A Fundamental Investigation of Motion of a Vertically Constrained Gyroscope

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Abstract

For a vertically constrained ship's gyroscope, which is affected by gravitation, the rotation of the earth, the speed and the course of the ship, the writer derived the equations of motion starting from the fundamental equation

Torque = Angular momentum × rate of precession,

and compared their solutions with the experimental data which were obtained by means of a simple apparatus. Both results coincided well within the limit of experimental error.

I. Theory

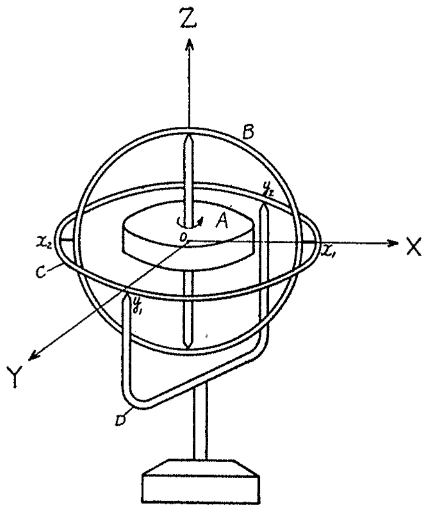
1. Undamped oscillation

Neglecting the friction at the bearings and resistances against the motion of the gyroscope, we investigate the motion of the axis of the gyroscope by taking into consideration the influences of the earth's rotation and of the speed and the course of the ship on which the gyroscope is carried.

(A) The equations of motion and their solutions :—The vertically constrained gyroscope is represented schematically by Fig. 1. Gyroscope A is held by a ring B at both ends of the axis of gyration, the ring B is supported by a ring C at the points x_1 and x_2, and the ring C by a ring D at the points y_1 and y_2. Straight lines $x_1 x_2$ and $y_1 y_2$ intersect at a point O in the horizontal plane XY. The centre of gravity of the gyroscope is below the point O, and when the axis of gyration incilines the gravity acts so as to keep it vertical. Let the fulcrum O of the oscillation be the origin of the rectangular coordinates X, Y and Z, the transversal axis of the ship the X-axis (positive to the portside), the longitudinal axis of the ship the Y-axis (positive to the bow) and the vertical direction the Z-axis (positive upwards). The meaning of the various notations used in the present paper are explained below :—

α : The component of the angle between the axis of the gyroscope and the OZ-axis in the ZX-plane (positive when the upper end of the

Fig. 1



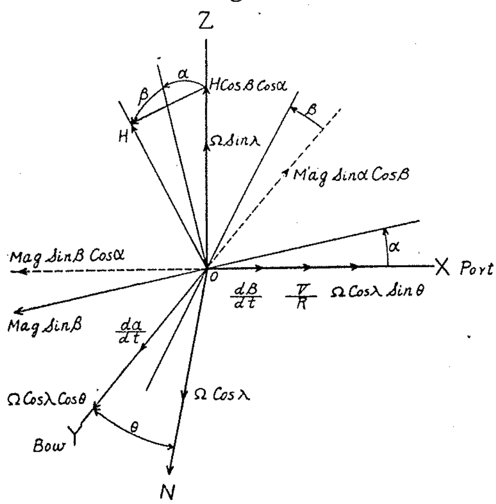
axis inclines to starboard). β: The component of the angle between the axis of the gyroscope and the OZ-axis in the ZY-plane (positive when the upper end of the axis inclines to the bow). θ: The course of the ship (measured from the north eastwards as positive). Ω: The angular velocity of the earth's rotation. λ: The latitude at the place where the gyroscope is. V: The speed of the ship. a: The distance between O and the centre of gravity of the gyroscope with the ring B. a': The distance between O and the centre of gravity of the gyroscope with the rings B and C. g: The acceleration of the earth's gravitation. R: The radius of the earth at the place where the gyroscope is. M: Mass of the gyroscope with the ring B. M': Mass of the gyroscope with the rings B and C. H: The angular momentum of the gyroscope (product of the number of revolution per second and the moment of inertia with respect to the axis of gyration).

