On Abel's Integral Equation

By

Tunezô Satô

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Introduction: In general the following singular integral equation, which is derived from the tautochrone problem in mechanics,

$$(1) \qquad \int_{+0}^{x} \frac{\varphi(s)}{(x-s)^{\alpha}} ds = f(x) \qquad (0<\alpha<1)$$

is called Abel's integral equation[1], in which $\varphi(s)$ means an unknown function and $f(x)$ a given real function of a real variable x, defined over the open interval (to the left) $0<x\leqq a$ which we call it I.

Now if we suppose that $f(x)$ be integrable in I, we have

$$(2) \qquad \varphi(x) = \frac{\sin \alpha\pi}{\pi} \frac{\partial}{\partial x} \int_{+0}^{x} \frac{f(z)}{(x-z)^{1-\alpha}} dz$$

as the unique and continuous solution.

But moreover if we allow the existence of the limit $f(+0)$ and the value of the integral:

$$\int_{+0}^{x} \frac{f'(z)}{(x-z)^{1-\alpha}} dz,$$

the above formula (2) will be written as follows:

$$(3) \qquad \varphi(x) = \frac{\sin \alpha\pi}{\pi} \left\{ \frac{f(+0)}{x^{1-\alpha}} + \int_{+0}^{x} \frac{f'(z)dz}{(x-z)^{1-\alpha}} \right\}.$$

This is a well-known formula, given by Mr. E. Goursat[2].

1. Maxime Bôcher; An Introduction To The Study of Integral Equations, Cambridge Trac. in Math. and Math. Phys., 1926, pp. 6-11.

2. Acta Math., Vol. 27 (1903), pp. 131-133.

Next, when the limits of $f(+0)$, $f'(+0)$,.........and $f^{(n)}(+0)$ exist, and also the integral $\int_{+0}^{x}(x-z)^{\alpha+n-1}f^{(n+1)}(z)dz$ exists, then the above formula (3) will take the following form

$$(4)\qquad \varphi(x)=\frac{\sin \alpha\pi}{\pi}\left\{\frac{f(+0)}{x^{1-\alpha}}+\frac{f'(+0)}{\alpha}x^{\alpha}+\frac{f''(+0)}{\alpha(\alpha+1)}x^{\alpha+1}+\ldots\ldots\right.$$
$$+\frac{f^{(n)}(+0)}{\alpha(\alpha+1)\ldots(\alpha+n-1)}x^{\alpha+n-1}$$
$$\left.+\frac{1}{\alpha(\alpha+1)\ldots(\alpha+n-1)}\int_{+0}^{x}(x-z)^{\alpha+n-1}f^{(n+1)}(z)dz\right\}.$$

Lately by generalizing the exponent α of $(x-s)^{-\alpha}$ in the integrand of (1), Mr. R. Rothe[1] discussed the functional equation

$$(5)\qquad \int_{+0}^{x}(x-s)^{\beta}\varphi(s)ds=f(x)\qquad\qquad(\beta>-1)$$

instead of (1). Evidently there will occur Abel's mechanical problem for the special case $\beta=-\frac{1}{2}$ in this formula. Mr. Rothe shows as another example Toricelli's law which holds the special case $\beta=\frac{1}{2}$, while before this Mr. N. Hirakawa[2] discussed the same case by deriving from another mechanical problem, but this method is quite different from that of the former.

Now taking such a number $\mu>-1$ as

$$(6)\qquad \mu+\beta+1=n\qquad\qquad (n \text{ being any non-negative integer}),$$

if we allow the existence of $\frac{\partial^{n+1}}{\partial s^{n+1}}\int_{+0}^{s}(s-x)^{\mu}f(x)dx$, we shall have the unique solution of (5):

$$(7)\qquad \varphi(s)=\frac{1}{\Pi(\mu)\Pi(\beta)}\,\frac{\partial^{n+1}}{\partial s^{n+1}}\int_{+0}^{s}(s-x)^{\mu}f(x)dx,$$

where $\Pi(x)$ expresses the Gaussian Pifunction for all the values of x, excepting only $x=-1, -2, -3, \ldots\ldots\ldots\ldots\ldots$.

1. In order to generalize the functional equation (1), we may consider the following formula:

1. Zur Abelschen Integralgleichung, Math. Zeitschr. Bd. 33 (1931) pp. 375-87.
2. On a Simple Integral Equation, The Tôhoku Math. Journ. Vol. 7 (1915) pp. 38-41.

$$(8)\qquad \int_{+0}^{x}\int_{+0}^{s_1}\cdots\int_{+0}^{s_{n-1}}\frac{\varphi(s_n)ds_n ds_{n-1}\ldots\ldots ds_1}{(x-s_1)^{\alpha_1}(s_1-s_2)^{\alpha_2}\ldots\ldots(s_{n-1}-s_n)^{\alpha_n}}=f(x)$$

under restrictions: $0<\alpha_s'<1$; $f(x)$ being given and $\varphi(x)$ an unknown function to be required. Conjecturing from this form, we may for the present call this the n-th repeated Abel's integral equation. And for the sake of convenience, we adopt the notation

$$(9)\qquad \int_{+0}^{x}(x-s)^{\beta}g(s)ds\equiv\mathfrak{M}_x(g,\ \beta)$$

after this.

