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Fuzzy Boundary Value Problems Concerning Fredholm Equations and Nagumo's Conditions

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1 Introduction

There have been many fruitful results on representations of fuzzy numbers, differentials and integrals of fuzzy functions. The authors established fundamental results concerning differentials (e.g., [5, 6, 7, 8, 9, 10, 19, 11, 12, 22]), integrals (e.g., [1, 20]), the existence and uniqueness of solutions for initial value problems of differential equations (e.g., [15, 16, 18, 23, 24]), the asymptotic behaviours of solutions (e.g., [3, 4, 13, 14, 17, 21]). In this study we introduce the parametric representation corresponding to the results due to Goetschel-Voxman so that it is easy to analyze fuzzy differential equations. By the representation we can discuss differential, integral of fuzzy functions and asymptotic behaviours of solutions for fuzzy differential equations in an analogous way to the theory of ordinary differential equations. In a similar way we treat fuzzy differential equations with fuzzy boundary conditions.

Our aim is to discuss the existence and uniqueness of solutions for the following boundary value problems of fuzzy differential equations:

$$x''(t) = f(t, x, x'), \quad x(a) = A, x(b) = B. \tag{1.1}$$

Here $J = [a, b] \subset \mathbf{R} = (-\infty, +\infty)$, $t \in J$, and fuzzy numbers $A, B \in \mathcal{F}_{\mathrm{b}}^{st}$,which is a set of fuzzy numbers with compact supports and strictly fuzzy convexity, and $f : J \times \mathcal{F}_{\mathrm{b}}^{st} \times \mathcal{F}_{\mathrm{b}}^{st} \to \mathcal{F}_{\mathrm{b}}^{st}$ is an $\mathcal{F}_{\mathrm{b}}^{st}$−valued function.

Denote $I = [0, 1]$. In what follows a fuzzy number x is characterized by a membership function μ_x which has four properties. We consider a set of fuzzy numbers with compact supports denoted by $\mathcal{F}_{\mathrm{b}}^{st}$:

Definition 1

$$\mathcal{F}_{\mathrm{b}}^{st} = \{\mu_x : \mathbf{R} \to I \ \textit{satisfying} \ \mathrm{(i)} - \mathrm{(iv)} \ \textit{below}\}.$$

(i) *There exists a unique* $m \in \mathbf{R}$ *such that* $\mu_x(m) = 1$*;*

(ii) $supp(\mu_x) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ *is bounded in* $\mathbf{R}$*;*

(iii) μ_x *is strictly fuzzy convex on* $supp(\mu_x)$*;*

(iv) μ_x *is upper semi-continuous on* $\mathbf{R}$.

A function μ_x is called strictly fuzzy convex (quasi-concave) on $supp(\mu_x)$ if

$$\mu_x(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu_x(\xi_1), \mu_x(\xi_2)] \tag{1.2}$$

for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in J$ such that $\xi_1 \neq \xi_2$. In usual case a fuzzy number x satisfies fuzzy convex on $\mathbf{R}$, *i.e.,*

$$\mu_x(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu_x(\xi_1), \mu_x(\xi_2)]$$

for $0 \leq \lambda \leq 1$ and $\xi_1, \xi_2 \in \mathbf{R}$. Denote $L_\alpha(\mu_x) = \{\xi \in \mathbf{R} : \mu_x(\xi) \geq \alpha\}$. When the membership function is fuzzy convex, then we have the following remarks.

Remark 1 *The following statements are equivalent eaxh other provided with (i) of Definition 1.*

(1) μ_x *is fuzzy convex on* $\mathbf{R}$;

(2) $L_\alpha(\mu_x)$ *is convex with respect to* $\alpha \in I$;

(3) $\mu_x(\alpha)$ *is non-decreasing in* $\alpha \in (-\infty, m]$ *and non-increasing in* $\alpha \in [m, +\infty)$, *respectively;*

(4) $L_\alpha(\mu_x) \subset L_\beta(\mu_x)$ *for* $\alpha > \beta$.