If a torque K acts upon a rotating body having an angular momentum H, and when the two vectors, K and H, are perpendicular to each other, then its axis of gyration follow the vector K by the shortest path. This motion is called precession, and if its angular velocity is Θ, then

$$K = H\Theta \quad \text{.........(1)}$$

Let us apply this relation to the precession of our gyroscope about the Y-axis (Fig. 2). The angular momentum

Fig. 2



is $H\cos\alpha\cos\beta$, the torque $Mag\ \sin\beta\cos\alpha$ and angular velocity of precession $\frac{d\alpha}{dt}+\Omega\cos\lambda\cos\theta$. Hence follows the equation of motion

$$Mag\sin\beta\cos\alpha=H\cos\alpha\cos\beta\left(\frac{d\alpha}{dt}+\Omega\cos\lambda\cos\theta\right).$$

Similarly the precession about the X-axis is given by

$$M'a'g\sin\alpha\cos\beta=-H\cos\alpha\cos\beta\left(\frac{d\beta}{dt}+\Omega\cos\lambda\sin\theta+\frac{V}{R}\right).$$

If α and β are so small, that they satisfy $\sin\alpha=\alpha$, $\cos\alpha=1$, $\sin\beta=\beta$ and $\cos\beta=1$, then

$$Mag\cdot\beta=H\left(\frac{d\alpha}{dt}+\Omega\cos\lambda\cos\theta\right),\ M'a'g\cdot\alpha=-H\left(\frac{d\beta}{dt}+\Omega\cos\lambda\sin\theta+\frac{V}{R}\right).\ (2)$$

Suppose that only α and β are the functions of time. Then eliminating β from equation (2), we obtain

$$X\frac{d^2\alpha}{dt^2}+Y\alpha+Z=0,$$

where $X\equiv\frac{H^2}{Mag}$, $Y\equiv M'a'g$, $Z\equiv H\left(\Omega\cos\lambda\sin\theta+\frac{V}{R}\right)$.

Equation (3) shows that α changes its value oscillatory with the lapse of time. X is the inertia of this motion, Y the coefficient of stability and Z the constant which determines the position of equilibrium. Substituting in equation (3) $\alpha=A\sin nt+c$ as a particular solution, we obtain

$$\left.\begin{aligned}&\alpha=A\sin\frac{2\pi}{T_0}t+\alpha_0,\qquad T_0=\frac{2\pi}{n}=\frac{2\pi H}{\sqrt{Mag\cdot M'a'g}},\\&\alpha_0=-\frac{H(\Omega\cos\lambda\sin\theta+V/R)}{M'a'g}.\end{aligned}\right\}\dots(4)$$

From the equation (2) and (4)

$$\beta=kA\cos\frac{2\pi}{T_0}t+\beta_0,\quad \beta_0=\frac{H\Omega\cos\lambda\cos\theta}{Mag},\quad k=\sqrt{\frac{M'a'}{Ma}}.\quad\dots(5)$$

The simultaneous equations (4) and (5) represent an ellipse (if $k>1$ its major axis is $2kA$ and if $k<1$ its major axis is $2A$) and its centre is at the distance of $\alpha=\alpha_0$ and $\beta=\beta_0$ from the origin (Fig. 3.) This ellipse is the projection on the horizontal plane of the locus traced by the upper end of the axis of the gyroscope. Specially when Ma is equal to $M'a'$ the locus becomes a circle. A and kA are determined from the initial position of the axis. If the gyroscope starts when its axis is at $\alpha=\alpha_0$, $\beta=\beta_0$ and $A=0$, the axis will keep the same position. As α_0 and β_0 are to be evaluated from the elements of the design of the gyroscope, the course and the speed of the ship, we can determine the vertical line in the ship by the position of the axis of the gyroscope even if the ship rolls or inclines.

Now we divide α_0 into two parts

$$\alpha_0=\alpha_0'+\alpha_0'', \quad \alpha_0'=-\frac{H\Omega\cos\lambda\sin\theta}{M'a'g}, \quad \alpha_0''-=\frac{H\cdot V/R}{M'a'g}. \qquad \ldots(6)$$

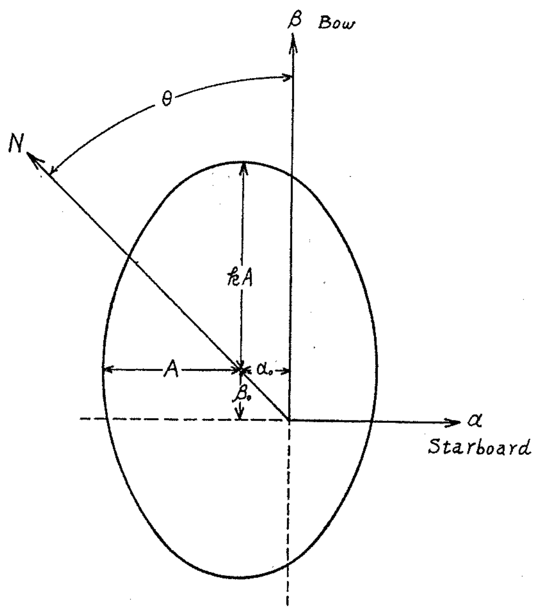
α_0 is determined by the latitude and the course of the ship and α_0'' by the speed. If $M'a'=500$ cm. g., $H=1{,}13\cdot10^8$ C. G. S., then