At first, we wish to study about the special case for $n=2$ in (8), which will be called the second repeated Abel's integral equation, that is,

$$(8')\qquad \int_{+0}^{x}\int_{+0}^{s_1}\frac{\varphi(s_2)ds_2ds_1}{(x-s_1)^{\alpha_1}(s_1-s_2)^{\alpha_2}}=f(x)\qquad (0<\alpha_i<1,\ i=1,\ 2).$$

For that purpose, put

$$(9)\qquad \int_{+0}^{s_1}\frac{\varphi(s_2)ds_2}{(s_1-s_2)^{\alpha_2}}=\varphi_1(s_1)\qquad (0<\alpha_2<1),$$

then the above equation (8′) will take the form

$$(8'')\qquad \int_{+0}^{x}\frac{\varphi_1(s_1)ds_1}{(x-s_1)^{\alpha_1}}=f(x)\qquad (0<\alpha_1<1).$$

Thus when we allow the existence of the limit $f(+0)$ and the integral $\mathfrak{M}_x(f',\ \alpha_1-1)$ in (8″), from (3) we obtain

$$\varphi_1(x)=\frac{\sin\alpha_1\pi}{\pi}\left\{\frac{f(+0)}{x^{1-\alpha_1}}+\int_{+0}^{x}\frac{f'(z)dz}{(x-z)^{1-\alpha_1}}\right\}.$$

By applying the formula (2) into the equation (9),

$$\varphi(x)=\frac{\sin\alpha_2\pi}{\pi}\ \frac{\partial}{\partial x}\int_{+0}^{x}\frac{\varphi_1(z)dz}{(x-z)^{1-\alpha_2}}.$$

Now replace the above results into φ_1, then we have

$$\int_{+0}^{x}\frac{\varphi_1(z)dz}{(x-z)^{1-\alpha_2}}=\frac{\sin\alpha_1\pi}{\pi}\left\{f(+0)\int_{+0}^{x}\frac{dz}{z^{1-\alpha_1}(x-z)^{1-\alpha_2}}+\right.$$
$$\left.+\int_{+0}^{x}f'(z_1)dz_1\int_{z_1}^{x}\frac{dz}{(x-z)^{1-\alpha_2}(z-z_1)^{1-\alpha_1}}\right\};$$

in the second member, if we put $z=xt$ into the first integral and $z=z_1+(x-z_1)t$ into the second one, and transform the integral variable z into t, we can evaluate as follows:

$$= \frac{\sin \alpha_1\pi}{\pi}\left\{\frac{\Pi(\alpha_1-1)\Pi(\alpha_2-1)}{\Pi(\alpha_1+\alpha_2-1)}f(+0)x^{\alpha_1+\alpha_2-1} + \frac{\Pi(\alpha_1-1)\Pi(\alpha_2-1)}{\Pi(\alpha_1+\alpha_2-1)}\mathfrak{M}_x(f', \alpha_1+\alpha_2-1)\right\}.$$

Hence we obtain

$$\varphi(x) = \frac{\sin \alpha_1\pi \sin \alpha_2\pi}{\pi^2}\frac{\Pi(\alpha_1-1)\Pi(\alpha_2-1)}{\Pi(\alpha_1+\alpha_2-2)}\left\{f(+0)x^{\alpha_1+\alpha_2-2} + \frac{1}{\alpha_1+\alpha_2-1}\frac{\partial}{\partial x}\mathfrak{M}_x(f', \alpha_1+\alpha_2-1)\right\}.$$

Consequently we can conclude as follows:

Theorem **1.** *When the limits* $f(+0), f'(+0), \ldots\ldots\ldots,$ *and* $f^{(\nu)}(+0)$ *and the integral* $\mathfrak{M}_x(f^{(\nu+1)}, \alpha_1+\alpha_2+\nu-2)$ *exist, the second repeated Abel's integral equation has one and only one solution, which is given by*

$$\begin{aligned}\varphi(x) = \frac{\sin \alpha_1\pi \sin \alpha_2\pi}{\pi^2}\frac{\Pi(\alpha_1-1)\Pi(\alpha_2-1)}{\Pi(\alpha_1+\alpha_2-2)}\Bigg\{&f(+0)x^{\alpha_1+\alpha_2-2} \\ &+ \frac{f'(+0)}{\alpha_1+\alpha_2-1}x^{\alpha_1+\alpha_2-1} + \\ &+ \frac{f''(+0)}{(\alpha_1+\alpha_2-1)(\alpha_1+\alpha_2)}x^{\alpha_1+\alpha_2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \frac{f^{(\nu)}(+0)x^{\alpha_1+\alpha_2+\nu-2}}{(\alpha_1+\alpha_2-1)(\alpha_1+\alpha_2)\ldots(\alpha_1+\alpha_2+\nu-2)} + \\ &+ \frac{1}{(\alpha_1+\alpha_2-1)\ldots(\alpha_1+\alpha_2+\nu-2)}\mathfrak{M}_x(f^{(\nu+1)}, \alpha_1+\alpha_2+\nu-2)\Bigg\},\end{aligned}$$

where $\alpha_1+\alpha_2-1 \neq 0$.

Thus we have at once the following corollary to Theorem **1.**

Corollary. *In order that the continuous solution of the second repeated Abel's integral equation* (8′), *defined over* I, *may hold the finite and determinate limit at the point* $x=+0$, *it is necessary and sufficient that* (i) $f(+0)=f'(+0)=0$ *for* $\alpha_1+\alpha_2-1<0$; (ii) $f(+0)=0$ *for* $\alpha_1+\alpha_2-1>0$, *where* $0<\alpha_i<0$ $(i=1,\ 2)$.

Next let us investigate on the particular case when $\alpha_1+\alpha_2-1=0$. For this case we see

$$\int_{+0}^{x}\frac{\varphi_1(z)dz}{(x-z)^{1-\alpha_2}} = \frac{\sin \alpha_1\pi}{\pi}\left\{\frac{\Pi(\alpha_1-1)\Pi(-\alpha_1)}{\Pi(0)}f(+0) + \frac{\Pi(\alpha_1-1)\Pi(-\alpha_1)}{\Pi(0)}\int_{+0}^{x}f'(z)dz\right\} = f(+0)+\int_{+0}^{x}f'(z)dz = f(x),$$

accordingly we have

$$\varphi(x)=\frac{\sin \alpha_2\pi}{\pi}\frac{\partial}{\partial x}\int_{+0}^{x}f(z)dz=\frac{\sin(1-\alpha_1)\pi}{\pi}f(x).$$