In the similar way as [9, 10] we consider the following parametric representation of $\mu_x \in \mathcal{F}_b^{st}$ such that

$$x_1(\alpha) = \min L_\alpha(\mu_x), \quad x_2(\alpha) = \max L_\alpha(\mu_x)$$

for $0 < \alpha \leq 1$ and that

$$x_1(0) = \min supp(\mu_x), \quad x_2(0) = \max supp(\mu_x).$$

We denote a fuzzy numbers x by (x_1, x_2), *i.e.*, $x = (x_1, x_2)$. Because of Condition (iii) and (iv), x_1, x_2 are functions defined on I. However, when the membership function is fuzzy convex, x_1, x_2 are not necessarily defined on I.

We treat fuzzy type of Nagumo's Condition to (1.1) and give the existence and uniqueness theorems to (1.1) by parametric representation of fuzzy numbers. Moreover we show applications of fuzzy type of Nagumo's condition to the Fredholm equation concerning (1.1) by applying the contraction principle and Schauder's fixed point theorem.

2 Parametric representation of fuzzy functions

In [21] we showed that a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve in the $\mathbf{R}^2$ space as follows.

Theorem 1 *Denote* $x = (x_1, x_2) \in \mathcal{F}_b^{st}$, *where* $x_1, x_2 : I \to \mathbf{R}$ *are the left and right end-points of the membership function* μ. *Then it follows that the following properties* (i)-(iii) *hold:*

(i) $x_i \in C(I), i = 1, 2$. *Here* $C(I)$ *is the set of all the continuous functions on* I;

(ii) *there exists a unique* $m \in \mathbf{R}$ *such that*

$$x_1(1) = x_2(1) = m, \; x_1(\alpha) \leq m \leq x_2(\alpha)$$

for $\alpha \in I$;

(iii) *one of the following statements* (a) *and* (b) *holds;*

(a) x_1 *is non-decreasing and* x_2 *is non-increasing. There exists a positive* $c \leq 1$ *such that* $x_1(\alpha) = m = x_2(\alpha)$ *for* $\alpha \in [c, 1]$ *and that* $x_1(\alpha) < x_2(\alpha)$ *for* $\alpha \in (0, c)$;

(b) $x_1(\alpha) = x_2(\alpha) = m$ *for* $\alpha \in (0, 1]$.

Conversely, under the above conditions (i) -(iii), *if we denote*

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \tag{2.3}$$

Then μ_x *is the membership function of* x, *i.e.*, $\mu \in \mathcal{F}_b^{st}$.

When the membership function is strictly fuzzy convex, the function is fuzzy convex and (2)-(4) of Remark1 hold.

Proof. (i) Let $x = (x_1, x_2) \notin \mathbf{R}$. Let $\lim_{n\to\infty} \alpha_n = \alpha_0$ for $\alpha_0 \in I$. Denote $A_1 = \liminf_{n\to\infty} x_1(\alpha_n)$. We shall prove that $A_1 \geq x_1(\alpha_0)$. Suppose that $A_1 < x_1(\alpha_0)$. Then for any sufficiently small $\varepsilon > 0$ there exist a number ℓ such that $A_1 - \varepsilon < x_1(\alpha_\ell) < A_1 + \varepsilon < x_1(\alpha_0)$. Denote

$$\begin{aligned} M &= \{\alpha \in I : x_1(\alpha) = x_2(\alpha) = m\}, \\ S(c) &= \{\alpha \in I : x_1(\alpha) = c \text{ on some interval}\} \text{ for } c \in \mathbf{R}. \end{aligned}$$

There are the three cases as follows;

(a) $\alpha_0 \in M$; (b) $\alpha_0 \in S(c)$ for some c ; (c) $\alpha_0 \notin M \cup S(c)$ for any c.
In case of (a) we consider two cases: (a1) α_0 is an interior point of M, *i.e.*, there exists a sufficiently small number $\delta >$ such that the neighborhood $U_\delta(\alpha_0) \subset M$; (a2) α_0 is a isolated point. In (a1) it follows that $m < A_1 + \varepsilon < m$,which leads o a contradiction. In (a2) there exist two integers $p < q$ such that

$$|x_1(\alpha_q) - A_1| < 1/q < |x_1(\alpha_p) - A_1| < 1/p.$$

Then $\min L_{\alpha_q}(\mu) = x_1(\alpha_q) < x_1(\alpha_p) = \min L_{\alpha_p}(\mu) < m$ and this means that $L_{\alpha_p}(\mu) \subset L_{\alpha_q}(\mu)$ and $L_{\alpha_p}(\mu) \neq L_{\alpha_q}(\mu)$. On the other hand $L_{\alpha_p}(\mu) \supset L_{\alpha_q}(\mu)$ because $\alpha_p < \alpha_q < 1$. This leads to a contradiction.