$\alpha_0'\doteqdot-(57\cos\lambda\sin\theta)'$, $\alpha_0''\doteqdot-(0.063V)'$ (V in knots).

If we substitute $\alpha=\alpha_0$ and $\beta=\beta_0$ as the initial conditions, and $\theta=0$ in the equations (4) and (5), then

$\alpha=\alpha_0''\doteqdot0$, $\beta=\dfrac{H\Omega\cos\lambda}{M'a'g}$, i. e. the axis rests inclining its upper

Fig. 3



end to the north by β, which may be called the latitude error. In such case the axis is really making a precessional motion so as to incline its upper end to the east by the torque $Mag\cdot\beta=H\Omega\cos\lambda$ and this precessional motion is cancelling the inclination due to the earth's rotation. Therefore if we lay a weight m on the case of the gyroscope at a point south of the centre of the axis, and make m satisfy the equation $mlg=H\Omega\cos\lambda$, where l is the distance between m and the centre of the axis, then the axis will rest vertically. Thus if we change ml properly according to the change of λ and turn the gyroscope by using a gyro-compass, so that the point of action of m on the case of the gyroscope is always to the south, then the equations (4) and (5) become

$$\alpha_0=0, \quad \beta_0=0.$$

These are obvious too from the equation of motion (1). When, by putting m, we supply to the gyroscope the torques

$mlg(\cos\theta\cos\beta+\sin\theta\sin\alpha\sin\beta)$ about X-axis, $mlg(\sin\theta\cos\alpha+\cos\theta\sin\alpha\sin\beta)$ about Y-axis, the equations of motion become

$$\frac{H^2}{Mag}\frac{d^2\alpha}{dt^2}+M'a'g\cdot\alpha+H\frac{V}{R}=0, \qquad \frac{H^2}{M'a'g}\frac{d^2\beta}{dt^2}+Mag\cdot\beta=0.$$

(B) The influence of the external forces:—As explained in the former article the end of the axis of the gyroscope traces the arc of an ellipse when it inclines for any cause. Disregarding this elliptic motion by assuming that the time interval during which the external force is acting on the gyroscope is negligibly small compared with the period of the elliptic motion, we consider only the inclination which is directly caused by the influence of the external forces.

(a) Change of the speed of the ship. When the speed of the ship increases or decreases the acceleration acts upon the gyroscope in a direction parallel to the longitudinal axis of the ship and the gyroscope makes a precessional motion parallel to the transversal axis. If we denote the acceleration by γ, then from equation (1)

$$-H\frac{d\alpha}{dt}Ma\gamma, \qquad \therefore \quad \alpha_1=-\frac{Ma}{H}\int_{t_1}^{t_2}\gamma dt,$$

where α_1 is the maximum inclination caused by the change of the speed. Supposing that $\gamma=\frac{V_2-V_1}{t_2-t_1}$[1], we obtain by integration

$$\alpha_1=-\frac{Ma}{H}(V_2-V_1), \quad \dots\dots\dots(7)$$ when the speed varies

from V_1 to V_2 in a time interval (t_2-t_1). For example, in the case of $Ma=500$ cm. g., $H=1{,}13\cdot10^8$ C. G. S., $V_2-V_1=20$ knots, we get $\alpha_1 \doteqdot 15'$.

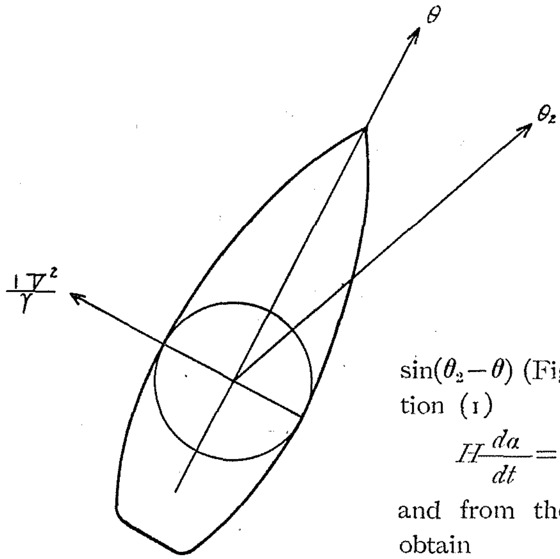
(b) Turning of the ship. Here we shall inquire into the influence of the centrifugal force which act upon the gyroscope during the turning under the following conditions.