Hence *when* $\alpha_1+\alpha_2-1=0$, *the second repeated Abel's integral equation has one and only one continuous solution. Such a solution is given by*

$$\varphi(x)=\frac{\sin(1-\alpha_1)\pi}{\pi}f(x).$$

Now we should like to look again at (8′) from another view-point. Apply the Dirichlet's formula for the double integral to the equation (8′), and since by using the transformation $s_1=s_2+(x-s_2)t$

$$\int_{+0}^{x}\varphi(s_2)ds_2\int_{s_2}^{x}\frac{ds_1}{(x-s_1)^{\alpha_1}(s_1-s_2)^{\alpha_2}}=\int_{+0}^{x}\frac{\varphi(s_2)ds_2}{(x-s_2)^{\alpha_1+\alpha_2-1}}\times$$

$$\int_{+0}^{1}t^{-\alpha_2}(1-t)^{-\alpha_1}dt=\frac{\Pi(-\alpha_1)\Pi(-\alpha_2)}{\Pi(1-\alpha_1-\alpha_2)}\int_{+0}^{x}(x-s)^{1-\alpha_1-\alpha_2}\varphi(s)ds,$$

the equation (8′) may be written

(8″) $$\int_{+0}^{x}(x-s)^{\beta}\varphi(s)ds=f_1(x) \qquad (1>\beta>-1),$$

where $$f_1(x)=\frac{\Pi(1-\alpha_1-\alpha_2)}{\Pi(-\alpha_1)\Pi(-\alpha_2)}f(x), \qquad \beta=1-\alpha_1-\alpha_2.$$

This is the same form as the equation (5), previously given by Mr. R. Rothe, hence on account of (7), we have the following

Theorem **2.** *For any choice of such a number* $\mu>-1$ *as* $\mu+\beta+1=m$ (m : *any given non-negative integer*), *if* $\frac{\partial^{m+1}}{\partial x^{m+1}}\mathfrak{M}_x(f,\ \mu)$ *exist, the second repeated Abel's integral equation has one and only one continuous solution in* I *and this solution is given by*

$$\varphi(x)=\frac{1}{\Pi(-\alpha_1)\Pi(-\alpha_2)\Pi(\alpha_1+\alpha_2+m-2)}\times$$

$$\frac{\partial^{m+1}}{\partial x^{m+1}}\int_{+0}^{x}(x-s)^{\alpha_1+\alpha_2+m-2}f(s)ds.$$

Particularly we seek a solution of (8″) which satisfies the limiting condition

(10) $$\lim_{s\to+0}s\cdot\varphi(s)=G \qquad (G \text{ being any fixed number})$$

when $0<\beta<1$. Then after Mr. Rothe[1] we can easily obtain

1. Loc. cit., p. 378.

$$(11)\qquad \varphi(x)=\frac{\Pi(1-\alpha_1-\alpha_2)}{(1-\alpha_1-\alpha_2)\pi}\frac{\sin(\alpha_1+\alpha_2)\pi}{\Pi(-\alpha_1)\Pi(-\alpha_2)}\left\{\frac{f'(+0)}{x^{1-\alpha_1-\alpha_2}}+\frac{f''(+0)}{\alpha_1+\alpha_2}x^{\alpha_1+\alpha_2}+\frac{1}{\alpha_1+\alpha_2}\int_{+0}^{x}(x-s)^{\alpha_1+\alpha_2}f'''(s)ds\right\}$$

and this is the unique solution (continuous in I) of (8″), accordingly of (8′). Furthermore at the same time the condition $G=0$ will necessarily follow.

Now if we put $\nu=2$ into Theorem **1**, we have

$$\varphi(x)=\frac{\sin\alpha_1\pi\cdot\sin\alpha_2\pi}{\pi^2}\frac{\Pi(\alpha_1-1)\Pi(\alpha_2-1)}{\Pi(\alpha_1+\alpha_2-2)}\left\{f(+0)x^{\alpha_1+\alpha_2-2}+\frac{f'(+0)}{\alpha_1+\alpha_2-1}x^{\alpha_1+\alpha_2-1}+\frac{f''(+0)}{(\alpha_1+\alpha_2-1)(\alpha_1+\alpha_2)}x^{\alpha_1+\alpha_2}+\frac{1}{(\alpha_1+\alpha_2-1)(\alpha_1+\alpha_2)}\mathfrak{M}_x(f''',\ \alpha_1+\alpha_2)\right\};$$

when the above $\varphi(x)$ satisfies the limiting condition (10), from $0>\alpha_1+\alpha_2-1>-1$, $f(+0)=0$ will be followed. Therefore

$$\varphi(x)=\frac{\sin\alpha_1\pi\cdot\sin\alpha_2\pi}{\pi^2}\frac{\Pi(\alpha_1-1)\Pi(\alpha_2-1)}{\Pi(\alpha_1+\alpha_2-1)}\left\{\frac{f'(+0)}{x^{1-\alpha_1-\alpha_2}}+\frac{f''(+0)}{\alpha_1+\alpha_2}x^{\alpha_1+\alpha_2}+\frac{1}{\alpha_1+\alpha_2}\int_{+0}^{x}(x-s)^{\alpha_1+\alpha_2}f'''(s)ds,\right.$$

whence the above thus obtained solution must be identically equal to the expression (11). But we can easily ascertain this identity by properties of the Pifunction, namely

$$\begin{aligned}&\frac{\Pi(1-\alpha_1-\alpha_2)\cdot\sin(\alpha_1+\alpha_2)\pi}{(1-\alpha_1-\alpha_2)\pi\cdot\Pi(-\alpha_1)\Pi(-\alpha_2)}\\&=\frac{\Pi(1-\alpha_1-\alpha_2)}{(1-\alpha_1-\alpha_2)\Pi(-\alpha_1)\Pi(-\alpha_2)\Pi(-\alpha_1-\alpha_2)\Pi(\alpha_1+\alpha_2-1)}\\&=\frac{(1-\alpha_1-\alpha_2)\Pi(1-\alpha_1-\alpha_2)}{(1-\alpha_1-\alpha_2)\Pi(-\alpha_1)\Pi(-\alpha_2)\Pi(\alpha_1+\alpha_2-1)\Pi(1-\alpha_1-\alpha_2)}\\&=\frac{1}{\Pi(-\alpha_1)\Pi(-\alpha_2)\Pi(\alpha_1+\alpha_2-1)}\\&=\frac{\Pi(\alpha_1-1)\sin\alpha_1\pi}{\pi}\frac{\Pi(\alpha_2-1)\sin\alpha_2\pi}{\pi}\frac{1}{\Pi(\alpha_1+\alpha_2-1)}.\end{aligned}$$