In case of (b) the point α_0 is an interior point of $S(c)$, *i.e.*, there exists a sufficiently small number $\delta >$ such that the neighborhood $U_\delta(\alpha_0) \subset S(c)$. Then $c = x_1(\alpha_\ell) < A_1 + \varepsilon < c$, which means a contradiction.

In case of (c), by Relation (3) of Remark1, $x_1(\alpha)$ is strictly monotonously incerasing in α. Consider a sequence $\{\varepsilon_n > 0\}$ such that $\varepsilon_n > \varepsilon_{n+1} > 0$ and that $\varepsilon_n \to +0$ as $n \to \infty$. Then

$$\alpha_\ell = \mu(x_1(\alpha_\ell)) < \mu(A_1 + \varepsilon_1) < \mu(x_1(\alpha_0)) = \alpha_0,$$

which contradicts with $\lim_{n\to\infty} \alpha_n = \alpha_0$. Therefore $A_1 \geq x_1(\alpha_0)$ and x_1 is lower semi-continuous. In the same way x_1 is upper semi-continuous and x_1 is continuous on I. It can be seen that $x_2(\alpha)$ is continuous on I by the same discussion.

(ii) It is clear that the uniquess of m and that $x_1(1) = m = x_2(1)$. Since the membership is fuzzy convex, it follows that $x_1(\alpha) \leq m \leq x_2(\alpha)$ for $\alpha \in I$.

(iii) Let M be defined in (i). In case that $M = (0,1]$, we have $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in (0,1]$. This means that (iiib) holds. In case that $M \neq (0,1]$, because of the continuity of x_1, x_2, denoting $c = \inf M$, it follows that $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in [c,1]$ and that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in (0,c)$, which means that (iiia) holds.

Conversely (2.3) means that the upper level set $L_\beta(\mu)$ satisfies $L_\beta(\mu) = [x_1(\beta), x_2(\beta)] \subset \mathbf{R}$ for $\beta \in I$. From (2.3) it follows that if $\xi \in [x_1(\alpha), x_2(\alpha)]$ then $\mu(\xi) \geq \alpha$ and that $\xi \notin [x_1(\mu(\xi)+\varepsilon), x_2(\mu(\xi)+\varepsilon)]$ for each $\varepsilon > 0$. Then it can be seen that $[x_1(\beta), x_2(\beta)] \subset L_\beta(\mu)$. When $\mu(\xi) = \beta$, from (2.3), it follows that a $\xi \in [x_1(\beta), x_2(\beta)]$. When $\mu(\xi) > \beta$, then there exists an $\alpha \in I$ such that $\xi \in [x_1(\alpha), x_2(\alpha)]$ and $\alpha \geq \beta$, which means that $\xi \in [x_1(\alpha), x_2(\alpha)] \subset [x_1(\beta), x_2(\beta)]$. Therefore we have $L_\beta(\mu) = [x_1(\beta), x_2(\beta)]$.

From (2.3) it is immediately seen that (i) and (ii) of Definition1 hold. The $\alpha-$ cut set $L_\alpha(\mu)$ is closed for $\alpha \in I$, i.e., the function μ is upper semi-continuous on $\mathbf{R}$. For $\alpha \in I$, $L_\alpha(\mu)$ is convex, i.e., the function μ is fuzzy convex on $\mathbf{R}$ See, e.g., [25].

From (2.1), $\mu(\xi) = \overline{\alpha}$ means that $\xi = a(\overline{\alpha})$ or $\xi = b(\overline{\alpha})$. If suppose that $a(\overline{\alpha}) < \xi < b(\overline{\alpha})$, which means that $\mu(\xi) > \overline{\alpha}$. Suppose that there exist $\xi_1, \xi_2 \in J$ and λ such that $\xi_1 \neq \xi_2, 0 < \lambda < 1$ and $\mu(\xi_3) = \mu(\overline{\xi})$, where $\xi_3 = \lambda\xi_1 + (1-\lambda)\xi_2$ and $\mu(\overline{\xi}) = \min[\mu(\xi_1), \mu(\xi_2)]$. Then we have $\xi_3 \neq \overline{\xi}$ and $\xi_3 = a(\mu(\overline{\xi}))$ or $\xi_3 = b(\mu(\overline{\xi}))$, i.e., $a^{-1}(\xi_3) = \mu(\overline{\xi})$ or $b^{-1}(\xi_3) = \mu(\overline{\xi})$. Thus we get, from (2.1), $\overline{\xi} = a(\mu(\overline{\xi})) = a(a^{-1}(\overline{\xi})) = \xi_3$ or $\xi_3 = b(b^{-1}(\overline{\xi})) = \overline{\xi}$. This leads to a contradiction. Therefor μ_x is strictly fuzzy convex.