1. The decrease of the speed during the turning is neglected.
2. The turning circle has a uniform radius r.
3. The time needed to turn is negligibly small compared with the period T_0.
4. $Ma=M'a'$.

When the ship turns at a speed V, the centrifugal force acts upon

1. If $t_2-t_2 \ll T_0$ then we can assume that γ is constant.

Fig. 4



the gyroscope parallel to the transversal axis of the ship. In our case the X- and Y-axes are fixed to the ship, and they turn with the ship. The component of the moment of the centri-fugal force in the final course θ_2 is $Ma\frac{V_2}{r}\sin(\theta_2-\theta)$ (Fig. 4), therefore from equation (1)

$$H\frac{d\alpha}{dt} = -Ma\frac{V^2}{r}\sin(\theta_2-\theta),$$

and from the conditions 1 and 2 we obtain

$$\alpha = -\frac{MaV}{H}\int_{\theta_1}^{\theta_2}\sin(\theta_2-\theta)d\theta = \frac{MaV}{H}\{\cos(\theta_2-\theta_1)-1\}, \quad \ldots(8a)$$

where θ_1 and θ_2 are the initial and final course respectively.. Similarly

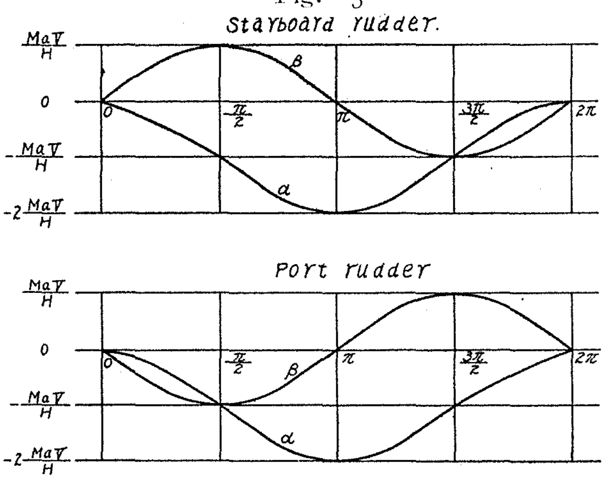
$$\beta = \frac{MaV}{H}\sin(\theta_2-\theta_1). \quad \ldots(8b)$$

The variations of α and β with the turning angle $\theta_2-\theta_1$ are as shown in Fig. 5. In the case of $Ma=500$ cm. g., $H=1{,}13\cdot10^8$ C. G. S., $V=30$ knots, the maximum values of α and β are

$$2\frac{MaV}{H} \doteqdot 46',$$

$$\frac{MaV}{H} \doteqdot 23'.$$

Fig. 5

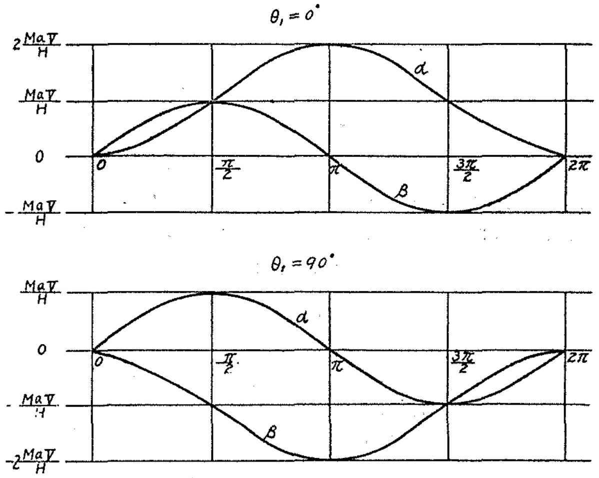


By the turning, if we rotate the gyroscope properly so that its axis of α-inclination coincides with the meridian, then

$$\alpha = \frac{MaV}{H}(\cos\theta_1 - \cos\theta_2), \quad \beta = \frac{MaV}{H}(\sin\theta_2 - \sin\theta_1) \quad \ldots(8')$$