When we suppose that limits $f(+0)$, $f'(+0)$,......, and $f^{(\nu)}(+0)$ and the integral $\mathfrak{M}_x(f^{(\nu+1)},\ \alpha_1+\alpha_2+\nu-2)$ may exist in Theorem **2**, then clearly Theorem **1** will be derived.

Now in general we consider the previously stated integral equation

$$(8)\qquad \int_{+0}^{x}\int_{+0}^{s_1}\cdots\int_{+0}^{s_{n-1}}\frac{\varphi(s_n)ds_n ds_{n-1}\cdots\cdots ds_1}{(x-s_1)^{\alpha_1}(s_1-s_2)^{\alpha_2}\cdots\cdots(s_{n-1}-s_n)^{\alpha_n}}=f(x).$$

First, put

$$\int_{+0}^{s_1}\int_{+0}^{s_2}\cdots\int_{+0}^{s_{n-1}}\frac{\varphi(s_n)ds_n ds_{n-1}\cdots\cdots ds_2}{(s_1-s_2)^{\alpha_2}(s_2-s_3)^{\alpha_3}\cdots\cdots(s_{n-1}-s_n)^{\alpha_n}}\equiv\varphi_1(s_1),$$

then (8) will take the following form

$$\int_{+0}^{x}\frac{\varphi_1(s_1)ds_1}{(x-s_1)^{\alpha_1}}=f(x)\qquad (0<\alpha_1<1);$$

while, the solution of the above equation is given uniquely by

$$\varphi_1(x)=\frac{\sin\alpha_1\pi}{\pi}\cdot\frac{\partial}{\partial x}\mathfrak{M}_x(f,\ \alpha_1-1).$$

Thus generally if we put

$$\int_{+0}^{s_i}\int_{+0}^{s_{i+1}}\cdots\int_{+0}^{s_{n-1}}\frac{\varphi(s_n)ds_n\cdots\cdots ds_{i+1}}{(s_i-s_{i+1})^{\alpha_{i+1}}\cdots\cdots(s_{n-1}-s_n)^{\alpha_n}}\equiv\varphi_i(s_i),$$

then we shall get

$$\int_{+0}^{x}\frac{\varphi_i(s)ds}{(x-s)^{\alpha_i}}=\varphi_{i-1}(x),\quad \varphi_0(x)\equiv f(x),\quad \varphi_n(x)\equiv\varphi(x)$$

$$i=1,\ 2,\ 3,\cdots\cdots\cdots\cdots\cdots,\ n;$$

and this solution is given uniquely by

$$\varphi_i(x)=\frac{\sin\alpha_i\pi}{\pi}\ \frac{\partial}{\partial x}\mathfrak{M}_x(\varphi_{i-1},\ \alpha_i-1).$$

Thus from these results, it is evident that if $\varphi_{i-1}(x)$ be continuous in I, $\varphi_i(x)$ will be also. Hence we have the following

Theorem **3.** *When* $\frac{\partial}{\partial x}\mathfrak{M}_x(f,\ \alpha_1-1)$ *is finite and continuous in* I, *the n-th repeated Abel's integral equation* (8) *has the unique continuous solution and that solution will be written*

$$\varphi(x)=\prod_{i=1}^{n}\frac{\sin\alpha_i\pi}{\pi}\ \frac{\partial}{\partial x}\int_{+0}^{x}\frac{dz_n}{(x-z_n)^{1-\alpha_n}}\ \frac{\partial}{\partial z_n}$$
$$\int_{+0}^{z_n}\frac{dz_{n-1}}{(z_n-z_{n-1})^{1-\alpha_{n-1}}}\cdots\cdots\frac{\partial}{\partial z_2}\int_{+0}^{z_2}\frac{f(z_1)dz_1}{(z_2-z_1)^{1-\alpha_1}}.$$

Next we look again at the same Abel's equation (8) from another point of view; namely by using the Dirichlet's transformation for the double integral

$$\int_{+0}^{s_{n-2}}\int_{+0}^{s_{n-1}}\frac{\varphi(s_n)ds_n ds_{n-1}}{(s_{n-2}-s_{n-1})^{\alpha_{n-1}}(s_{n-1}-s_n)^{\alpha_n}}$$

$$=\int_{0+}^{s_{n-2}}\varphi(s_n)ds_n\int_{s_n}^{s_{n-2}}\frac{ds_n}{(s_{n-2}-s_{n-1})^{\alpha_{n-1}}(s_{n-1}-s_n)^{\alpha_n}}$$

$$=\frac{\Pi(-\alpha_{n-1})\Pi(-\alpha_n)}{\Pi(1-\alpha_{n-1}-\alpha_n)}\int_{+0}^{s_{n-2}}(s_{n-2}-s_n)^{1-\alpha_{n-1}-\alpha_n}\varphi(s_n)ds_n,$$

hence we have

$$\int_{+0}^{s_{n-3}}\int_{+0}^{s_{n-2}}\int_{+0}^{s_{n-1}}\frac{\varphi(s_n)ds_n ds_{n-1}ds_{n-2}}{(s_{n-3}-s_{n-2})^{\alpha_{n-2}}(s_{n-2}-s_{n-1})^{\alpha_{n-1}}(s_{n-1}-s_n)^{\alpha_n}}$$

