On the Group of Zero-Sequence

By

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1. Given a sequence of an infinite number of real numbers, $\{u_\nu\}$, we rearrange its terms by the substitution

$$S \equiv \begin{pmatrix} u_1 & u_2 & \cdots\cdots\cdots & u_\nu & \cdots\cdots\cdots \\ u_{s_1} & u_{s_2} & \cdots\cdots\cdots & u_{s_\nu} & \cdots\cdots\cdots \end{pmatrix},$$

or more simply by

$$S \equiv \begin{pmatrix} 1 & 2 & \cdots\cdots\cdots & \nu & \cdots\cdots\cdots \\ s_1 & s_2 & \cdots\cdots\cdots & s_\nu & \cdots\cdots\cdots \end{pmatrix}$$

where the transformed sequence is $\{u_{s_\nu}\}$. If the series $\sum_{\nu=1}^{\infty}(u_\nu - u_{s_\nu})$ be absolutely convergent, then we say the sequence $\{u_\nu\}$ admits the substitution S. The totality of all such substitutions forms the group of the sequence $\{u_\nu\}$. If the sequence be a zero-sequence, i. e., if $\lim_{\nu\to\infty} u_\nu = 0$, then the group of the sequence $\{u_\nu\}$ is also called the group of the series (convergent or divergent)

$$\sum u_\nu \equiv u_1 + u_2 + \cdots\cdots + u_\nu + \cdots\cdots$$ [1]

In this note we shall consider only zero-sequences. A group is said to be admitted by a sequence provided all its substitutions are admitted by the sequence. Such a group is a subgroup of the group of the sequence.

2. If $\sum u_\nu$ be absolutely convergent, then it admits all possible substitutions, i. e., the symmetric group. Conversely

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1. Toshizô Matsumoto. Group-theory of semi-convergent series. These Memoirs, X, No. 5 (1927)
   " Group-theory of sequences of numbers. *Ibid*, XI, No. 1 (1928)
   " On the decomposition of substitutions of groups of sequences, *Ibid*, XIII, No. 2 (1930)

If the sequence $\{u_\nu\}$ admits the symmetric group, then the series $\sum u_\nu$ is absolutely convergent.

I enunciated this theorem in my first paper without proof. When the sequence $\{u_\nu\}$ has an infinite number of terms of both signs, it is evident.[1] In the other case, we may suppose $u_\nu > 0$, $\nu = 1, 2, \ldots\ldots$ without loss of generality. Since $u_\nu \to 0$, it gives a partial sequence $\{u_{n_\nu}\}$ such that $\sum u_{n_\nu}$ is convergent. Let $\{u_{m_\nu}\}$ be the sequence with terms remaining in $\{u_\nu\}$ after the rejection of u_{n_ν}, $\nu = 1, 2, \ldots.$ Then $\sum u_{m_\nu}$ must be divergent, or else $\sum u_\nu$ would be convergent. Now by the substitution

$$(m_1\ n_1)(m_2\ n_2)\ldots\ldots(m_\nu\ n_\nu)\ldots\ldots$$

the series of the differences of the absolute values of the corresponding terms will be

$$2\sum |u_{m_\nu} - u_{n_\nu}| \geqslant 2(\sum u_{m_\nu} - \sum u_{n_\nu}) = \infty.$$

This contradicts the hypothesis; so that the series $\sum u_\nu$ must be convergent.

If $\{u_\nu\}$ is not a zero-sequence, the theorem is not true.[2]

3. If the sequence $\{u_\nu\}$ admits the substitution $S \equiv \binom{\nu}{s_\nu}$ i. e., if the series $\sum |u_\nu - u_{s_\nu}|$ eb convergent, the series $\sum \big|\, |u_\nu| - |u_{s_\nu}| \,\big|$ is *a fortiori* convergent. Therefore the sequence $\{|u_\nu|\}$ admits the substitution S. Hence the group of $\{u_\nu\}$ is a subgroup of that of $\{|u_\nu|\}$.

We shall consider in the following the *positive zero-sequences*, i. e., those whose terms are all positive and tend to zero.

4. *Let $\{u_\nu\}$ be a positive zero-sequence, then the group of $\{u_\nu\}$ is a proper subgroup (divisor) of that of the sequence $\{u_\nu^\rho\}$, $(\rho > 1)$,* in so far as the series $\sum u_\nu$ is divergent.