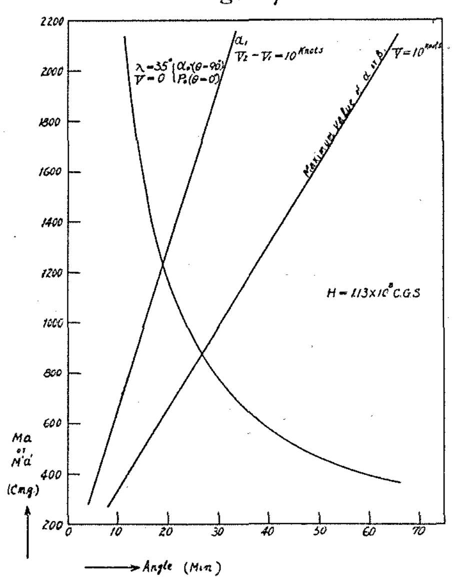
The variations of α and β with the turning angle in the cases of $\theta_1 = 0^0$ and $\theta_1 = 90^0$ are as shown in Fig. 6. If we compare these two cases, we see that in the former α and β depend upon the turning angle and are independent of the initial and final courses, but in the latter they vary with them, and that in both cases they depend only upon the speed and not at all upon the radius of turning or its angular velocity.

Fig. 6



(c) Rolling and pitching and vibration. If the amplitude of rolling, pitching or vibration is strictly a sine or cosine function of time, then the larger the period of the proper os cillation of the gyroscope in comparison with their period, the smaller their influence on the gyroscope. Actually the relationship is not so simple, but their influence is different according to the construction of the gyroscope, and it would appear to be possible to avoid their influence by properly designing the mechanism of the gyroscope. The most important constants of the gyroscope are its angular momentum and the coefficient of moment of gravitation (Ma and $M'a'$). We know from equation (3) that $M'a'g$ is the coefficient

Fig. 7



of stability in regard to Y-axis (α-inclination). Hence the greater Ma and $M'a'$, the smaller α_0 and β_0, i. e. the gyroscope becomes more stable. But equations (7), (8) and (8') show that the inclination caused by a change of speed or course is proportional to Ma or $M'a'$. The period of the oscillation varies with Ma and $M'a'$, and as before stated, it is affected by the rolling or the vibration. Therefore the determination of the value of Ma, $M'a'$ and H is very important when we make use of a gyroscope as an artificial horizon installation. The curves in Fig. 7 represent the variation of α_0, β_0, α_1 and β_1 with Ma and $M'a'$.

(C) Comparison of our gyroscope with the gyro-compass :—The axis of the gyroscope of a gyro-compass is horizontally constrained and its period of undamped oscillation is given by

$$T_0' = 2\pi\sqrt{\frac{H^2}{Ma\,g\cdot\Omega\cos\lambda}}.$$

Comparing T_0' with our $T_0 = 2\pi\sqrt{\dfrac{H^2}{Ma\,g\cdot M'a'}}$ we see that T_0' changes its value with the latitude λ but T_0 does not.

Next let us compare simply the inclinations of the two gyroscopes, which are caused by the same external forces.

Latitude. Though α_0' (in equation (6)) seems to correspond to the latitude error of the Sperry gyro-compass (the Anschütz gyro-compass avoids this error), it is essentially different, because the latitude error of the gyro-compass $-\left(\dfrac{D}{Ma\,g}\right)\tan\lambda$ is proportional to the damping factor D, and vanishes when $D=0$.

Speed. Both α_0'' (in equation (6)) and the speed error δ of the gyro-compass contain a factor of speed V. It is evident from the equations $\alpha_0'' = \dfrac{H\cdot V}{Ma\,g\cdot R}$ and $\delta = \dfrac{V\cos\theta}{R\Omega\cos\lambda}$ that α_0'' varies with the constants Ma and H and is independent of the course, and δ is independent of the constants and varies with the course.

Change of speed and centrifugal force. The acceleration error and the centrifugal force error of the gyro-compass result from the torque of acceleration with respect to the NS-axis. Therefore they vary with the course of the ship. But with our gyroscope these errors are independent of the course.

Rolling and pitching. In the rolling and pitching errors of the gyro-compass there is a quadrantal error[2], but not with our gyroscope.

2. When the angular moment a with respect to the NS-axis and the EW-axis are not equal, the rolling and pitching errors are maximum in the courses NE, NW, SE and SW, and minimum in the courses N, S, E and W.