Q.E.D.

Let a metirc in $\mathcal{F}_{\mathrm{b}}^{st}$ be

$$d(x,y) = \sup_{\alpha \in I}(|x_1(\alpha) - y_1(\alpha)| + |x_2(\alpha) - y_2(\alpha)|)$$

for $x = (x_1, x_2), y = (y_1, y_2)$.

Theorem 2 *The metric space $(\mathcal{F}_{\mathrm{b}}^{st}, d)$ is complete.*

Proof. Let a Cauchy sequence $\{x_k = (x_1^{(k)}, x_2^{(k)}) \in \mathcal{F}_{\mathrm{b}}^{st} : k = 1, 2, \cdots\}$. It suffieces that there an fuzzy number $x_0 \in \mathcal{F}_{\mathrm{b}}^{st}$ such that $\lim_{n\to\infty} d(x_n, x_0) = 0$. Since $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, from the well-known the Cauchy's theorem in Calculus, there exists an limit $x_0 = (x_1^{(0)}, x_2^{(0)}) \in C(I) \times C(I)$ such that the following properties(i)-(iv) hold.

(i) $\lim_{k\to\infty} d(x_k, x_0) = 0$;

(ii) $x_1^{(0)}$ and $x_2^{(0)}$ are non-decreasing, non-increasing on I, respectively;

(iii) $x_1^{(0)}(\alpha) \leq m \leq x_2^{(0)}(\alpha)$ for $\alpha \in I$ and $x_1^{(0)}(1) = m = x_2^{(0)}(1)$.

Suppose that there exists a number $n \neq m$ such that $x_1(1) = x_2(1) = n$. This contraicts with the uniform convergence of the Cauchy's squence. Thus a unique $m \in \mathbf{R}$ satisfies Theorem1(ii). Denote $C = \{\alpha \in I : x_1^{(0)}(\alpha) = x_2^{(0)}(\alpha) = m$ and $\alpha > 0\}$. In case when $C = (0,1]$, we get $x_1^{(0)}(\alpha) = x_2^{(0)}(\alpha) = m$ for $0 < \alpha \leq 1$, which means that Theorem1(iiia) holds. In case $C \neq (0,1]$, by the continuity of x_1, x_2, there exists a real number c such that $0 < c \leq 1$ and that c satisfies the following statements (1) and (2) hold.

(1) $x_1(\alpha) = x_2(\alpha)$ for $\alpha \in [c, 1]$; (2) $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in (0, c)$.

This means that Theorem1(iiib) holds. Therefore, $x_0 \in \mathcal{F}_{\mathbf{b}}^{st}$ and the metric space $(\mathcal{F}_{\mathbf{b}}^{st}, d)$ is complete. **Q.E.D.**

Denote a fuzzy function $x = (x_1, x_2) : J \to \mathcal{F}_{\mathbf{b}}^{st}$ has a variable $t \in J$ and the parameter $\alpha \in I$ such that x_1, x_2 are functions defined on $J \times I$ to $\mathbf{R}$. A fuzzy function $x = (x_1, x_2)$ is called differentiable at t if there exists an fuzzy number $\eta \in \mathcal{F}_{\mathbf{b}}^{st}$ such that (d1) $x(t+h) = x(t) + h\eta + o(h)$ and (d2) $x(t) = x(t-h) + h\eta + o(h)$ as $h \to +0$. Here $o(h) = (o_1(h), o_2(h))$, *i.e.*, $\lim_{|h| \to 0} \frac{d(o(h),0)}{|h|} = 0$. Denote $x'(t) = \eta$. It's called a differential coefficient of the Hukuhara-differentiation (See [19]). It can be seen that $x = (x_1, x_2)$ is called differentiable at t if and only if $x_1(\cdot, \alpha), x_2(\cdot, \alpha)$ are differentiable at t for any $\alpha \in I$ and there exists $\eta \in \mathcal{F}_{\mathbf{b}}^{st}$ satisfying the above (d1) and (d2).