Let the substitution $S \equiv \binom{\nu}{s_\nu}$ be any substitution admitted by the sequence $\{u_\nu\}$, it can easily be seen that the sequence $\{u_\nu^\rho\}$ also admits S. For let A be the upper bound of the sequence $\{u_\nu\}$, then

$$|u_\nu^\rho - u_{s_\nu}^\rho| < \rho A^{\rho-1} |u_\nu - u_{s_\nu}|, \quad (\rho > 1).$$

Hence the series $\sum |u_\nu^\rho - u_{s_\nu}^\rho|$ is convergent, since by hypothesis the series $\sum |u_\nu - u_{s_\nu}|$ is convergent.

To prove that there is a substitution admitted by $\{u_\nu^\rho\}$, but not by $\{u_\nu\}$, $\sum u_\nu$ being divergent, at first we note that if $A_1, A_2, \ldots A_n$ be positive, then for any number $\lambda > 1$, we have the inequality

1. *Loc. cit.*, the first paper, p. 220.
2. *Loc. cit.*, the second paper, p. 12.

$$A_1^\lambda + A_2^\lambda + \dots + A_n^\lambda < (A_1 + A_2 + \dots + A_n)^\lambda.$$

Now choose some terms of $\{u_\nu\}$ (in the order of increasing indeces) such that

$$1 \geqslant u_{11} + u_{12} + \dots + u_{1k_1} > \frac{1}{2},$$

where $u_{11}, u_{12}, \dots, u_{1k_1}$ are certain terms of $\{u_\nu\}$, only the notations of the indeces being changed. This is always possible, since $u_\nu \to 0$ and $\sum u_\nu = \infty$. Again we may choose some terms from the remaining terms of $\{u_\nu\}$ such that

$$\frac{1}{2} \geqslant u_{21} + u_{22} + \dots + u_{2k_2} > \frac{1}{3},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{1}{n} \geqslant u_{n1} + u_{n2} + \dots + u_{nk_n} > \frac{1}{n+1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

In choosing these terms we may assume that

$$k_1 \leqslant k_2 \leqslant \dots\dots \leqslant k_n \leqslant \dots\dots$$

For after $u_{11}, u_{12}, \dots, u_{1k_1}$ is chosen, we choose the next group of such terms $u_{21}, u_{22}, \dots, u_{2k_2}$ among the remaining terms of $\{u_\nu\}$ as

$$k_1 u_\nu \leqslant \frac{1}{2}.$$

This is possible since $u_\nu \to 0$. Then we have

$$u_{21} + u_{22} + \dots + u_{2k_1} \leqslant k_1 \times \frac{1}{2k_1} = \frac{1}{2};$$

and we add some terms if necessary so that

$$\frac{1}{2} \geqslant u_{21} + u_{22} + \dots + u_{2k_2} > \frac{1}{3}.$$

This is possible since $u_\nu \to 0$ and $\sum u_\nu = \infty$. Therefore

$$k_2 \geqslant k_1.$$

This consideration is quite general. So we may suppose that

$$k_1 \leqslant k_2 \leqslant \dots\dots \leqslant k_n \leqslant \dots\dots$$

Now put for simplicity

$$U_n \equiv u_{n1} + u_{n2} + \dots + u_{nk_n},$$

$$W_n \equiv u_{n1}^\rho + u_{n2}^\rho + \dots + u_{nk_n}^\rho, \quad n = 1, 2, \dots\dots$$

Then by the inequality remarked above

$$U_n^\rho = (u_{n1} + u_{n2} + \ldots + u_{nk_n})^\rho$$

$$> u_{n1}^\rho + u_{n2}^\rho + \ldots + u_{nk_n}^\rho = W_n.$$

Since $\frac{1}{n} \geqslant U_n$ and hence $\frac{1}{n^\rho} \geqslant U_n^\rho$, we have

$$\frac{1}{n^\rho} > W_n.$$

Thus we see the series $\sum_{n=1}^{\infty} U_n$ is divergent while the series $\sum_{n=1}^{\infty} W_n$ is convergent.

Now since $U_n \to 0$, we may take a partial sequence $\{U_{n_i}\}$ in order from $\{U_n\}$ such that the series

$$U_{n_1} + U_{n_2} + \ldots + U_{n_i} + \ldots\ldots$$

is convergent. Here we may suppose that

$$n_1 > 1,\ n_2 > n_1 + 1, \ldots,\ n_{i+1} > n_i + 1, \ldots ;$$

for the sequence $\{U_n\}$ is monotone decreasing, and to select a convergent series, a smaller U_n, i. e., U_n with a greater index is more effective. Let $\{U_{m_i}\}$, $(m_1 = 1)$, be the remaining sequence in order, then the series

$$U_{m_1} + U_{m_2} + \ldots + U_{m_i} + \ldots\ldots$$

is divergent, since $\sum U_n = \sum U_{m_i} + \sum U_{n_i} = \infty$. By our choice of n_1, $n_2, \ldots\ldots$, we have