If the speed of the ship changes from V_1 to V_2 the change of the settling point of the axis of the gyroscope is $-\frac{H(V_2-V_1)}{Mag\cdot R}$ (in equation (6)) and the acceleration is $-\frac{Ma}{H}(V_2-V_1)$ (in equation (7)), and their directions are the same. Now if we equate them, then we have $\left(\frac{H}{Ma}\right)^2=Rg$, moreover if $Mag=M'a'g$, then

$T_0=2\pi\frac{H}{Mag}$, i. e. $T_0=2\pi\sqrt{\frac{R}{g}}$. If T_0 is equal to the period of a simple pendulum whose length is equal to the earth's radius, then the inclinations a_1 and a_0'' cancel each other (in this case T_0 is about 85 minutes), as in the case of the gyro-compass.

2. Damped oscillation

If we neglect the friction at the bearings and subjoin the other damping arrangement, so as to make a torque, proportional to a, with respect to the transversal axis acting on the axis of the gyroscope, then the equations (2) become

$$P'a=-H\left(\frac{d\beta}{dt}+Q\right),\quad P\beta-Da=H\left(\frac{da}{dt}+S\right),\qquad \text{where } D=\text{const.},$$

$P=Mag$, $P'=M'a'g$, $Q=\Omega\cos\lambda\sin\theta+V/R$, $S=\Omega\cos\lambda\cos\theta$. If $D^2<4PP'$, then

$$a=A_1e^{-\frac{D}{2H}t}\sin\frac{2\pi}{T}t+a_0,\quad \beta=kA_1e^{-\frac{D}{2H}t}\cos\left(\frac{2\pi}{T}t-\xi\right)+\beta_0,$$

$$T=\frac{2\pi H}{\sqrt{Mag\cdot M'a'g-D^2/4}},\quad k=\sqrt{\frac{M'a'}{Ma}},$$

$$\xi=\cos^{-1}\sqrt{1-\frac{D^2}{4Mag\cdot M'a'g}},\quad a_0=-\frac{H(\Omega\cos\lambda\sin\theta+V/R)}{M'a'g},$$

$$\beta_0=\frac{H\Omega\cos\lambda\cos\theta}{Mag}-\frac{DH(\Omega\cos\lambda\sin\theta+V/R)}{Mag\cdot M'a'g}.$$

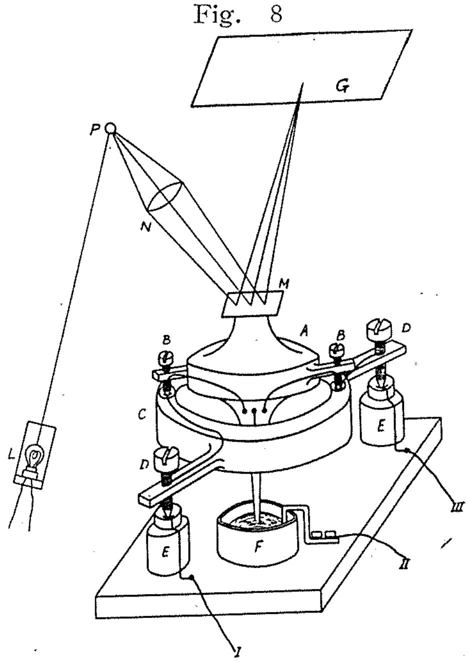
If we compare these results with those of the undamped oscillation, we see that a_0 is the same in both cases, β_0 is smaller in the present case by $\frac{DH(\Omega\cos\lambda\sin\theta+V/R)}{Mag\cdot M'a'g}$, the period becomes greater and there is the phase difference ξ between a and β.