A fuzzy function $x = (x_1, x_2)$ is called integrable over $[t_1, t_2]$ if x_1, x_2 are integrable over $[t_1, t_2]$ for any $\alpha \in I$. Define

$$\int_{t_1}^{t_2} x(s)ds = \{(\int_{t_1}^{t_2} x_1(s,\alpha)ds, \int_{t_1}^{t_2} x_2(s,\alpha)ds)^T \in \mathbf{R}^2 : \alpha \in I\}.$$

3 Fredholm equations arising from fuzzy boundary problems

Assume that $f : J \times \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_{\mathbf{b}}^{st}$ is continuous. Consider the following Fredholm equation

$$x(t) = w(t) + \int_a^b G(t,s) f(s, x(s), x'(s)) ds$$

for $t \in J$. Here a fuzzy function $w \in C(J)$ and an $\mathbf{R}$-valued function $G \in C(\mathbf{R}^2)$ with $G(t,s) \geq 0$ such that

$$w(t) = \frac{A(b-t) + B(t-a)}{b-a}, \tag{3.4}$$

$$G(t,s) = \begin{cases} \frac{(b-t)(s-a)}{b-a} & (a \leq t \leq s \leq b) \\ \frac{(b-s)(t-a)}{b-a} & (a \leq s \leq t \leq b) \end{cases} \tag{3.5}$$

Then we get $w''(t) \equiv 0$ and also $w(a) = A$, $w(b) = B$. It follows that

$$\int_a^b G(t,s)ds \leq \frac{(b-a)^2}{8}, \quad \int_a^b \frac{\partial G}{\partial t}(t,s)ds \leq \frac{b-a}{2}.$$

In the same way in theory to boundary value problems of ordinary differential equation the following proposition are shown immediately.

Proposition 1 *Fuzzy function x is a continuously differentiable solution of* (1.1) *if and only if x is a fixed point of $T : C^1(J; \mathcal{F}_{\mathbf{b}}^{st}) \to C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$ such that*

$$[T(x)](t) = w(t) + \int_a^b G(t,s) f(s, x(s), x'(s)) ds.$$

Here $C^1(J; \mathcal{F}_{\mathbf{b}}^{st})$ is the set of continuously differentiable functions defined on J to $\mathcal{F}_{\mathbf{b}}^{st}$, etc.

In the same way in applying the contraction principle [17] gets the existence and uniquess theorem of (1.1).

Theorem 3 *Suppose that There exist positive numbers K, L such that*

$$d(f(t,x,y), f(t,u,v)) \leq Kd(x,u) + Ld(y,v) \tag{3.6}$$

for $t \in J$ and $x, y, u, v \in \mathcal{F}_b^{st}$ and that

$$\frac{K(b-a)^2}{8} + \frac{L(b-a)}{2} < 1. \tag{3.7}$$

Then (1.1) *has one and only one solution in $C^2(J; \mathcal{F}_b^{st})$.*

We illustrate the above theorem as follows.

Example 1 *Let fuzzy numbers $k = (k_1, k_2), \ell = (\ell_1, \ell_2)$ in $\mathcal{F}_b^{st}$ with $k_1(\alpha) \geq 0, \ell_1(\alpha) \geq 0$ for $\alpha \in I$ and $k_2(0) \leq K, \ell_2(0) \leq L$, respectively. Assume that positive real numbers K, L satisfy the inequality* (3.6) *and $p_i \geq 0, q_i \geq 0$ for $i = 1, 2$. We consider fuzzy functions $f = (f_1, f_2)$ of $(t, x, y) \in J \times \mathcal{F}_b^{st} \times \mathcal{F}_b^{st}$ with $x = (x_1, x_2).y = (y_1, y_2)$ such that*

$$f_i(t,x,y,\alpha) = k_i(\alpha)e^{-p_i t}x_i(\alpha) + \ell_i(\alpha)e^{-q_i t}y_i(\alpha)$$

for $\alpha \in I, i = 1, 2$. Then, for any boundary values $(A, B) \in \mathcal{F}_b^{st} \times \mathcal{F}_b^{st}$, there exists a unique solution for (1.1).

4 Fuzzy type of Nagumo's condition

Assume that the following properties (i) -(iii).