$$m_i < n_i, \quad i = 1,\ 2, \ldots\ldots$$

Therefore the number of terms in U_{m_i} is not greater than that in U_{n_i} i. e.,

$$k_{m_i} \leqslant k_{n_i}, \quad i = 1,\ 2, \ldots\ldots$$

Now put

$$U'_{n_i} \equiv u_{n_i 1} + u_{n_i 2} + \ldots + u_{n_i k_{m_i}},$$

then since

$$U'_{n_i} \leqslant U_{n_i}, \quad i = 1,\ 2, \ldots\ldots$$

the series $U'_{n_1} + U'_{n_2} + \ldots + U'_{n_i} + \ldots\ldots$

is also convergent. Now consider the substitutions

$$T_i \equiv (u_{m_i 1},\ u_{n_i 1}) \ldots\ldots (u_{m_i k_{m_i}},\ u_{n_i k_{m_i}}),$$

$$(i = 1,\ 2, \ldots\ldots)$$

$$T \equiv T_1 T_2 \ldots\ldots T_i$$

The series of the absolute values of differences of corresponding terms of the sequence $\{u_\nu\}$ and its transformed one by T is

$$2\sum_{i=1}^{\infty}\{|u_{m_i 1}-u_{n_i 1}|+\ldots+|u_{m_i k_{m_i}}-u_{n_i k_{m_i}}|\}$$
$$\geqq 2\sum_{i=1}^{\infty}(U_{m_i}-U'_{n_i})=\infty.$$

Thus the sequence $\{u_\nu\}$ does not admit the substitution T.

On the other hand

$$2\sum_{i=1}^{\infty}\{|u^{\rho}_{m_i 1}-u^{\rho}_{n_i 1}|+\ldots+|u^{\rho}_{m_i k_{m_i}}-u^{\rho}_{n_i k_{m_i}}|\}$$
$$<2\sum_{i=1}^{\infty}(W_{m_i}+W_{n_i}).$$

This is convergent, since $\sum_{n=1}^{\infty}W_n$ is convergent.

Thus we see there is at least a substitution T in the group of the sequence $\{u^{\rho}_\nu\}$ which is not admitted by the sequence $\{u_\nu\}$. We conclude therefore the group of $\{u_\nu\}$ is a proper subgroup of that of $\{u^{\rho}_\nu\}$, $(\rho>1)$.

5. We would conjecture that the group of $\{u_\nu\}$ is a normal subgroup of that of $\{u^{\rho}_\nu\}$, $(\rho>1)$. But this is not true. For as we have proved there is a partial sequence, say, $\{u_{n_i}\}$ of $\{u_\nu\}$ such that $\sum u_{n_i}$ is divergent while $\sum u^{\rho}_{n_i}$ is convergent. Hence suppose from the beginning that $\sum u_\nu$ is divergent while $\sum u^{\rho}_\nu$ is convergent, $(\rho>1)$, then clearly it gives a cycle, say

$$C\equiv(\ldots\ldots c_{-2}\ c_{-1}\ c_1\ c_2\ldots\ldots)$$

admitted by the sequence $\{u_\nu\}$ where c_1, c_{-1}, c_2, c_{-2},......are as a whole equal to 1, 2, 3, 4,...... Also we can easily construct a cycle, say

$$D\equiv(\ldots\ldots d_{-2}\ d_{-1}\ d_1\ d_2\ldots\ldots)$$

not admitted by the sequence $\{u_\nu\}$, where d_1, d_{-1}, d_2, d_{-2},......are as a whole equal to 1, 2, 3, 4,...... Now put

$$T\equiv\begin{pmatrix} c_1 & c_{-1} & c_2 & c_{-2}\ldots\ldots\ldots \\ d_1 & d_{-1} & d_2 & d_{-2}\ \ldots\ldots \end{pmatrix}.$$

Since $\sum u^{\rho}_\nu$ is convergent, $\{u^{\rho}_\nu\}$ admits T, and

$$T^{-1}CT=D$$

which is not admitted by $\{u_\nu\}$. It is clear that T is not admitted by $\{u_\nu\}$ or else D should be admitted by it. Therefore the group of $\{u_\nu\}$ cannot be a normal subgroup of that of $\{u^{\rho}_\nu\}$, $(\rho>1)$.