$$=\frac{\Pi(-\alpha_{n-1})\Pi(-\alpha_n)}{\Pi(1-\alpha_{n-1}-\alpha_n)}\int_{+0}^{s_{n-3}}\int_{+0}^{s_{n-2}}\frac{\varphi(s_n)ds_n}{(s_{n-3}-s_{n-2})^{\alpha_{n-2}}(s_{n-2}-s_n)^{\alpha_{n-1}+\alpha_{n-1}}}$$

$$=\frac{\Pi(-\alpha_{n-1})\Pi(-\alpha_n)}{\Pi(1-\alpha_{n-1}-\alpha_n)}\cdot\frac{\Pi(-\alpha_{n-2})\Pi(1-\alpha_{n-1}-\alpha_n)}{\Pi(2-\alpha_{n-2}-\alpha_{n-1}-\alpha_n)}\times$$

$$\int_{+0}^{s_{n-3}}(s_{n-3}-s_n)^{2-\alpha_{n-2}-\alpha_{n-1}-\alpha_n}.\varphi(s_n)ds_n.$$

Thus repeating these processes, we arrive finally at

$$\frac{\Pi(-\alpha_1)\Pi(-\alpha_2)\dots\Pi(-\alpha_n)}{\Pi(n-1-\alpha_1-\alpha_2-\dots-\alpha_n)}\int_{+0}^{x}(x-s)^{n-1-\alpha_1-\alpha_2-\dots-\alpha_n}.\varphi(s)ds=f(x).$$

This has the same meaning as (8). Then in the above equation, by changing in notations

$$f_1(x)=\frac{\Pi(\beta)}{\Pi(-\alpha_1)\Pi(-\alpha_2)\dots\Pi(-\alpha_n)}f(x)$$

$$\beta=n-1-\alpha_1-\alpha_2-\dots\dots-\alpha_n=n-1-\sum_{i=1}^{n}\alpha_i,$$

Abel's integral equation (8) will become consequently

$$(8''')\qquad \int_{+0}^{x}(x-s)^{\beta}\varphi(s)ds=f_1(x),$$

where $\quad n-1>\beta>-1$[1].

Therefore corresponding to Theorem **2,** we have

Theorem **4.** *For any choice of a number $\mu>-1$ which satisfies the equation $\mu+\beta+1=m$, where m being a certain non-negative integer and given suitably, when $\frac{\partial^{m+1}}{\partial x^{m+1}}\mathfrak{M}_x(f,\mu)$ exists, then the n-th repeated Abel's integral equation* (8) *has one and only one continuous solution and the solution is given by*

1. For since $0<\sum_{i=1}^{n}\alpha_i<n$, we see at once $n-1>n-1-\sum_{i=1}^{n}\alpha_i>-1$.

$$\varphi(x)=\frac{1}{\Pi(-a_1)\Pi(-a_2)\ldots\Pi(-a_n)\Pi(\mu)}\frac{\partial^{m+1}}{\partial x^{m+1}}\int_{+0}^{x}(x-s)^{\mu}\cdot f(s)ds,$$

where $\beta=n-1-\sum_{i=1}^{n}a_i.$

Remark. *Since* $n-1>\beta>-1$, $\mu+n>\mu+\beta+1=m>\mu$. *Hence for any* μ $n>m-\mu>0$. *While* $\text{Max}(m-\mu)<m+1$, *therefore* $n>m+1$. From this remark, we have the following

Theorem **5.** *In the previously stated Theorem* **4**, *if* $\mathfrak{M}_x(f,\mu)$ *be any polynomial of the higher order than the n-th, there will never exist any trivial solution.*

2. We consider a new integral equation, which contains the classical Abel's integral equation (1) as a special case, as follows:

$$(12)\qquad \int_{+0}^{x}\frac{\varphi(s)ds}{[\tau(x)-\tau(s)]^{\alpha}}=f(x)\qquad (0<a<1),$$

where $\tau(x)$ being defined over I to be monotone increasing and continuous with its first derivative, furthermore the limits $\tau(+0)$ and $\tau'(+0)$ exist. We begin with multiplying both sides of the equation by $\dfrac{\tau'(x)dx}{[\tau(z)-\tau(x)]^{1-\alpha}}$, integrating with respect to x from $+0$ to z, so we have

$$\int_{+0}^{z}\int_{+0}^{x}\frac{\varphi(s)\tau'(x)dxds}{[\tau(z)-\tau(x)]^{1-\alpha}[\tau(x)-\tau(s)]^{\alpha}}=\int_{+0}^{z}\frac{f(x)\tau'(x)dx}{[\tau(z)-\tau(x)]^{1-\alpha}}.$$

Apply the Dirichlet's transformation for the double integral to the left-side, then we have

$$\int_{+0}^{z}\varphi(s)ds\cdot\int_{s+0}^{z}\frac{\tau'(x)dx}{[\tau(z)-\tau(x)]^{1-\alpha}[\tau(x)-\tau(s)]^{\alpha}}.$$

And besides if we transform the integral variable x into t by

$$(\mathrm{T})\qquad \begin{cases}\tau(x)=\tau(s)+[\tau(z)-\tau(s)]t\\ \tau'(x)dx=[\tau(z)-\tau(s)]dt,\end{cases}$$

the above integral will be written

$$\int_{+0}^{1}\frac{dt}{(1-t)^{1-\alpha}t^{\alpha}}\Pi(-a)\cdot\Pi(a-1)=\frac{\pi}{\sin a\pi}.$$

