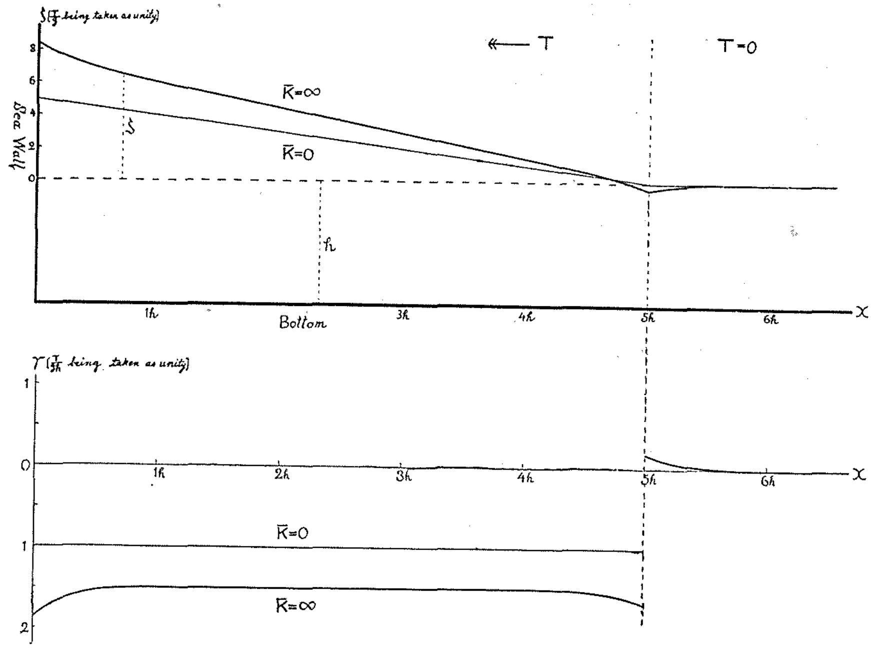
K̄=0
← T
K̄=0
← T
x=l
T=0
Surface
Sea Wall
Bottom
Bottom

Fig. 3



Conclusion

Let us enumerate the important points which can be seen from the above Figs.

1) Near the coast ($x=0$) or the place ($x=l$) where the wind traction changes abruptly from value T to zero, water particles circulate in the vertical section of the water basin. The horizontal velocity u vanishes at the coast itself, but it changes continuously at the place $x=l$. Thus our principal object for the boundary value problem has been attained.

2) A perceptible vertical motion is limited in the immediate vicinity of the coast or of the place of the abruptly changing wind; a little apart the current becomes much the same as that in Ekman's theory. Indeed the horizontal range where some correction to Ekman's theory will be needed is only a half or equal to the depth of the sea.

3) Regarding the surface elevation from the mean sea level, our solution is much the same as that of Ekman's theory. When $\overline{K}=0$, no correction is needed and when $\overline{K}=\infty$, a slight correction will be wanted as seen in Fig. 3, in which the correction value is exagerated

to ten times the calculated values. The surface slope, however, must be corrected somewhat near the coast and $x=l$.

4) After all, from the practical purpose our result indicates no marked difference from Ekman's theory, except the generation of the circulation near the coast and $x=l$, but makes some advance from the theoretical standpoint.

In a similar manner, we may obtain formally the solution for a rotating sea; but from the above reason we find no necessity to discuss it in detail.

In conclusion, the writer wishes to express his sincere thanks to Prof. T. Nomitsu for his kind advice, and also to Assistant Prof. T. Namekawa for his valuable suggestions.