6. We may extend the theorem proved in N°4 to the sequences $\{\varepsilon_\nu u_\nu\}$ and $\{\varepsilon_\nu u_\nu^\rho\}$, $(\rho>1)$ where $\{u_\nu\}$ is as before a positive zero-sequence and ε_ν is $+1$ or -1. It is clear that the group of $\{\varepsilon_\nu u_\nu\}$ is a subgroup of that of $\{\varepsilon_\nu u_\nu^\rho\}$. For by $S\equiv\binom{\nu}{s_\nu}$, let $\sum|\varepsilon_\nu u_\nu-\varepsilon_{s_\nu}u_{s_\nu}|$ be convergent, then since $u_\nu\to 0$, considering only the terms that u_ν, u_{s_ν} are sufficiently small, $(u_\nu,\ u_{s_\nu}<e^{-1})$, the series $\sum|\varepsilon_\nu u_\nu^\rho-\varepsilon_{s_\nu}u_{s_\nu}^\rho|$ is also convergent. This can easily be shown by the method of maximum and minimum Since $\sum u_\nu$ is supposed to be divergent, there is a partial sequence $\{\varepsilon_{n_i}u_{n_i}\}$ of $\{u_\nu\}$ whose terms are of the same sign and $\sum\varepsilon_{n_i}u_{n_i}$ is divergent. Applying the theorem of N°4 to the sequences $\{\varepsilon_{n_i}u_{n_i}\}$ and $\{\varepsilon_{n_i}u_{n_i}^\rho\}$ we may conclude that

The group of the zero-sequence $\{\varepsilon_\nu u_\nu\}$ is a proper subgroup of that of the sequence $\{\varepsilon_\nu u_\nu^\rho\}$, $(\rho>1)$.

This subgroup is not a normal subgroup. The following example is to be noted.

$$\{\varepsilon_\nu u_\nu\}\equiv\left\{(-1)^{\nu-1}\frac{1}{\nu}\right\},$$

$$\{\varepsilon_\nu u_\nu^\rho\}\equiv\left\{(-1)^{\nu-1}\frac{1}{\nu^\rho}\right\},\quad(\rho>1)$$

$$S\equiv(1\ \ 3)(4\ \ 6)(7\ \ 9)\ldots\ldots(3n-2\ \ 3n)\ldots\ldots\ldots,$$
$$T\equiv(1\ \ 2)(4\ \ 5)(7\ \ 8)\ldots\ldots(3n-2\ \ 3n-1)\ldots\ldots\ldots$$

Since the series $\sum(-1)^{\nu-1}\frac{1}{\nu^\rho}$ is absolutely convergent, it admits the symmetric group; while the sequence $\left\{(-1)^{\nu-1}\frac{1}{\nu}\right\}$ admits the substitution S but not the substitution T. Now

$$\begin{aligned}T^{-1}ST&=(1\ \ 2)(1\ \ 3)(1\ \ 2)\times(4\ \ 5)(4\ \ 6)(4\ \ 5)\times(7\ \ 8)(7\ \ 9)(7\ \ 8)\\&\quad\times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&\quad\times(3n-2\ \ 3n-1)(3n-2\ \ 3n)(3n-2\ \ 3n-1)\times\ldots\ldots\\&=(2\ \ 3)(5\ \ 6)(8\ \ 9)\ldots(3n-1\ \ 3n)\ldots\ldots\ldots\end{aligned}$$

and this is not admitted by $\left\{(-1)^{\nu-1}\frac{1}{\nu}\right\}$.

7. *Let $\{u_\nu\}$ and $\{v_\nu\}$ be two zero-sequences (not necessarily positive), such that for any two indeces ν, μ we have either $u_\nu\geqq u_\mu$ and $v_\nu\geqq v_\mu$ at the same time or $u_\nu\leqq u_\mu$ and $v_\nu\leqq v_\mu$ at the same time, then the set of all substitutions admitted by $\{u_\nu\}$ and $\{v_\nu\}$ at the same time forms a group.*

For let $S \equiv \binom{\nu}{s_\nu}$ be admitted by $\{u_\nu\}$ and $\{v_\nu\}$ at the same time, i. e., $\sum |u_\nu - u_{s_\nu}|$ and $\sum |v_\nu - v_{s_\nu}|$ be convergent, then since

$$\sum |(u_\nu + v_\nu) - (u_{s_\nu} + v_{s_\nu})|$$
$$= \sum \{ |u_\nu - u_{s_\nu}| + |v_\nu - v_{s_\nu}| \},$$

the sequence $\{u_\nu + v_\nu\}$ also admits the substitution S. Conversely any substitution admitted by $\{u_\nu + v_\nu\}$ is also admitted by $\{u_\nu\}$ and $\{v_\nu\}$ at the same time. The set of all substitutions admitted by $\{u_\nu + v_\nu\}$ forms a group. Hence the set of all substitutions common to both groups of $\{u_\nu\}$ and $\{v_\nu\}$ forms a group.