(i) Function $f = (f_1, f_2) : J \times \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} \to \mathcal{F}_b^{st}$ is continuous. Here (f_1, f_2) is the parametric representation of f.

(ii) Let $r_i > 0, i = 1, 2$. There exists a function $h_i : [0, \infty) \to [0, \infty)$ such that

$$|f_i(t,x,y,\alpha)| \leq h_i(|y_i(\alpha)|)$$

for $t \in J, \alpha \in I, i = 1, 2$, and $|x_i(\alpha)| \leq r_i$, $y = (y_1, y_2) \in \mathcal{F}_b^{st}$. Here $x = (x_1, x_2), y = (y_1, y_2)$ are parametric representations of x, y, respectively.

(iii) Assume that h_i, $i = 1, 2$, satisfy

$$\int_{+0}^{\infty} \frac{\eta d\eta}{h_i(\eta)} > 2r_i.$$

The above condition is applied to the fuzzy boundary value problem (1.1) in the same way as [2].

Lemma 1 *Assume that $f = (f_1, f_2)$ satisfies fuzzy type of Nagumo's condition. Let $r_i > 0$, $i = 1, 2$, be in fuzzy type of Nagumo's condition and a solution $x = (x_1, x_2) \in C^2(J; \mathcal{F}_b^{st})$ of* (1.1) *satisfy $|x_i(t, \alpha)| \leq r_i$ for $i = 1, 2, t \in J, \alpha \in I$.*

There exist numbers $N_i > 0, i = 1, 2$ such that $|x_i'(t, \alpha)| \leq N_i$ for $t \in J, \alpha \in I$.

Fuzzy type of Nagumo's condition concerning $x'' = 0$

In what follows we consider fuzzy type of Nagumo's condition concerning $x'' = 0$. Assume that the following properties (i) -(iii).

(i) $f = (f_1, f_2) : J \times \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} \to \mathcal{F}_b^{st}$ is continuous.

(ii) Let $r_i > 0, i = 1,2$, and w is the function in (3.4). There exists a function $h_i : [0,\infty) \to [0,\infty)$ such that

$$|f_i(t,x,y,\alpha)| \le h_i(|y_i(\alpha) - w_i'(t,\alpha)|)$$

for $t \in J, \alpha \in I, i = 1,2$, and

$$|x_i(\alpha) - w_i(t,\alpha)| \le r_i,\ y = (y_1,y_2) \in \mathcal{F}_{\mathbf{b}}^{st}.$$

Here $x = (x_1,x_2), y = (y_1,y_2)$ are parametric representations of x, y, respectively.

(iii) Assume that $h_i,\ i = 1,2$, satisfy

$$\int_{+0}^{\infty} \frac{\eta d\eta}{h_i(\eta)} > 2r_i. \tag{4.8}$$

Lemma 2 *Assume that there exist functions $h_i, i = 1,2$, satisfy (4.8) and (4.9). Let $r_i > 0,\ i = 1,2$, be in (4.9) and a solution $x = (x_1,x_2) \in C^2(J;\mathcal{F}_{\mathbf{b}}^{st})$ of (1.1) satisfy*

$$|x_i(t,\alpha) - w_i(t,\alpha)| \le r_i$$

for $i = 1,2, t \in J$ and $\alpha \in I$.

There exist numbers $N_i > 0, i = 1,2$, such that

$$|x_i'(t,\alpha) - w_i'(t,\alpha)| \le N_i$$

for $t \in J, \alpha \in I$.

In cases where $h_i(\eta) = \eta, h_i(\eta) = \eta^2$ for $\eta \ge 0$ it suffices that N_i satisfies $N_i > 2r_i, N_i > 0$, for (4.9), respectively.

5 Applications of fuzzy type of Nagumo's condition

In this section we show the existence of solutions for (1.1) by applying Schauder's fixed point theorem as well as we give the existence and uniqueness of solutions by applying the contraction principle under assumption that Nagumo's condition concerning $x'' = 0$. Let $r = (r_1,r_2)$ and $N = (N_1,N_2)$. Denote

$$S_w(r,N) = \{(x,y) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} : |x_i(\alpha) - w_i(t,\alpha)| \le r_i, |y_i(\alpha) - w_i'(t,\alpha)| \le N_i, \text{ for } i = 1,2, t \in J, \alpha \in I\}.$$

Theorem 4 *Assume that the same conditions of Lemma 2 hold. Let*

$$|f_i(t,x,y,\alpha)| \le \min(\frac{2N_i}{b-a}, \frac{8r_i}{(b-a)^2})$$

for $t \in J, (x,y) \in S_w(r,N), i = 1,2, \alpha \in I$.

