# An Integral Equation with a Parameter

By

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Mr. A. Kawaguchi derived the integral equation

$$\varphi(x)=\lambda A(x)\varphi(x)+\lambda\int_a^b K(x, y)\varphi(y)dy, \tag{1}$$

where $K(x, y)$ being finite and continuous in the square-domain $D$ $a\leqq x\leqq b$, $a\leqq y\leqq b$, moreover real and symmetric; $\lambda$ being a parameter $[|\lambda|<+\infty]$.

In this paper, I will prove the following

Theorem. *When $A(x)$ has the same sign throughout the interval and $|A(x)|$ always small values, there exists at least a solution, which does not vanish identically, of the integral equation*

$$\varphi(x)=\lambda A(x)\varphi(x)+\lambda\int_a^b K(x, y)\varphi(y)dy.$$

For the purpose of simplifying the demonstration, we denote Fredholm's determinant of $A(x, y)$ by

$$D[A]=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_a^b\cdots\int_a^b A\begin{pmatrix} z_1, z_2, \ldots\ldots, z_n \\ z_1, z_2, \ldots\ldots, z_n\end{pmatrix}dz_1dz_2\ldots\ldots dz_n.^{1}$$

Let

$$L(x, y; \lambda)=\frac{\lambda K(x, y)}{1-\lambda A(x)}, \tag{2}$$

then the integral equation (1) will take the form

$$\varphi(x)=\lambda\int_a^b L(x, y; \lambda)\,\varphi(y)dy, \tag{3}$$

therefore our problem is reduced to the investigation about zeros of

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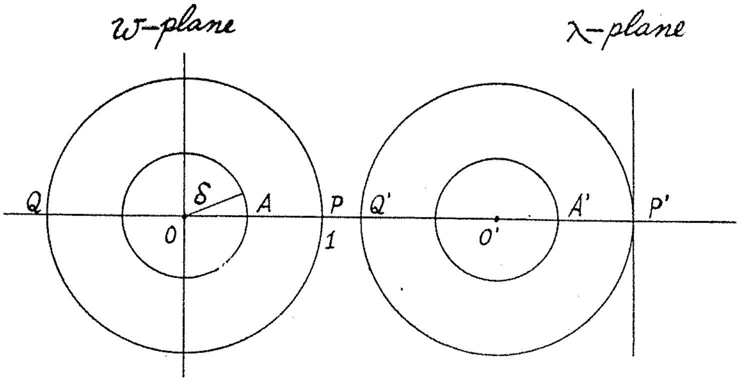
1. It is usual to be denoted by $D_A$.

Fredholm's determinant $D[\lambda L]$ i. e. $D\left[\frac{\lambda K(x, y)}{1-\lambda A(x)}\right]$. In order that the integral or some iterated integral of $L(x, y; \lambda)$ in $D$ for any $\lambda$ may exist, the set of $\lambda$ which satisfies the equation $1-\lambda A(x)=0$ for any $x$ must be excluded from the all $\lambda$-plane.

Now suppose $A(x)>0$ for any $x$. From giving such a number $\delta>0$ independent on $x$ and $\lambda$ that $1-\lambda A(x_0)=w$, $|w|\geqq\delta$ for $x=x_0$, we consider their correspondence (circle-circle) between $w$- and $\lambda$-plane. Then a general state of these results will appear as Fig. 1 in which dashes show each pair of corresponding points.

Fig. 1




$$\left.\begin{aligned} &OA=\delta, \\ &O'A'=\delta/A_0, \\ &OP=1, \\ &O'P'=1/A_0, \\ &OQ=1, \\ &P'Q'=2/A_0, \\ &A(x_0)\equiv A_0. \end{aligned}\right\}$$

Now when we denote the closed domain bounded by a circle with its radius $\delta/A_0$, having the point $O'$ as its center by $\Omega(x_0)$, and the compliment $\Omega'(x_0)$ in the $\lambda$-plane, then the function $L(x_0, y; \lambda)$ will be regular with respect to $\lambda$ for any value of $y$. As the point $x_0$ moves throughout the interval $(a, b)$, a certain domain will be swept by the circular domain $\Omega(x_0)$, say $E$, then $\Omega(x)\subset E$ for all values of $x$. Let

$$\beta\geqq A(x)\geqq\alpha>0,$$

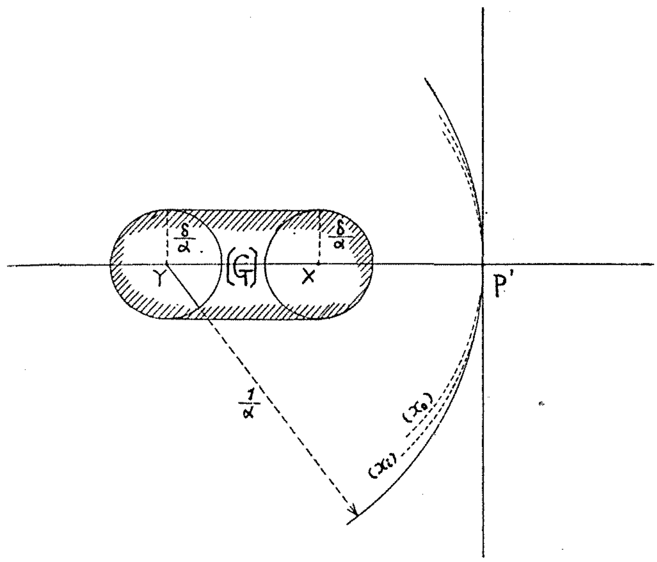
then evidently

$$\frac{\delta}{\beta} \leqq \frac{\delta}{A(x)} \leqq \frac{1}{\alpha}, \quad \frac{1}{\beta} \leqq \frac{1}{A(x)} = O'P' \leqq \frac{1}{\alpha}, \quad \text{and}$$

$$\frac{2}{\beta} \leqq \frac{2}{A(x)} = P'Q' \leqq \frac{2}{\alpha}$$

$$\begin{cases} XP' = \dfrac{1}{\beta}, \\ YP' = \dfrac{1}{\alpha}. \end{cases}$$

Fig. 2




Next when we denote the domain swept by the circular domain with the radius $\delta/\alpha$, as the center goes from $Y$ to $X$, by $G$, then we have $E \leqq G$.