Thus the present problem may be reduced to the dependence on the existence of the integral in the second member of the above equation; While, from the mean-valued theorem

$$\tau(x)-\tau(z)=(x-z)\cdot\tau'(\xi) \qquad (z\leqq\xi\leqq x).$$

Hence the following equivalent relation will take place

$$\lim_{x\to+0}\int_{+0}^{x}\frac{f(z)\tau'(z)dz}{[\tau(x)-\tau(z)]^{1-\alpha}}\sim\lim_{x\to+0}[\tau'(x)]^{\alpha}\int_{+0}^{x}\frac{f(z)dz}{(x-z)^{1-\alpha}}$$
$$=[\tau'(+0)]^{\alpha}\lim_{x\to+0}\mathfrak{M}_x(f,\ \alpha-1),\quad \tau'(x)\geqq 0 \text{ in } I.$$

Therefore we have

Theorem **6**. *If* $\mathfrak{M}_x(f,\ \alpha-1)$ *exist, the integral equation* (12) *has one and only one solution and the solution is given by*

$$(13)\qquad \varphi(x)=\frac{\sin\alpha\pi}{\pi}\ \frac{\partial}{\partial x}\int_{+0}^{x}\frac{f(z)\tau'(z)}{[\tau(x)-\tau(z)]^{1-\alpha}}dz.$$

Now for avoiding complexities of expressions, we establish the following notations:

$$(\mathrm{U})\qquad \begin{cases} f_i(z)\equiv\dfrac{f'_{i-1}(z)}{\tau'(z)}, \qquad i=1,\ 2,\ 3,\ldots\ldots\ldots\ldots, \\ f'_0(z)\equiv f'(z), \\ \tau_q(z)\equiv[\tau(z)-\tau(+0)]^q\tau'(z), \\ f_1(+0)=\lim\limits_{z\to+0}\dfrac{f'(z)}{\tau'(z)}=\dfrac{f'(+0)}{\tau'(+0)},\quad f_i(+0)=\lim\limits_{z\to+0}\dfrac{f'_{i-1}(z)}{\tau'(z)}, \end{cases}$$

then we obtain a parallel result to (4).

Theorem **7**. *When the limits* $f(+0)$, $f_1(+0),\ldots\ldots\ldots$, *and* $f_n(+0)$, *and moreover* $\mathfrak{M}_x(f'_n,\ \alpha+n-1)$ *are permissible to exist, then* (13) *will be expressible as follows*:

$$(14)\qquad \varphi(x)=\frac{\sin\alpha\pi}{\pi}\Big\{f(+0)\tau_{\alpha-1}+\frac{f_1(+0)}{\alpha}\tau_\alpha+\frac{f_2(+0)}{\alpha(\alpha+1)}\tau_{\alpha+1}+\ldots$$
$$+\frac{f_n(+0)}{\alpha(\alpha+1)\ldots(\alpha+n-1)}\tau_{\alpha+n-1}+$$
$$\frac{\tau'(x)}{\alpha(\alpha+1)\ldots(\alpha+n-1)}+\int_{+0}^{x}f'_n(z)[\tau(x)-\tau(z)]^{\alpha+n-1}dz\Big\}.$$

Particularly when we take $\tau(x)=kx$ $(k>0)$, then (U) becomes

$$\begin{cases} f_1(z)=\dfrac{1}{k}f'(z),\quad f_i(z)=\dfrac{f'_{i-1}(z)}{k}=\dfrac{f''_{i-2}(z)}{k^2}=\ldots=\dfrac{f^{(i)}(z)}{k^i}, \\ \tau(+0)=0,\quad \tau'(+0)=k,\quad \tau_q=k^{q+1}z^q, \\ f_1(+0)=\dfrac{f'(+0)}{k},\quad f_i(+0)=\dfrac{f^{(i)}(+0)}{k^i} \end{cases}$$

hence instead of (14) we have

$$(14') \qquad \varphi(x)=\frac{\sin a\pi}{\pi}k^{\alpha}\Big\{\frac{f(+0)}{x^{1-\alpha}}+\frac{f'(+0)}{a}x^{\alpha}+\frac{f'''(+0)}{a(a+1)}x^{\alpha+1}+\dots$$

$$+\frac{f^{(n)}(+0)}{a(a+1)\dots(a+n-1)}x^{\alpha+n-1}$$

$$+\frac{1}{a(a+1)\dots(a+n-1)}\int_{+0}^{x}f^{(n+1)}(z)(x-z)^{\alpha+n-1}dz\Big\}.$$

This is the previous formula (4) itself.

Now we prove briefly Theorem **7**: From integrating the second member of (13) by parts

$$\int_{+0}^{x}\frac{f(z)\tau'(z)dz}{[\tau(x)-\tau(z)]^{1-\alpha}}=\frac{1}{a}[\tau(x)-\tau(+0)]^{\alpha}\cdot f(+0)$$

$$+\frac{1}{a}\int_{+0}^{x}f'(z)[\tau(x)-\tau(z)]^{\alpha}dz.$$

Accordingly

$$\varphi(x)=\frac{\sin a\pi}{\pi}\Big\{f(+0)\tau_{\alpha-1}(x)+\tau'(x)\int_{+0}^{x}\frac{f'(z)dz}{[\tau(x)-\tau(z)]^{1-\alpha}}\Big\};$$

while, in the last integral on the right,

$$\int_{+0}^{x}\frac{f'(z)dz}{[\tau(x)-\tau(z)]^{1-\alpha}}=\int_{+0}^{x}\frac{f'(z)}{\tau'(z)}\,\frac{\tau'(z)dz}{[\tau(x)-\tau(z)]^{1-\alpha}}$$

$$=-\frac{1}{a}\int_{+0}^{x}f_1(z)\frac{\partial}{\partial z}[\tau(x)-\tau(z)]^{\alpha}dz,$$

again making use of the integration by parts

$$=\frac{f_1(+0)}{a}[\tau(x)-\tau(+0)]^{\alpha}+\frac{1}{a}\int_{+0}^{x}f_1'(z)[\tau(x)-\tau(z)]^{\alpha}dz.$$