Then (1.1) has at least one solution x such that $(x(t),x'(t)) \in S_w(r,N)$ for $t \in J$ and any $A, B \in \mathcal{F}_{\mathbf{b}}^{st}$.

[2] show Nagumo's condition of $\mathbf{R}^n$, but they give no theorems of existence of solutions for boundary value problems.

In the following theorem we get the existence and uniqueness of solutions for (1.1).

Theorem 5 *Assume that the same conditions of Theorem 3 hold. Assume that there exist integrable functions $p_1,p_2 : J \to [0,\infty)$ such that for $t \in J, i = 1,2, (x,y), (u,v) \in S_w(r,N)$,*

$$|f_i(t,x,y,\alpha) - f_i(t,u,v,\alpha)| \le p_i(t)(d(x,u) + d(y,v))$$

and

$$\lambda = \sup_{t\in J}\int_a^b G(t,s)p_1(s)ds + \sup_{t\in J}\int_a^b \frac{\partial G}{\partial t}(t,s)p_2(s)ds < 1. \tag{5.9}$$

Then (1.1) has one and only one solution in $C^2(J;\mathcal{F}_{\mathbf{b}}^{st})$ such that $(x(t),x'(t)) \in S_w(r,N)$ for $t \in J$ and any $A, B \in \mathcal{F}_{\mathbf{b}}^{st}$.

Example 2 *Denote fuzzy numbers* $k = (k_1, k_2), \ell = (\ell_1, \ell_2), m = (m_1, m_2), n = (n_1, n_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ *such that all* $k_1(0), \ell_1(0), m_1(0), n_1(0)$ *are non-negative. Denote integrable and non-negative functions* a_i, b_i, c_i, d_i *defined on* $[0, \infty)$ *for* $i = 1, 2$. *Assume that* $a_1(t) \leq a_2(t), b_1(t) \leq b_2(t), c_1(t) \leq c_2(t), d_1(t) \leq d_2(t)$ *for* $t \in J$. *Let*

$$\begin{aligned} f_i(t, x, y, \alpha) &= k_i(\alpha)a_i(t)|x_1(\alpha) - w_1(t, \alpha)| + \ell_i(\alpha)b_i(t)|x_2(\alpha) - w_2(t, \alpha)| \\ &\quad + m_i(\alpha)c_i(t)[y_1(\alpha) - w_1'(t, \alpha)]^2 + n_i(\alpha)d_i(t)|y_2(\alpha) - w_2'(t, \alpha)| \end{aligned}$$

for $t \in J, x = (x_1, x_2), y = (y_1, y_2) \in S_w(r, N), i = 1, 2$.

Assume that r_i, N_i *for* $i = 1, 2$ *satisfy the following conditions* (i) - (ii).

(i) *There exist* p_1, p_2 *such that* (5.10) *and that*

$$\max(k_1(1)a_1(t), \ell_1(1)b_1(t), 2N_1m_1(1)c_1(t), n_1(1)d_1(t)) \leq p_1(t)),$$
$$\max(k_2(0)a_2(t), \ell_2(0)b_2(t), 2N_2m_2(0)c_2(t), n_2(0)d_2(t)) \leq p_2(t)$$

for $t \in J$.

(ii) *Suppose that*

$$\sup_{t\in J} p_i(t)(r_1 + r_2 + N_1^2 + N_2) \leq \min(\frac{8r_i}{b-a}, \frac{2N_i}{(b-a)^2})$$

for $i = 1, 2$ *and that* $N_1 > 0, N_2 > 2r_2$.

We get $h_1(\eta) = \eta^2, h_2(\eta) = \eta$ *for* $\eta \geq 0$. *It follows that* $\int_{+0}^{\infty}(\eta/h_1(\eta))d\eta = \int_{+0}^{\infty}(\eta/h_2(\eta))d\eta = \infty$. *Then conditions of Theorem 4 are satisfied. Therefore, by Theorem* 4, (1.1) *has one and only one solution in* $S_w(r, N)$ *for any* $(A, B) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$.

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