Thus for any point $(x, y)$ in $D$ the function $L(x, y; \lambda)$ becomes regular with respect to $\lambda$ everywhere in the compliment $G'$ of $G$. Let $L_n(x, y; \lambda)$ be the $n$-th iterated kernel of $L(x, y; \lambda)$, and $l(x, y; \lambda, \mu)$ the reciprocal kernel; namely

$$l(x, y; \lambda, \mu) = -\sum_{h=0}^{\infty} \mu^h L_{h+1}(x, y; \lambda).$$

In view of the construction of these iterated kernels, it is clear that $L_{h+1}(x, y; \lambda)$ will be a regular function of $\lambda$ in $G'$ for any point $(x, y)$

in $D$, therefore $l(x, y; \lambda, \mu)$ will also be a regular function of $\lambda$ for any point $(x, y)$ in $D$ and within the convergence-circle of $\mu$.

Now for a provisional fixed value of $\lambda$ in $G'$, we have

$$\frac{\frac{d}{d\mu}D[\mu L]}{D[\mu L]} = -\sum_{h=0}^{\infty}\mu^h l_{h+1}(\lambda)$$

$$l_{h+1}(\lambda) = \int_a^b L_{h+1}(x, x; \lambda)dx,$$

where $D[\mu L]$ expresses a transcendental integral function of $\mu$ for a provisional fixed point $\lambda$ in $G'$, and at the same time a regular function of $\lambda$ for a provisional fixed point $\mu$.

Thus we can prove that *the kernel $L(x, y; \lambda)$ has at least one characteristic constant in the neighbourhood of $\lambda = 0$.* In other words it is sufficient for us that the spur $l_4(\lambda)$ does not vanish in the neighbourhood of $\lambda = 0$. Since

$$l_4(\lambda) = \iiiint L(x, u; \lambda)L(u, t; \lambda)L(t, v; \lambda)L(v, x; \lambda)du\, dv\, dt\, dx$$

and $L(x, y; \lambda)$ is regular with respect to $\lambda$ in $G'$, on account of (2),

$$l_4(0) = \iiiint K(x, u)\, K(u, t)\, K(t, v)\, K(v, x)\, du\, dv\, dt\, dx$$

$$= \iint [K_2(x, t)]^2 dt\, dx.$$

Therefore $l_4(0) \geqq 0$. Now if $l_4(0) = 0$, then for any $x$ and $t$

$$0 = K_2(x, t) = \int_a^b K(x, u)\, K(u, t)du; \quad \text{accordingly}$$

$$0 = \int_a^b [K(x, u)]^2 du.$$

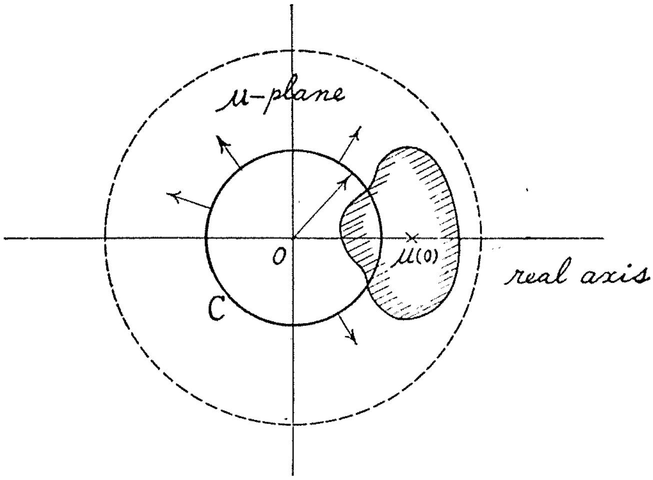
Hence we have $K(x, u) \equiv 0$ for any $x$ and $u$. This is absurd.

Consequently we can conclude that for any value of $\lambda$ within a circle $C$, having the point $\lambda = 0$ as its center, there exists always the zero $\mu(\lambda)$ which satisfies the equation $D[\mu L] = 0$. Especially when we take the one with the least modulus as $\mu(\lambda)$, the function $\mu(\lambda)$ is finite and regular in $C$ from the above stated properties of $D[\mu L]$; namely the existence-domain of $\mu(\lambda)$ is the circular domain which is bounded by $C$. Since $\mu(0)$ is evidently a characteristic constant of the symmetric kernel $K(x, y)$, it is a real number.

Now let $g$ be a certain real number, which $|g|$ may be taken so great that

$$|g\lambda| \geqq |\mu(\lambda)| \quad \text{on } C.$$

Fig. 3




Thus when we consider a new kernel $\frac{1}{g}K(x, y)$ instead of $K(x, y)$, we see at once that there will exist a corresponding characteristic constant $\mu_1(\lambda)$ to $\mu(\lambda)$, and evidently $\mu_1(\lambda)=\frac{1}{g}\mu(\lambda)$. Hence we have $|\lambda| \geqq |\mu_1(\lambda)|$ on $C$.

On the other hand, let $\beta$ be so small sufficiently that the domain $G$ will be pushed out of the circle $C$, then the above stated circular domain will be entirely involved into $G'$. Thus our theorem has been proved. Similarly when $A(x)$ has the negative sign through the interval, the theorem will also hold.

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