II. Experiments

1. Arrangement

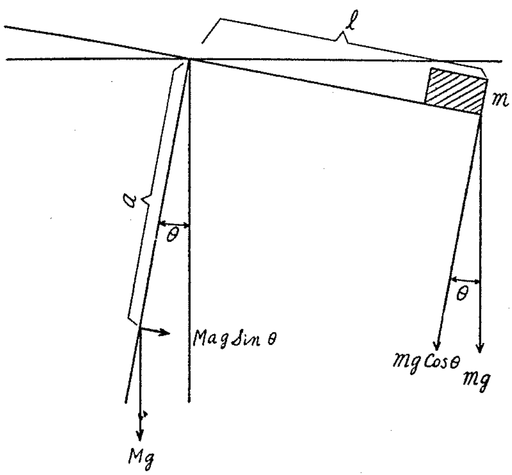
Fig. 8 is a sketch which shows the arrangements. The gyroscope in the case A is vertically held by two ball-bearings, the upper and the lower. The case A is held by the ring C, being pivoted by the end points of the steel screws B, B and the sapphire bearings. The ring C is also held by the props E, E, being pivoted by the end points of the steel screws D, D and the sapphire bearings. An exciting coil is fixed inside the case A and an induction coil is fixed to the gyroscope. Through the exciting coil the three phase current of 330 cycles and 120 volts passes. The current from I, II and III are supplied to the exciting coil by means of mercury contained in the concave parts of the sapphire bearings D, D, B, B and a glass vessel F. The light ray from a lamp L is reflected by a sphere of glass full of mercury (it acts as a point source). The reflected ray is projected by a lens N as a light spot on the lower surface of a glass plate G, after being reflected by a plain mirror M on the top of the case of the gyroscope.

Fig. 8



The motion of the projected light spot is recorded on a sheet of semi-transparent section paper placed on the upper surface of G. Thus we can investigate the variation of the inclination of the axis of the gyroscope. The distance between M and G is 111 cm., therefore 1 mm. on the section paper corresponds to 1.5 minutes of the angle of inclination of the axis. The values

Fig. 9.

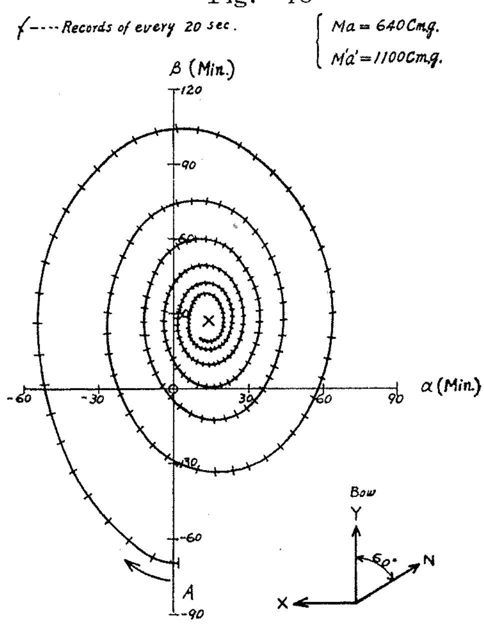


of Ma and $M'a'$ can be adjusted by means of the screws B, B and D, D. They are measured as follows. Having stopped the revolution of the gyroscope, we place a small body of known weight on the case (or on the ring C) and measure the inclination of the axis of the gyroscope by the movement of the light spot on the plate G. Let the weight of the small body be m, the distance of the small body from the axis of inclination be l and the angle of the inclination thus produced be θ (Fig. 9). Then $mlg\cos\theta = Mag\sin\theta$, $\therefore$ $Ma = ml\cot\theta$. From the above equation we can obtain Ma (or $M'a'$).

2. Results

After the revolution of the gyroscope has become steady, we give an inclination to its axis by pushing it with a finger, and record the position of the light spot on the section paper at equal time intervals. Fig. 10 is an example of the records. The light spot starts from A and moves along the curve in the direction indicated by the arrow and comes to rest at the point X.

Fig. 10



Now we compare the experimental results thus obtained with those theoretically derived, by taking Ma = 640 cm. g., $M'a'$ = 1100 cm. g. which are measured by the above-mentioned method.

(A) Period :—In Fig. 10 the points on the spiral represent the positions of the light spot at every twenty seconds. With this spiral the period is measured as about 840 seconds. Now from the equation

$$T = \frac{2\pi H}{\sqrt{Mag.M'a'g - D^2/4}}$$

the period is calculated as 865 seconds (by our gyroscope $H = 1.13 \cdot 10^8$ C. G. S.). Fig. 11 shows the values of the empirical period and the curve obtained from the theoretical period.