Thus the above $\varphi(x)$ may be written as follows:

$$\varphi(x)=\frac{\sin a\pi}{\pi}\Big\{f(+0)\tau_{\alpha-1}+\frac{f_1(+0)}{a}\tau_{\alpha}+$$

$$\frac{\tau'(x)}{a}\int_{+0}^{x}f_1'(z)[\tau(x)-\tau(z)]^{\alpha}dz\Big\}.$$

In the like manner, making use of

$$\int_{+0}^{x}f_1(z)\cdot[\tau(x)-\tau(z)]^{\alpha}dz=\int_{+0}^{x}\frac{f_1'(z)}{\tau'(z)}[\tau(x)-\tau(z)]^{\alpha}\tau'(z)dz$$

$$=-\frac{1}{a+1}\int_{+0}^{x}f_2(z)\frac{\partial}{\partial z}[\tau(x)-\tau(z)]^{\alpha+1}dz$$

$$=\frac{f_2(+0)}{\alpha+1}[\tau(x)-\tau(+0)]^{\alpha+1}+\frac{1}{\alpha+1}\int_{+0}^{x}f_2'(z)[\tau(x)-\tau(z)]^{\alpha+1}dz,$$

then the above just obtained $\varphi(x)$ becomes

$$\varphi(x)=\frac{\sin\alpha\pi}{\pi}\Big\{f(+0)\tau_{\alpha-1}+\frac{f_1(+0)}{\alpha}\tau_\alpha+\frac{f_2(+0)}{\alpha(\alpha+1)}\tau_{\alpha+1}+$$
$$+\frac{\tau'(x)}{\alpha(\alpha+1)}\int_{+0}^{x}f_2'(z)[\tau(x)-\tau(z)]^{\alpha+1}dz.$$

Hence we can conclude that the equality (14) will be true in general.

Q. E. D.

In the second place we are going to generalize the integral equation (12) after Mr. Rothe; namely we compute

$$(15)\qquad \int_{+0}^{x}[\tau(x)-\tau(s)]^{\beta}\varphi(s)ds=f(x),\qquad (\beta>-1).$$

for the present case, multiply both sides of (15) by $[\tau(z)-\tau(x)]^{\mu}\cdot\tau'(x)$ and integrate with respect to x, then

$$\int_{+0}^{z}[\tau(z)-\tau(x)]^{\mu}\tau'(x)dx\int_{+0}^{x}[\tau(x)-\tau(s)]^{\beta}\cdot\varphi(s)ds$$
$$=\int_{+0}^{z}[\tau(z)-\tau(x)]^{\mu}\tau'(x)f(x)dx.$$

Let us take the Dirichlet's transformation for the double integral on the left, and evaluate by the transformation (T), and besides choose $\mu>-1$ as $\mu+\beta+1=n$ (n being any given non-negative integer), then finally (15) may be written

$$(15')\qquad \int_{+0}^{z}\frac{[\tau(z)-\tau(s)]^{n}}{n\,!}\varphi(s)ds$$
$$=\frac{1}{\Pi(\mu)\Pi(\beta)}\int_{+0}^{z}[\tau(z)-\tau(x)]^{\mu}\tau'(x)f(x)dx.$$

For simplifying the preceding computation, we make the following notations

$$(\mathrm{V})\qquad \begin{cases} f(z,\mu)=\displaystyle\int_{+0}^{z}[\tau(z)-\tau(x)]^{\mu}\tau'(x)f(x)dx, \\ f_i(z,\mu)=\dfrac{1}{\tau'(z)}\dfrac{\partial}{\partial z}f_{i-1}(z,\mu), \\ f_1(z,\mu)\equiv\dfrac{1}{\tau'(z)}\dfrac{\partial}{\partial z}f(z,\mu), \qquad i=1,\ 2,\ 3,\dots\dots,\ n. \end{cases}$$

Whence, we can show that

Theorem **8.** *If we take it for granted that* $\frac{\partial}{\partial z}f_n(z, \mu)$ *exist, then the integral equation* (15) *has a unique continuous solution which is given by*

$$(16) \qquad \varphi(x)=\frac{1}{\Pi(\mu)\Pi(\beta)}\frac{\partial}{\partial x}f_n(x, \mu),$$

where $\qquad \mu+\beta+1=n.$

The proof is very easy; for, by differentiating (15′) up to k-times

$$\int_{+0}^{z}\frac{[\tau(z)-\tau(s)]^{n-k}}{(n-k)!}\varphi(s)ds=\frac{1}{\Pi(\mu)\Pi(\beta)}f_k(z, \mu).$$

In the above equation, particularly put $\tau(x)=x$, then $\tau'(x)=1$, accordingly it follows that $f_k(z, \mu)=\frac{\partial}{\partial z}f_{k-1}(z, \mu)=\frac{\partial^k}{\partial z^k}f(z, \mu)$. Hence (16) becomes

$$(16') \qquad \varphi(x)=\frac{1}{\Pi(\mu)\Pi(\beta)}\frac{\partial^{n+1}}{\partial x^{n+1}}f(x, \mu), \qquad \mu+\beta+1=n.$$

We see that this represents (7).

Now we desire to show a certain example.

Example 1. In the study of radiation the following functional equation will take place

$$(A) \qquad I(\varphi)=2NdS\int_0^{\psi_0}f(\theta)\frac{r-\rho\cos\varphi\cos\psi}{[r^2+\rho^2-2r\rho\cos\varphi\cos\psi]^{\frac{3}{2}}}d\psi,$$

where $f(\theta)$ is unknown; moreover the following relations

$$\cos\varphi\cos\psi_0=\frac{\rho}{r}\equiv a \qquad (const.)$$

$$\cos\theta=-\frac{\rho-r\cos\varphi\cos\psi}{[r^2+\rho^2-2\rho r\cos\varphi\cos\psi]^{\frac{1}{2}}}$$

are given.

For the first instance let $\cos\varphi\cos\psi=\mu$ be, then $d\mu=-\cos\varphi\sin\psi d\psi$, $\frac{r-\rho\cos\varphi\cos\psi}{[r^2+\rho^2-2\rho r\cos\varphi\cos\psi]^{\frac{3}{2}}}=\frac{r-\rho\mu}{[r^2+\rho^2-2r\rho\mu]^{\frac{3}{2}}}\equiv K(\mu)$, and $\cos\theta=-\frac{\rho-r\mu}{[\rho^2+r^2-2\rho r\mu]^{\frac{1}{2}}}$. Therefore since θ is expressible to be a function of μ, if we denote $G(\mu)\equiv f(\theta)$, then the equation (A) may be reduced to the form

(A′) $$-F(\varphi)\cos\varphi=\int_{\cos\varphi}^{\cos\varphi\cos\psi_0}G(\mu)K(\mu)\frac{1}{\sin\psi}d\mu,$$

where $F(\varphi)\equiv I(\varphi)/2NdS$.

If we put $\cos\varphi=t$ here, then moreover (A′) may be written

(A″) $$L(t)=\int_a^t\frac{M(\mu)d\mu}{\sqrt{t^2-\mu^2}},$$

while $L(t)\equiv F(\varphi)$ and $M(\mu)\equiv G(\mu)K(\mu)$.

Consequently our present problem under consideration will be reduced to the discussion on the existence of the solution of the following integral equation

(17) $$\int_{+0}^{x}\frac{\varphi(s)ds}{\sqrt{x^2-s^2}}=f(x).$$

We see at once that this equation is none but the special case when $\alpha=\frac{1}{2}$, $\tau(x)=x^2$ in (12).

Hence by means of (13),

$$\varphi(x)=\frac{2}{\pi}\frac{\partial}{\partial x}\int_{+0}^{x}\frac{f(z)zdz}{\sqrt{x^2-z^2}}$$

must become a unique continuous solution of (17).

Remark: *At this instant it is worthy to remark that there does not ever exist the limit $f_1(+0)$ in Theorem 7.*

3. We proceed now to the following integral equation

(18) $$\int_{+0}^{\tau(x)}\frac{\varphi(s)ds}{[\tau(x)-s]^\alpha}=f(x),\quad 0<\alpha<1,\quad \tau(+0)=0.$$

$\tau(x)$ has the same meaning as previously stated; accordingly $\tau(x)$ is a one-valued function in I.

Change the integral variable by the transformation $s=\tau(u)$ e. g. $s\to u$. Then (18) becomes

$$\int_{+0}^{x}\frac{\varphi[\tau(u)]\tau'(u)du}{[\tau(x)-\tau(u)]^\alpha}=f(x).$$

And besides by simplifying this,

(18′) $$\int_{+0}^{x}\frac{\Phi(u)du}{[\tau(x)-\tau(u)]^\alpha}=f(x),$$

where $\Phi(u)\equiv\tau'(u)\varphi[\tau(u)]$.

Thus a unique solution of (18) may be given by

$$(19) \qquad \varphi(x)=\frac{1}{\tau'[\rho(x)]}\Phi[\rho(x)],$$

where Φ satisfies the integral equation (18'), and $\rho(x)$ is the inverse function of $\tau(x)$.

Next by generalizing the above (18) after Mr. Rothe again, we have

$$(20) \qquad \int_{+0}^{\tau(x)}[\tau(x)-s]^{\beta}\cdot\varphi(s)ds=f(x) \qquad (\beta>-1).$$

Since this is also treated in the same manner as (18), a unique continuous solution of (20) becomes

$$(21) \qquad \varphi(x)=\frac{1}{\tau'[\rho(x)]}\Phi[\rho(x)];$$

while Φ on the right satisfies the following integral equation

$$(22) \qquad \int_{+0}^{x}[\tau(x)-\tau(s)]^{\beta}\Phi(s)ds=f(x).$$

Remark: *It is known that* $f(x)$ *in* (18) *may be expressed by* $f[\tau(x)]$.
Example 2. The following integral equation has been treated by Mr. Nakagorô Hirakawa[1]

$$(23) \qquad kmx=\int_{+0}^{mx}\sqrt{mx-\xi}\cdot\varphi(\xi)d\xi.$$

The above integral equation (23) will be reduced to the special case when in (20) $f(x)=kmx$, $\tau(x)=mx$, $\tau'(x)=m$, accordingly $\rho(x)=x/m$, $\beta=1/2$, hence from (21) the unique continuous solution of (23) becomes

$$(24) \qquad \varphi(x)=\frac{1}{m}\Phi\left(\frac{x}{m}\right),$$

where Φ satisfies (22), that is,

$$\int_{+0}^{x}[mx-ms]^{\frac{1}{2}}\Phi(s)ds=kmx.$$

A unique continuous solution of the above just stated integral equation by (16)

$$\Phi(x)=\frac{1}{\Pi\left(\frac{1}{2}\right)\Pi\left(\frac{1}{2}\right)}\frac{\partial}{\partial x}f_2\left(x,\frac{1}{2}\right),$$

1. Loc. cit.

where $2=n=\beta+\mu+1=\frac{1}{2}+\mu+1$, i. e. $\mu=\frac{1}{2}$.

From (V)

$$\begin{cases} f_1\left(z, \frac{1}{2}\right)=km^{2+\frac{1}{2}}\int_{+0}^{z}(z-x)^{\frac{1}{2}}x\,dx=\frac{2}{5}\cdot\frac{2}{3}km^{\frac{5}{2}}z^{\frac{5}{2}}, \\ f_2\left(z, \frac{1}{2}\right)=\frac{1}{m}\frac{\partial}{\partial z}f_1\left(z, \frac{1}{2}\right)=km^{\frac{1}{2}}z^{\frac{1}{2}}. \end{cases}$$

Replacing these results into Φ,

$$\Phi(z)=\frac{2}{\pi}km^{\frac{1}{2}}z^{-\frac{1}{2}}.$$

Moreover replace the above function into (24), then

$$\varphi(x)=\frac{2k}{\pi}\frac{1}{\sqrt{x}}.$$

This obtained result coincides very well with Mr. Hirakawa's one.

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