(B) α_0 **and** β_0 :—The point O in Fig. 10 represents the position of the axis when it is exactly vertical, which is determined from the position of the light spot when the gyroscope is at rest. The point X

Fig. 11

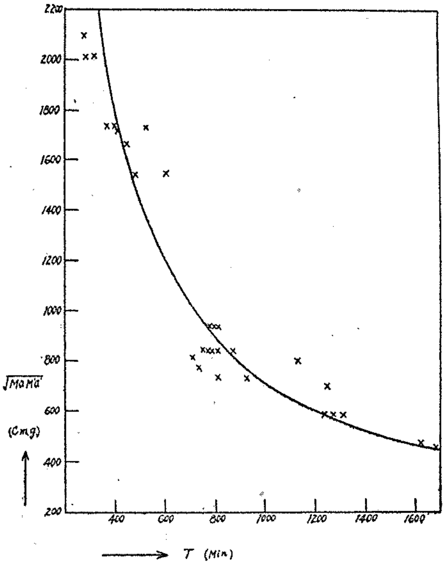
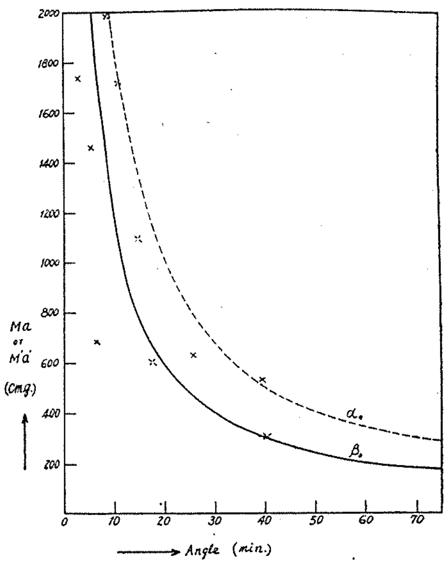


Fig. 12



(i. e. α_0 and β_0), the point of rest of the light spot, is measured as $\alpha_0 = 14.5'$ and $\beta_0 = 26.5'$, O being the origin. On the other hand the values calculated from the equation (9) are $\alpha_0 = 18.4'$ and $\beta_0 = 22.0'$. The comparison of the measured values of α_0 and β_0 with the curves obtained with the theoretical values of α_0 and β_0 is shown in Fig. 12.

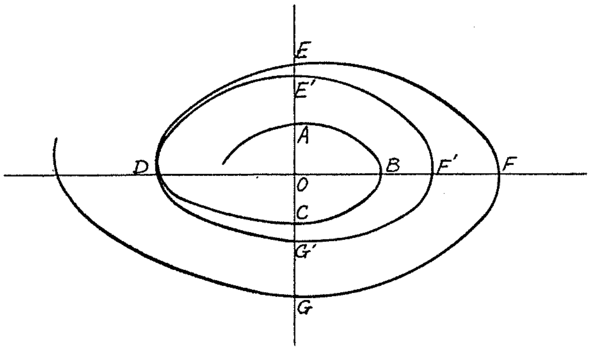
(C) ξ **and** D :—The ratio of the successive amplitudes of α is

$$\left| \frac{\alpha_{n+1}}{\alpha_n} \right| = e^{-\frac{D}{2H}(t_{n+1}-t_n)}$$ by equation (9). Therefore

$$D = -\frac{2H}{t_{n+1}-t_n} \log \left| \frac{\alpha_{n+1}}{\alpha_n} \right|.$$ Measuring $t_{n+1}-t_n$ and $\left| \frac{\alpha_{n+1}}{\alpha_n} \right|$ in Fig. 10 and substituting $H = 1{,}13 \cdot 18^8$ C. G. S., we obtatin $D = 1{,}17 \cdot 10^5$. By substitution of D in equation (9) we also obtain $\xi = 9.8$ sec. The measurement in Fig. 10. gives $\xi = 12$ sec.

(D) k :—In Fig. 13 the spiral ABC... is the damping curve and E', F' and G' are defined as $EE' = \frac{1}{4} AE$, $FF' = \frac{1}{2} BF$ and $GG' = \frac{3}{4} CG$. If we measure $DO/E'O$, $F'O/G'O$, etc. in Fig. 10, assuming that $DE'F'G'$ is an ellipse of the undamped oscillation, then their mean

Fig. 13



value k measures 1,31. By calculation from equation (9) we get $k=1,31$. But as $DO/E'O$, $F'O/G'O$, etc. are the values of k in the case where the damping rate is proportional to the lapse of time, the above comparison is no more than a rough estimate. The results are shown in following table.

	α_0	β_0	ξ	T	k
Measured values.	14.5	26.5	12.0 sec.	840 sec.	1.31
Calculated values.	18.4	22.0	9.6 sec.	865 sec.	1.31

In conclusion, the writer wishes to express his sincere thanks to Prof. T. Nomitsu of Kyoto Imperial University for the interest he has taken in the above research.